CHAPTER 7
DIFFERENTIAL EQUATIONS
Tutorial Solutions

Basic Mastery Questions

1. Solve the equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+x^2$.

$$x\frac{\mathrm{d}y}{\mathrm{d}x}=1+x^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}+x \quad \text{for } x\neq 0$$
$$y=\ln|x|+\frac{1}{2}x^2+C$$

2. Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-y$, given that $y<1$ and that $y=0$ when $x=0$.

$$\int\frac{1}{1-y}\,\mathrm{d}y=\int\mathrm{d}x$$
$$-\ln|1-y|=x+C$$
$$1-y=\mathrm{e}^{-x-C}, \quad \text{since } 1-y>0$$
$$1-y=A\mathrm{e}^{-x} \quad \text{where } A=\mathrm{e}^{-C}$$

When $x = 0$, $y = 0$ then $A = 1$.

$$y=1-\mathrm{e}^{-x}$$

3. Use the substitution $y=vx$, where v is a function of x, to solve the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x}=3x+y$, given that $y=0$ when $x=2$. Prove that, in the general case, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{3}{x}$.

$$x\frac{\mathrm{d}y}{\mathrm{d}x}=3x+y \quad -- \quad (1)$$

Let $y = vx \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}v}{\mathrm{d}x}x+v$

Substitute into (1): $x^2\dfrac{\mathrm{d}v}{\mathrm{d}x}+xv=3x+vx$

When $x = 2$, $y = 0$: $0=3\ln 2+C$

$\Rightarrow C=-3\ln 2$

$\therefore \quad y=3x\ln|x|-3x\ln 2$

$\qquad =3x\ln\left|\dfrac{x}{2}\right|$

From (1), $x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}=3+\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$\therefore \quad \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{3}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3}{x}$$
$$\int \mathrm{d}v = \int \frac{3}{x}\mathrm{d}x$$
$$v = 3\ln|x| + C$$
$$\therefore \quad \frac{y}{x} = 3\ln|x| + c$$

4. When a body moves freely under gravity, we know that $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -g$, where s is the height of the body above the ground at time t and g is the acceleration due to gravity. Assuming that $g = 10\ m\ s^{-2}$, find the general solution for the second-order differential equation. Given that the initial velocity is $5\ m\ s^{-1}$ and initial height is 3 m, find the particular solution for the differential equation.

$$\frac{d^2 s}{dt^2} = -10$$

(Velocity) $\frac{ds}{dt} = -10t + c$ ---(1)

$\therefore s = -5t^2 + ct + d$ ---(2) (Ans.)

When t = 0, $\frac{ds}{dt} = 5$:

$\Rightarrow c = 5$ (from (1))

When t = 0, s = 3 :

$\Rightarrow d = 3$ (from (2))

Particular solution: $s = -5t^2 + 5t + 3$