Anglo-Chinese School

(Independent)

PRELIMINARY EXAMINATION 2022

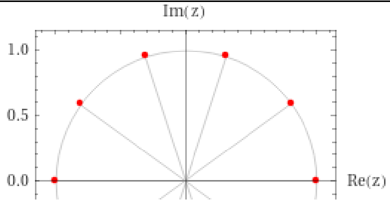
YEAR 6 IB DIPLOMA PROGRAMME

MATHEMATICS : ANALYSIS AND APPROACHES

HIGHER LEVEL

Paper 3

Qn	Solution
1.	
(a)	$\omega^{10}=\left(e^{i\frac{\pi}{5}}\right)^{10}=e^{i2\pi}=\cos 2\pi+i\sin 2\pi=1$ Hence $\omega=e^{i\frac{\pi}{5}}$ is a root of the above equation
	Comments: Generally well done.
(b)	$(\omega^n)^{10}=\left(\omega^{10}\right)^n=1$ so ω^n is a root of $z^{10}=1$ Any root $\omega^n=\omega^{10p+k}=\left(\omega^{10}\right)^p\omega^k=\omega^k$ Any integer power of ω can be reduced exclusively to integer power of k only. Since there are 10 values of k, this leads to 10 distinct complex number arguments.
	Comments: poorly done. Many did not comprehend the question.

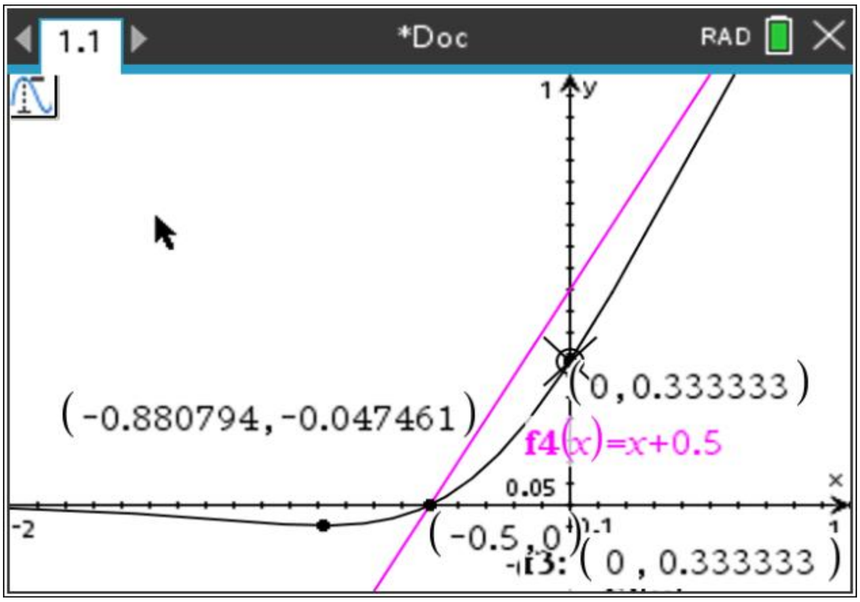
Qn	Solution
(c)	 Using the Argand diagram, it can be observed that there is a total of 6 equally spaced roots and thus $\alpha = \frac{\pi}{5}$,
	Comments: Some draw argand diagram with wrong angles and lack of symmetry. Some did not label correctly according to what is required in the question.
(d)	Let $\sin\alpha = b$, $\sin 2\alpha = c$, $\cos\alpha = d$ and $\cos 2\alpha = e$ Considering the symmetry of the Argand Diagram about the Imaginary axes, it can be observed that $\sin 4\alpha = b$, $\sin 2\alpha = c$, $\cos 4\alpha = -d$ and $\cos 3\alpha = -e$ Therefore $\sin 4\alpha = \sin\alpha$, $\cos 4\alpha = -\cos\alpha$ and $\sin 3\alpha = \sin 2\alpha$. Hence $\tan 3\alpha \tan 4\alpha = \frac{\sin 3\alpha}{\cos 3\alpha} \times \frac{\sin 4\alpha}{\cos 4\alpha}$ $= \frac{\sin 2\alpha}{-\cos 2\alpha} \times \frac{\sin\alpha}{-\cos\alpha}$ $= \frac{2\sin\alpha\cos\alpha}{-\left(2\cos^2\alpha - 1\right)} \times \frac{\sin\alpha}{-\cos\alpha}$ $= \frac{2\sin^2\alpha}{2\cos^2\alpha - 1}$ $= \frac{2\tan^2\alpha}{2 - \sec^2\alpha}$ $= \left(\frac{2\tan\alpha}{1 - \tan^2\alpha}\right)\tan\alpha$ $= \tan\alpha\tan 2\alpha$

Qn	Solution
	Alternative Solution for proving $\tan 3\alpha \tan 4\alpha = \tan\alpha \tan 2\alpha$ Using the Argand Diagram Let gradient $\tan\alpha = m_1$ then by symmetry, $\tan 4\alpha = -m_1$ Similarly, let gradient $\tan 2\alpha = m_2$ then $\tan 3\alpha = -m_2$ $\tan 3\alpha \tan 4\alpha = (-m_1)(-m_2)$ $= m_1 \times m_2$ $= \tan\alpha \tan 2\alpha$
	Comments: Surprising number did not utilize the argand diagram effectively and gave long winded workings to arrive at same result. Some misread the question and miss out the need to prove the trigo identities and lost marks then. Students generally quite weak in trigonometry. A few just used GDC to give approximate value and not the exact value needed for proof.
(e)	DE = $\Phi - 1$ **Or using** $\frac{CD}{ED} = \frac{AB}{ED} = \Phi$ $ED = \frac{1}{\Phi}$ Since AB = 1 **(alternative form of solution accepted)**
	Comments: Generally well done.

Qn	Solution
(f)	$\frac{1}{\Phi}=\Phi-1$ (resulting from property of Golden Rectangle given in question or from the fact that DE can be expressed in the two forms above) $\Phi^2-\Phi-1=0$ Using quadratic formula, $\Phi=\frac{1\pm\sqrt{1-4(-1)}}{2}$ $=\frac{1\pm\sqrt{5}}{2}$ Since Φ is the ratio of lengths, we reject the negative answer and therefore $\Phi=\frac{1+\sqrt{5}}{2}$ (proven)
	Comments: Common careless mistake in not rejecting the negative root.
(g)	Using compound angle formula, double angle formula and appropriate simple trigonometric identities, Starting from $\sin 3\alpha=\sin 2\alpha$, $\sin 3\alpha=\sin 2\alpha$ $\sin(2\alpha+\alpha)=2\sin\alpha\cos\alpha$ $\sin 2\alpha\cos\alpha+\cos 2\alpha\sin\alpha=2\cos\alpha\sin\alpha$ $2\sin\alpha\cos^2\alpha+(2\cos^2\alpha-1)\sin\alpha-2\cos\alpha\sin\alpha=0$ Since $\sin\alpha\neq 0$ $2\cos^2\alpha+(2\cos^2\alpha-1)-2\cos\alpha=0$ and rearranging, $4\cos^2\alpha-2\cos\alpha-1=0$ (shown)
	Comments: Some careless mistakes in proof or students. Many divided by the sin function which is not a good practice. Some missed out showing that $\sin\alpha\neq 0$ and hence lost a mark here.

Qn	Solution
(h)	Using quadratic formula (or otherwise) $\cos\alpha = \dfrac{2 \pm \sqrt{4-4(4)(-1)}}{8}$ $= \dfrac{2 \pm 2\sqrt{5}}{8}$ $= \dfrac{1 \pm \sqrt{5}}{4}$ As $\alpha = \dfrac{\pi}{5}$ is an acute angle $\cos\alpha = \dfrac{1+\sqrt{5}}{4} = \dfrac{\Phi}{2}$, since $\Phi = \dfrac{1+\sqrt{5}}{2}$.
	Comments: Some did not see the connection between this and the preivous part, did not use quadratic to simiplify and obtain the answer. A number of students again missed out showing that the negative angle is rejected and lost marks here.
2.	
(a)	$\dfrac{dy}{dx} = u$ $\dfrac{du}{dx} = -u + 2y$ $\dfrac{d^2y}{dx^2} \quad = \dfrac{du}{dx} \quad = -u + 2y$ $\dfrac{d^2y}{dx^2} + u - 2y = 0$ $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$
	Comments: Students must recognize that the solution of 2^{nd} order differential equations is not in the syllabus- and therefore they cannot try to solve it using 1^{st} order Differential Equations method. The question is teaching you how to find solutions to 2^{nd} order DE. Students must learn how to show a proof correctly and state clearly what you set out to prove.
(b)	$y = e^{rx}$ $\dfrac{dy}{dx} = re^{rx}$ $\dfrac{d^2y}{dx^2} = r^2e^{rx}$ $e^{rx}\left(r^2 + r - 2\right) = 0,$ Since $e^{rx} \neq 0$, $r^2 + r - 2 = 0$
	Comments: Do not simply cancel off e^{rx} Need to state $e^{rx} \neq 0$ or $e^{rx}>0$.

Qn	Solution
(c)	$(r+2)(r-1)=0$ $Therfore, r_1=-2$ or $r_2=1$
	Comments: Students should just simply factorise and not apply quadratic eqn formula or even sketch the graph.
(d)	$y=Ae^{-2x}+Be^{x}$ $\frac{dy}{dx}=-2Ae^{-2x}+Be^{x}$ $\frac{d^2y}{dx^2}=4Ae^{-2x}+Be^{x}$ $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y$ $=4Ae^{-2x}+Be^{x}+\left(-2Ae^{-2x}+Be^{x}\right)-2\left(Ae^{-2x}+Be^{x}\right)$ $=(4A-4A)e^{-2x}+(1+1-2)Be^{x}$ $=0$
	Comments: Need to state clearly what you set out to prove. Some students write the equation throughout .
(e)	$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$ $k^2e^{kx}-2ke^{kx}+e^{kx}=0$ $e^{kx}\left(k^2-2k+1\right)=0,$ $e^{kx}\neq 0$ $(k-1)^2=0$ $k=1$
	Need to write $e^{kx}\neq 0$
(f)	$y=(A+Bx)e^{x}$ $\frac{dy}{dx}=Be^{x}+(A+Bx)e^{x}=(A+B+Bx)e^{x}$ $\frac{d^2y}{dx^2}=(A+B+Bx)e^{x}+Be^{x}=(A+2B+Bx)e^{x}$ $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y$ $=(A+2B+Bx)e^{x}-2(A+B+Bx)e^{x}+(A+Bx)e^{x}$ $=(A+2B+Bx-2A-2B-2Bx+A+Bx)e^{x}=0$
	Comments: Need to show clearly what you set out to prove.

Qn	Solution
(g)	$\dfrac{(A+Bx)e^{kx}}{Ae^{r_1 x}+Be^{r_2 x}}=\dfrac{(1+2x)e^{x}}{e^{-2x}+2e^{x}}$ 
	Comments: Need to show clearly the shape of the graph, there is no straight line. Label clearly the intercepts, min point, state the equation of the horizontal asymptote y=0 (not x=0) .

Qn	Solution
(h)	Graphically, there seems to an oblique asymptote As $x \to \infty,\ y \to x + \frac{1}{2}$ $\lim\limits_{x\to\infty} \frac{(1+2x)e^{x}}{e^{-2x}+2e^{x}}$ $= \lim\limits_{x\to\infty} \frac{(1+2x)}{e^{-3x}+2}$ $= \frac{\left(1+2x\right)}{0+2}$ $= x + \frac{1}{2}$ $y = x + \frac{1}{2}$ **Alternative Solution using long division** $\begin{array}{r} x+\frac{1}{2} \\ 2+e^{-3x}\,\big)\overline{2x+1} \\ \underline{2x+xe^{-3x}} \\ 1-xe^{-3x} \\ \underline{1+\frac{1}{2}e^{-3x}} \\ \left(-\mathrm{x}-\frac{1}{2}\right)e^{-3x} \end{array}$ $\lim\limits_{x\to\infty}\left[x+\frac{1}{2}+\frac{\left(-\mathrm{x}-\frac{1}{2}\right)e^{-3x}}{2+e^{-3x}}\right] = x + \frac{1}{2}$
	Comments: Students need to show understanding of oblique asymptote, as x tends to infinity, y tends towards x+1/2.
(i)	$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0,$ $4m^2 - 4m + 1 = 0$ $\left(2m-1\right)^2 = 0$ $2m = 1$ $m = \frac{1}{2}$ Therefore the general solution is $y = \left(A + Bx\right)e^{\frac{1}{2}x}$
	Comments: This is quite poorly done- most students do not seem to understand the intend of the question, that is, teaching you how to solve a second order differential equation from the earlier parts- very few got it correct.