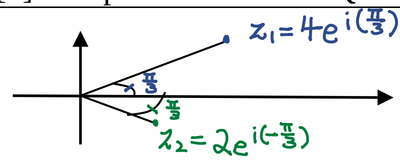
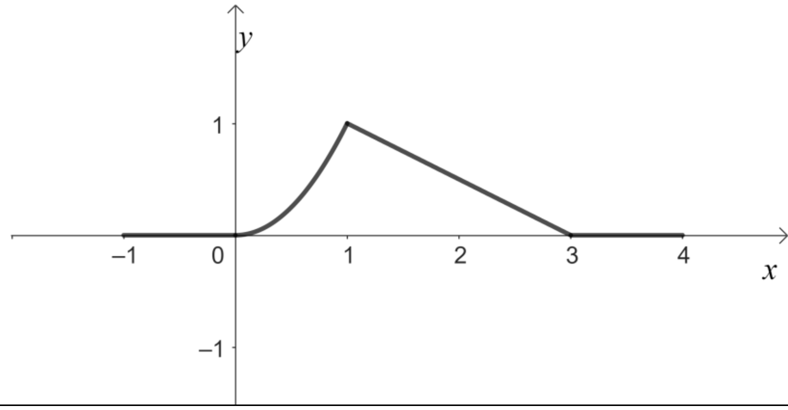
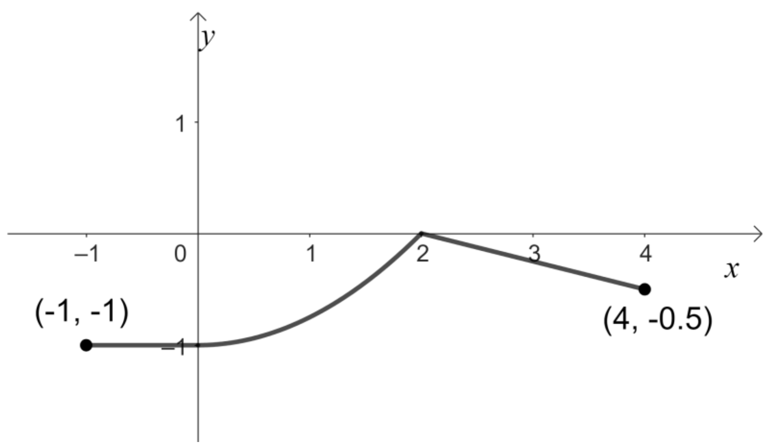
2023 H2MA Prelim Paper 2

1	Solution [5] Complex Numbers P2 Q1	
(i)	 $\lvert z_1 - z_2\rvert$ $=$ Distance between z_1 and z_2 $=\sqrt{4^2+2^2-2(4)(2)\cos\left(\frac{\pi}{3}+\frac{\pi}{3}\right)}$ - Using Cosine Rule $=\sqrt{28}$	Many students converted the z=re$^{i\theta}$ form of the complex number to the cartesian form z=x+iy first, before plotting out the points on the Argand Diagram. This probably shows that the student is not conversant with the z=re$^{i\theta}$ form. Students must be familiar with all forms of complex numbers given by a question. The r from the z=re$^{i\theta}$ form must not be seen as just a value obtained by applying a formula. Students ought to know that it measures the "distance" of complex number z from the Origin. Similarly, the θ represents the angle measured from the Real axis. Majority of students do not know how to handle complex numbers $(z_1 - z_2)$, suggesting that students lack preparation for linking complex numbers on argand diagrams to simple Vectors addition/subtraction concept.
(ii)	$z_1 = az_2 \Rightarrow a = \frac{z_1}{z_2}$	Same thing, many students solved for a by converting everything to cartesian form, not knowing that they could have just made a

	$a=\dfrac{4e^{\mathrm{i}\frac{\pi}{3}}}{2e^{-\mathrm{i}\frac{\pi}{3}}}=2e^{\mathrm{i}\frac{\pi}{3}-\left(-\mathrm{i}\frac{\pi}{3}\right)}$ $=2e^{\mathrm{i}\frac{2\pi}{3}}$ Either: z_1 is the **scaling of** z_2 **by factor of 2** and **rotating** z_2 $\dfrac{2\pi}{3}$ **anti-clockwise** about the Origin, Or: z_2 is the **scaling of** z_1 **by factor of** $\dfrac{1}{2}$ and **rotating** z_1 $\dfrac{2\pi}{3}$ **clockwise** about the Origin,	the subject immediately and working with z=r $e^{i\theta}$ form. Some students do not understand the phrase "geometrically… relationship…", leaving this part blank. For those who understand, tried answering using phrases such as "reflection", "symmetry", "half of the complex number", etc. Note that students should use words such as suggested by the solution.

2	Solution [6] P2 Sequence	
(i)	When $n=0$, $u_1-u_0=-\frac{4}{3}\left(\frac{1}{3}\right)^0+Q\left(\frac{2}{3}\right)^0$ $\frac{4}{3}-3=-\frac{4}{3}\left(\frac{1}{3}\right)^0+Q\left(\frac{2}{3}\right)^0$ $Q=\frac{8}{3}-3=-\frac{1}{3}$ and $u_2-u_1=-\frac{4}{3}\left(\frac{1}{3}\right)^1-\frac{1}{3}\left(\frac{2}{3}\right)^1$ $u_2=-\frac{4}{3}\left(\frac{1}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{4}{3}=\frac{2}{3}$	Many students made arithmetic mistakes while solving for Q and u_2. No evidence to show that students do not understand how to start this part of the question.
(ii)	$\left\|u_{n+1}-u_n\right\|=\left\|-\frac{4}{3}\left(\frac{1}{3}\right)^n-\frac{1}{3}\left(\frac{2}{3}\right)^n\right\|$ $\left\|u_{n+1}-u_n\right\|=\frac{1}{3}\left\|4\left(\frac{1}{3}\right)^n+\left(\frac{2}{3}\right)^n\right\|\le\frac{1}{3}\left\|4\left(\frac{2}{3}\right)^n+\left(\frac{2}{3}\right)^n\right\|$ $\left\|u_{n+1}-u_n\right\|\le\frac{5}{3}\left(\frac{2}{3}\right)^n$ Thus $\left\|u_{n+1}-u_n\right\|\le\varepsilon$ when $\frac{5}{3}\left(\frac{2}{3}\right)^n\le\varepsilon$ $\left(\frac{2}{3}\right)^n\le\frac{3\varepsilon}{5}$ $n\ge\frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)}$ n_0 can be $\left\lceil\frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)}\right\rceil$ or celling of $\frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)}$.	For the first half of the question, most students do not know what the question is asking for. The question can be rephrased as "...show that there can be a **maximum** value ϵ (indicated by the inequality sign), in which $\|u_{n+1}-u_n\|$ is bounded to. Subsequently, many students went on to solve for the second half of the question by using GC and listing out integer values which the inequality is satisfied.

	When $\varepsilon = 0.001$, $n \geq \dfrac{\ln\left(\dfrac{3\varepsilon}{5}\right)}{\ln\left(\dfrac{2}{3}\right)} =$ 18.29 n_0 can be 19.	

3	Solution [7] Transformation of graphs	
(i)	Graph of $y=\mathrm{f}\left(x\right)$ for $-1\le x\le 4$  	Some students still do not know that for each piece of graph, the coordinate of end-points must be clearly indicated. Otherwise, most students scored full credit for this part of the question.
(ii)	Graph of $y=\mathrm{f}\left(\frac{1}{2}x\right)-1$, for $-1\le x\le 4$  The end-points are at $(-1,-1)$ and $(4,-0.5)$ Note: When $x=4$ $y=\mathrm{f}\left(\frac{1}{2}x\right)-1=\mathrm{f}\left(2\right)-1=-\frac{1}{2}(2)+\frac{3}{2}-1=-\frac{1}{2}$	Many students show understanding for the transformation required for this question (scaling in x and translation in y). However, many students can be observed to use inappropriate approaches (such as translating and scaling each piece of graph), when they could have actually just transformed their original graph and cut off the graph at end-points.
(b)	Scale parallel to *y*-axis by factor 3. Translate 2 units in the negative *x*-direction. Reflect about the *y*-axis. OR Reflect about the *y*-axis. Translate 2 units in the positive *x*-direction.	Many students have difficulty handling the index (that is, the index 3) in the logarithm, suggesting transformation steps such as scaling by $\sqrt[3]{x}$, etc. Some students used transformation phrases such as “flipping” when they mean reflection, etc.

4	Solution [10] P2 3D Vectors	
(a)	$l_1:\mathbf{r}=\begin{pmatrix}0\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$ Let $\overrightarrow{OB}=\begin{pmatrix}0\\0\\3\end{pmatrix}$. Given that $\overrightarrow{OA}=\begin{pmatrix}1\\0\\4\end{pmatrix}$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}0\\0\\3\end{pmatrix}-\begin{pmatrix}1\\0\\4\end{pmatrix}=\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ Let $\mathbf{n}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $\pi_1:\mathrm{r}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=3$ $\pi_1:x-y-z=-3$	Generally well done. Some mistook the dot product form or the parametric form as the cartesian equation. Some included point P in their calculation of the equation leading to a wrong answer while some used Vector OA in the cross product to find normal which leads to the wrong answer.
(ii)	Q is the foot of perpendicular of P on l_1 $\overrightarrow{OQ}=\begin{pmatrix}0\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ for some λ $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}0\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}-3\\4\\5\end{pmatrix}$ $\overrightarrow{PQ}=\begin{pmatrix}3\\-4\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\overrightarrow{PQ}$ perpendicular to the line l_1 $\overrightarrow{PQ}\cdot\mathbf{d}=0$	Many are able to do this part using the zero dot product method or the projection method. Many who used the projection method misused the vector OP in their projection vector which gives a wrong answer. The projection should be from a point on the line to point P projected on the line. Some answers used the idea that the line is in the plane so Q is the foot of the perpendicular from P to the

	$\left[\begin{pmatrix}3\\-4\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=0$ $-1+2\lambda=0$ $\lambda=\dfrac{1}{2}$ $\overrightarrow{OQ}=\begin{pmatrix}0\\0\\3\end{pmatrix}+\dfrac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0.5\\0.5\\3\end{pmatrix}$	plane which is not point Q but rathe point N found in part (iii).
(iii)	l_2 parallel to π_1 $\Rightarrow \mathbf{d}_2$ is perpendicular to $\mathbf{n}_1$ $\Rightarrow \mathbf{d}_2\cdot\mathbf{n}_1=0$ $\Rightarrow \begin{pmatrix}3\\1\\m\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=0$ $\Rightarrow -3+1+m=0$ $\Rightarrow m=2$ ——— $h=$distance between l_2 and π_1 $h=$distance between $P(-3,4,5)$ and $\pi_1:\mathrm{r}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=3$ Let N be the foot of perpendicular of P on π_1. $l_{PN}:\mathbf{r}=\begin{pmatrix}-3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ --- (1) $\pi_1:\mathrm{r}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=3$ --- (2) To find N, sub (1) into (2): $\left[\begin{pmatrix}-3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=3$	Many able to find m but there are some who did not realise the fact that a line being parallel to the plane means the line is perpendicular to the normal of the plane. Many mistook point Q as the foot of the perpendicular from P to the plane. Many also mistook the length as the cross product between vector AP and $\hat{\mathbf{n}}$. Interestingly, points A, Q and N are colinear thus we get a situation where using $\overrightarrow{AP}\times\widehat{\overrightarrow{AQ}}$ gives the correct answer coincidentally although the method does not work otherwise unless point N is found.

$$\begin{pmatrix}-3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=3$$

$12+3\lambda=3$

$\lambda=-3$

$$\overrightarrow{ON}=\begin{pmatrix}-3\\4\\5\end{pmatrix}-3\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$h=\overrightarrow{PN}=\begin{pmatrix}-3\\4\\5\end{pmatrix}-3\begin{pmatrix}-1\\1\\1\end{pmatrix}-\begin{pmatrix}-3\\4\\5\end{pmatrix}=-3\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$PN=\left|-3\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|=3\left|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|=3\sqrt{3}$$

Alternatively

Note that $A(1,0,4)$ is a point on the plane

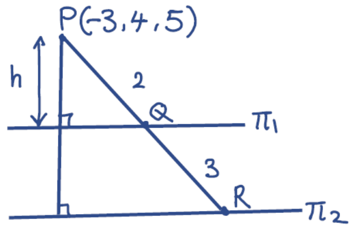
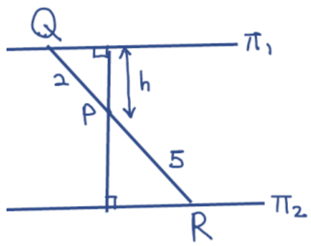
Dist of $P(-3,4,5)$ from $\pi_1:\mathrm{r}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=3$ is h.

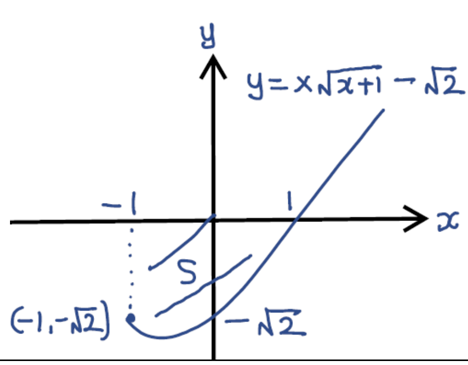
$h=\left|\overrightarrow{AP}\cdot\hat{\mathbf{n}}\right|$ where $\overrightarrow{AP}=\begin{pmatrix}-3\\4\\5\end{pmatrix}-\begin{pmatrix}1\\0\\4\end{pmatrix}=\begin{pmatrix}-4\\4\\1\end{pmatrix}$

$$h=\left|\begin{pmatrix}-4\\4\\1\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|$$

$$=\frac{9}{\sqrt{3}}$$

$$=3\sqrt{3}$$

(iv)	$PQ:PR=2:5$ Case 1: Q is between P and R Distance between π_1 and $\pi_2 = \frac{3h}{2}$  Case 2: P is between Q and R Distance between π_1 and $\pi_2 = \frac{7h}{2}$ 	Many did not manage to find the correct answer through a change in perspective as most unsuccessful solutions are searching for the answer through the use of vectors. Some managed to find one of the correct answer but did not manage to correctly identify the other situation.

5	Solution [12] Integration	
(i)	$u^2 = x+1$ when $x=-1, u=0$ $2u\frac{\mathrm{d}u}{\mathrm{d}x}=1$ when $x=a-1, u=\sqrt{a}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2u}$ $\int_{-1}^{a-1} x\sqrt{x+1}\,\mathrm{d}x$ $=\int_0^{\sqrt{a}} \left(u^2-1\right)(u)(2u)\,\mathrm{d}u$ $=2\int_0^{\sqrt{a}} u^4-u^2\,\mathrm{d}u$ $=2\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_0^{\sqrt{a}}$ $=2\left[\left(\frac{1}{5}\sqrt{a}^5-\frac{1}{3}\sqrt{a}^3\right)-\left(\frac{1}{5}(0)^5-\frac{1}{3}(0)^3\right)\right]$ $=\frac{2}{5}a^{\frac{5}{2}}-\frac{2}{3}a^{\frac{3}{2}}$	Most candidates are able to obtain the derivative $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2u}$ and use it for further substitution. However, a sizeable number of students forget to find the respective u values when $x=-1$ and $a-1$, resulting in incorrect final answer.
(ii)	C: $y = x\sqrt{x+1}$ Translate curve C, $\sqrt{2}$ units in the negative y direction, to obtain curve D. D: $y = x\sqrt{x+1}-\sqrt{2}$ Let S denote the region bounded by curve D, x-axis and $x=-1$. Volume obtained by revolving region R, 2π radians about the line $y=\sqrt{2}$, is the same as the volume obtained by revolving region S, 2π radians about the x-axis. 	Most candidates are aware that they need to translate the curve $\sqrt{2}$ units in the negative y direction and relate that the required volume can be obtained by rotating the region S (as shown) 2π radians about the x-axis. Most candidates are able to state the integral with the correct limits $\pi\int_{-1}^{1}\left(x\sqrt{x+1}-\sqrt{2}\right)^2\,\mathrm{d}x$ Some candidates are unable to proceed after expanding $\int_{-1}^{1}\left(x\sqrt{x+1}-\sqrt{2}\right)^2\,\mathrm{d}x$ to

	From GC, x-intercept happens at $x=1$. Volume of solid $=\pi\int_{-1}^{1}\left(x\sqrt{x+1}-\sqrt{2}\right)^2\ \mathrm{d}x$ $=\pi\int_{-1}^{1}x^2\left(x+1\right)-2\sqrt{2}\left(x\sqrt{x+1}\right)+2\ \mathrm{d}x$ $=\pi\int_{-1}^{1}x^3+x^2+2\ \mathrm{d}x-2\sqrt{2}\pi\int_{-1}^{1}x\sqrt{x+1}\ \mathrm{d}x$ $=\pi\left[\frac{x^4}{4}+\frac{x^3}{3}+2x\right]_{-1}^{1}-2\sqrt{2}\pi\left[\frac{2}{5}(2)^{\frac{5}{2}}-\frac{2}{3}(2)^{\frac{3}{2}}\right]$ $=\pi\left[\left(\frac{1}{4}+\frac{1}{3}+2\right)-\left(\frac{1}{4}-\frac{1}{3}-2\right)\right]-2\sqrt{2}\pi\left[\frac{8}{5}\sqrt{2}-\frac{4}{3}\sqrt{2}\right]$ $=\frac{14}{3}\pi-\frac{16}{15}\pi$ $=\frac{18}{5}\pi$ units^3	$\int_{-1}^{1}x^3+x^2+2\ \mathrm{d}x-2\sqrt{2}\pi\int_{-1}^{1}x\sqrt{x+1}\ \mathrm{d}x$ Common mistakes includes: (1): Wrongly equating $\int_{-1}^{1}\left(x\sqrt{x+1}-\sqrt{2}\right)^2\ \mathrm{d}x$ as $\int_{-1}^{1}\left(x\sqrt{x+1}\right)^2\ \mathrm{d}x$ $-\int_{-1}^{1}\left(\sqrt{2}\right)^2\ \mathrm{d}x$ (2) Unable to relate $\int_{-1}^{1}x\sqrt{x+1}\,\mathrm{d}x$ to the result in part (i) and apply accordingly i.e. $\int_{-1}^{1}x\sqrt{x+1}\,\mathrm{d}x$ $=\int_{-1}^{2-1}x\sqrt{x+1}\,\mathrm{d}x$ $=\frac{2}{5}(2)^{\frac{5}{2}}-\frac{2}{3}(2)^{\frac{3}{2}}$ (3): Unable to simplify the final expression adequately to $\frac{18}{5}\pi$

6 Two fair six-sided dice are thrown and the highest score X is recorded.

(i) State the probability distribution for X. [1]

(ii) Find the expected value and variance for X. [3]

<table>
<tr><td>(i)</td><td>

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Note highest of 2 identical scores from two throws, say 1 and 1 is 1.

However, when 2 identical scores are obtained from two throws, say 1 and 1, neither is higher than the other, therefore there is no "higher score" in this scenario.
Thus, after removing all the cases where the 2 scores are tied in the 2 throws of the dice. There are 30 outcomes when we let X denote the higher of the scores.
Reference the following table for the solution for such a definition of X.</td><td></td></tr>
<tr><td>(ii)</td><td>

$\mathrm{E}(X)$

$= (1)\left(\frac{1}{36}\right) + (2)\left(\frac{3}{36}\right) + (3)\left(\frac{5}{36}\right) + (4)\left(\frac{7}{36}\right) + (5)\left(\frac{9}{36}\right) + (6)\left(\frac{11}{36}\right)$

$= \frac{161}{36} \approx 4.472$

$\mathrm{E}(X^2) = (1)^2\left(\frac{1}{36}\right) + (2)^2\left(\frac{3}{36}\right) + (3)^2\left(\frac{5}{36}\right) + (4)^2\left(\frac{7}{36}\right) + (5)^2\left(\frac{9}{36}\right) + (6)^2\left(\frac{11}{36}\right)$

$= \frac{791}{36}$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

$= \frac{791}{36} - \left(\frac{161}{36}\right)^2$

$= \frac{2555}{1296} \approx 1.97$</td><td>There are some candidates who are unaware that $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

Candidates ought to know that for questions involving the calculation of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$, the examiners award mainly accuracy marks, as t5he concept is relatively easy to grasp.

1 marks each awarded for</td></tr>
</table>

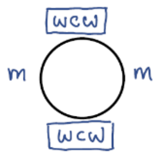
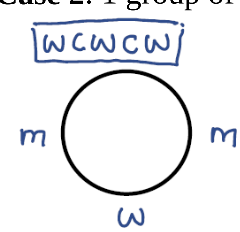
		correct Prob Distribution Table, $\mathrm{E}(X)$, $\mathrm{E}(X)^2$ and $\mathrm{Var}(X)$.

6 Two fair six-sided dice are thrown and the higher score X is recorded.

(i) State the probability distribution for X. [1]

(ii) Find the expected value and variance for X. [3]

6	Solution [5] DRV	
(i)	<table><tr><td>x</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>P($X = x$)</td><td>$\frac{2}{30}$</td><td>$\frac{4}{30}$</td><td>$\frac{6}{30}$</td><td>$\frac{8}{30}$</td><td>$\frac{10}{30}$</td></tr></table>	
(ii)	$\mathrm{E}(X)=(2)\left(\frac{2}{30}\right)+(3)\left(\frac{4}{30}\right)+(4)\left(\frac{6}{30}\right)+(5)\left(\frac{8}{30}\right)+(6)\left(\frac{10}{30}\right)$ $=\frac{14}{3}=4\frac{2}{3}$ $\mathrm{E}(X^2)=(2^2)\left(\frac{2}{30}\right)+(3^2)\left(\frac{4}{30}\right)+(4^2)\left(\frac{6}{30}\right)+(5^2)\left(\frac{8}{30}\right)+(6^2)\left(\frac{10}{30}\right)$ $=\frac{700}{30}=23\frac{1}{3}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2$ $=\frac{70}{3}-\left(\frac{14}{3}\right)^2$ $=\frac{14}{9}\approx 1.56$	

7	Solution [7] P&C P2 Q7	
(i)	Required number of ways to have WCW, with M all separated = All W separated $=(4-1)!\times 4!$ $=144$ ______________________________ Alternatively **Case 1**: 2 groups of WCW.  Number of ways to choose first group of WCW $=\binom{4}{2}\binom{2}{1}(2!)$ Number of ways to permute 2 men amongst the 2 groups of WCW $=2!$ When seating the 4 units around a circle - Let a WCW unit be seated first - This set the position of the second WCW unit - The 2 men are then slotted in between the WCW units in 2! Ways. Number of ways to arrange 4 units: 2 men and 2 groups of WCW in a circle $=2!$ Note there are 3 groups of CWC, so divide by 2! So as not to overcount Number of ways $=\dfrac{\binom{4}{2}\binom{2}{1}(2!)(2!)}{(2!)}(2)=48$ **Case 2**: 1 group of WCWCW  	For P&C, students should consider drawing a diagram to see what is going on / what is needed, before starting to apply methods or formulae. Many students tried solving by cases, and did not solve properly.

	Number of ways $=\binom{4}{3}(3!)(2!)(2!)=96$ Total number of ways = 48 + 96 = 144 Alternatively **Case 1**: 2 groups of WCW. There are $(4!)(2!)$ ways to get 2 groups of WCW. Number of ways $=\frac{2!}{2!}(4!)(2!)=48$ **Case 2**: 1 group of WCWCW There are $(4!)(2!)$ ways to get 1 group of WCWCW and a W. Number of ways $=\frac{2!}{2!}(4!)(2!)(2!)=96$ Total number of ways = 48 + 96 = 144	
(ii)	Number of ways = $\frac{8!}{4!}=1680$ since 4! Ways to arrange women if there is no restriction on them.	Many students did not read the question properly / assumed the wrong scenario. Assumptions such as the 4 women are grouped or separated were made, and thus starting the question badly.
(iii)	Number of ways = $\frac{\binom{2}{1}\binom{4}{2}\binom{2}{1}}{2!}=\frac{24}{2}=12$	Majority of students did not visualise properly that if they do not divide by a 2!-term, then they would have obtained over counting in forming their groups. Example: Suppose we have W1 to W4, C1 and C2, M1 and M2, then possible teams formed could be: #1: C1W1M1 and C2W2M2

		#2: C2W2M2 and C1W1M1 and etc. Notice that #1 and #2 are actually the same team formation.

8	Solution [7] Probability	
(i)	$\mathrm{P}(\text{Get all Bots} \cap \text{Get all number sets})$ $= \mathrm{P}(\text{box contain } \alpha \text{ and } \mathbb{R}) \times \mathrm{P}(\text{box contain } \beta \text{ and } \mathbb{Z})$ $\times \mathrm{P}(\text{box contain } \gamma \text{ and } \mathbb{Q}) \times \mathrm{P}(\text{box contain } \omega \text{ and } \mathbb{N})$ $+ \mathrm{P}(\text{box contain } \alpha \text{ and } \mathbb{Z}) \times \mathrm{P}(\text{box contain } \beta \text{ and } \mathbb{Q})$ $\times \mathrm{P}(\text{box contain } \gamma \text{ and } \mathbb{R}) \times \mathrm{P}(\text{box contain } \omega \text{ and } \mathbb{N})$ $= \left(\frac{1}{8} \times \frac{1}{8} \times \frac{1}{8} \times \frac{1}{10}\right)(4!) + \left(\frac{1}{8} \times \frac{1}{8} \times \frac{1}{8} \times \frac{1}{10}\right)(4!)$ $= \frac{24}{2560}$ $= \frac{3}{320}$ (shown)	Many students shown difficulty understanding the question, probably due to lack of exposure of probability questions phrased in this manner (given table of probabilities).
(ii)	$\mathrm{P}(\text{Get all bots} \mid \text{get all number sets})$ $= \dfrac{\mathrm{P}(\text{Get all Bots} \cap \text{Get all number sets})}{\mathrm{P}(\text{Get all number sets})}$ $= \dfrac{\frac{3}{320}}{\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times 4!}$ $= \frac{1}{10}$	Most students managed to see conditional probability and applying previous answer from (i). However, due to lack of understanding of the question, most did not manage to evaluate the probability in the denominator.
(c) (iii)	$\mathrm{P}(\text{get all bots}) = 0.3 \times 0.3 \times 0.3 \times 0.1 \times 4!$ $= 0.0648$ $\neq 0.1$ $= \mathrm{P}(\text{get all bots} \mid \text{get all number sets})$ The event that he gets all the bots and the event that he get all the number sets are not independent. OR $\mathrm{P}(\text{Get all bots} \cap \text{Get all number sets}) = \frac{3}{320}$ $\mathrm{P}(\text{Get all bots}) \times \mathrm{P}(\text{Get all number sets})$ $= (0.3^3 \times 0.1 \times (4!))\left(\frac{1}{4^4} \times (4!)\right) = 0.006048$	A handful of students made a gamble by stating independence or non-independency by "feel", as opposed to using mathematical proof and evaluating suitable probability values.

	$\mathrm{P}\left(\text{Get all bots} \cap G\text{et all number sets}\right)$ $\neq \mathrm{P}\left(\text{Get all bots }\right) \times \mathrm{P}\left(G\text{et all number sets}\right)$ The event that he gets all the bots and the event that he get all the number sets are not independent	

9	Solution [8] Binomial Dist	
(i)	The probability of a student achieving distinction in the course is a constant OR The event that a student achieves distinction is independent of the event that another student achieves distinction.	Some students stated that 'the *probability* of a student achieving distinction is independent of the *probability* of another student achieving distinction' which is wrong as it is only events that are independent.
(ii)	Mode of X is 7. $\mathrm{P}(X=6)<\mathrm{P}(X=7)$ $\binom{10}{6}\left(\frac{p}{100}\right)^6\left(1-\frac{p}{100}\right)^4<\binom{10}{7}\left(\frac{p}{100}\right)^7\left(1-\frac{p}{100}\right)^3$ $\frac{10!}{6!4!}\left(\frac{p}{100}\right)^6\left(1-\frac{p}{100}\right)^4<\frac{10!}{7!3!}\left(\frac{p}{100}\right)^7\left(1-\frac{p}{100}\right)^3$ $7\left(1-\frac{p}{100}\right)<4\left(\frac{p}{100}\right)$ $p>\frac{700}{11}$ and $\mathrm{P}(X=7)>\mathrm{P}(X=8)$ $\frac{10!}{7!3!}\left(\frac{p}{100}\right)^7\left(1-\frac{p}{100}\right)^3>\frac{10!}{8!2!}\left(\frac{p}{100}\right)^8\left(1-\frac{p}{100}\right)^2$ $8\left(1-\frac{p}{100}\right)>3\left(\frac{p}{100}\right)$ $p<\frac{800}{11}$ $\Rightarrow \frac{700}{11}<p<\frac{800}{11}$ $\Rightarrow 63.6<p<72.7\ (3sf)$	Many students overlooked that it was p% chance of achieving distinction and stated that $X\sim B(10,p)$ wrongly and thus getting the wrong answer of $\frac{7}{11}<p<\frac{8}{11}$. Some students only worked on $\mathrm{P}(X=6)<\mathrm{P}(X=7)$ or $\mathrm{P}(X=7)>\mathrm{P}(X=8)$ only and thus did not obtain the full inequality answer as needed. There were also some students who wrongly presented the mode of X = 7 by having the following: $\mathrm{P}(X=7)>1-\mathrm{P}(X\neq 7)$

<table>
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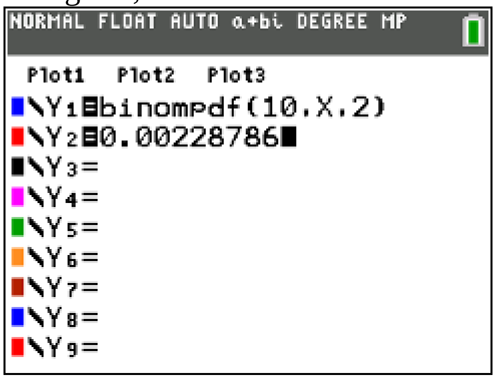
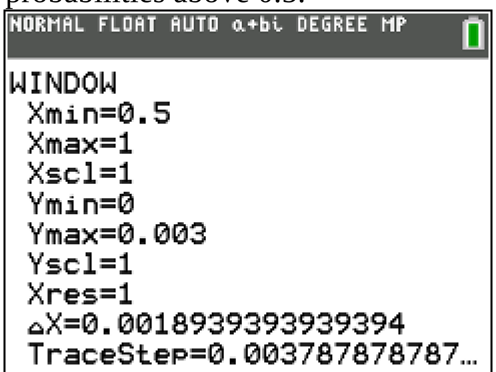
<td>(iii)</td>
<td>

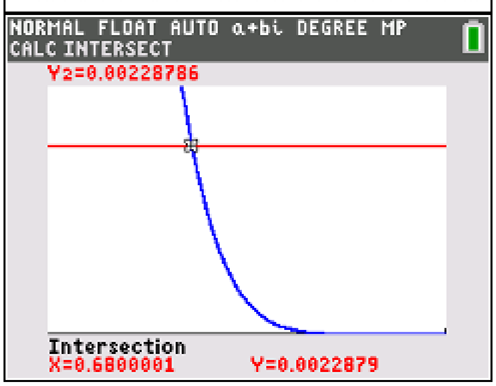
$P(X=2)=0.00228786$

Using GC,



Note that since $63.6 < p < 72.7$, we should look for probabilities above 0.5.





$\therefore \frac{p}{100}=0.68$

$\Rightarrow p=68$

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The most common error for this part is the follow through error from part (ii) that $p\%$ was overlooked again and students obtained $p=0.68$ instead of the correct $p=68$.

Another common wrong answer was $p=73.4$ as p needed to be such that $63.6 < p < 72.7$ from part (ii).

It is not correct to assume p to be a whole number and try listing the value of the binomial pdf function $y=\text{binompdf}\left(10,\frac{x}{100},2\right)$ and check for value x such that prob= 0.00228786.

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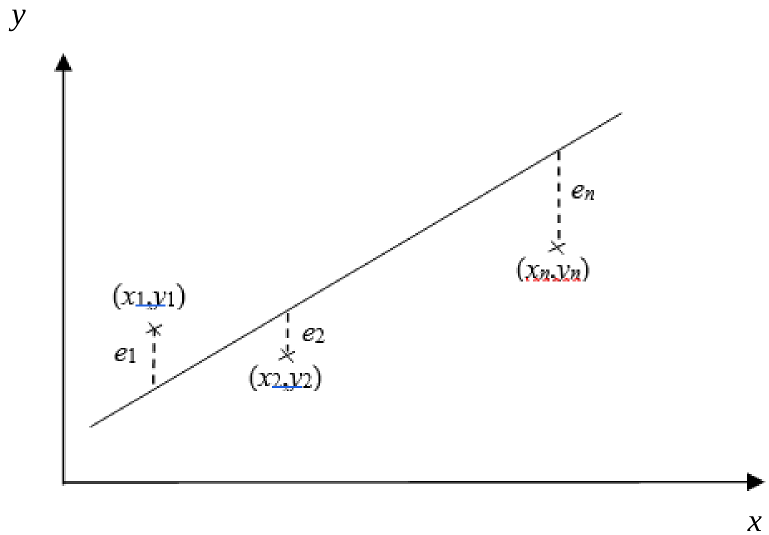
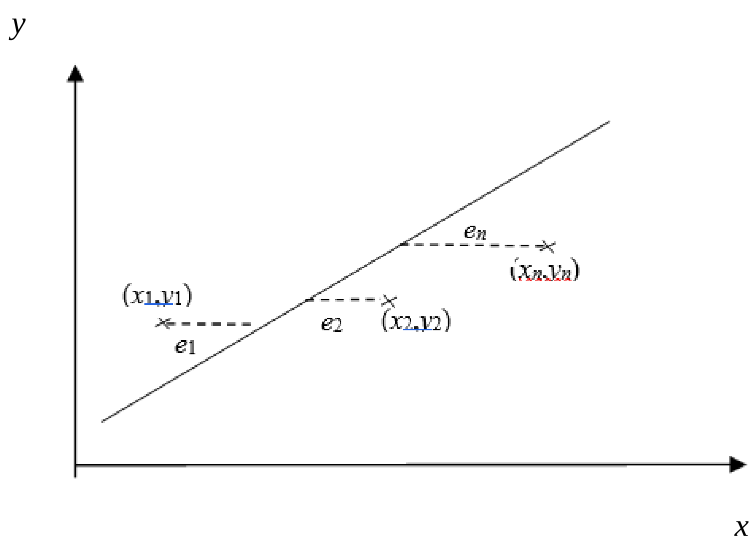
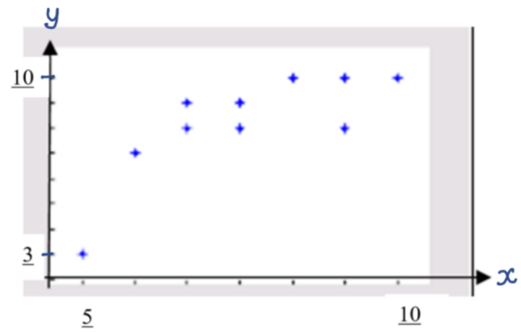
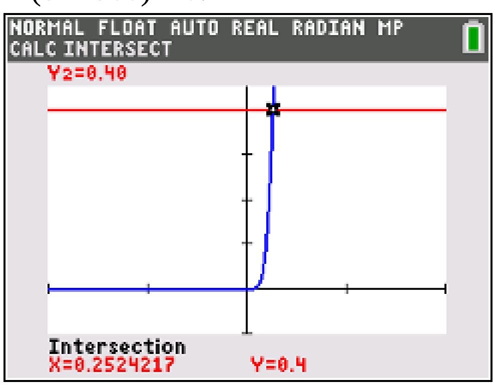
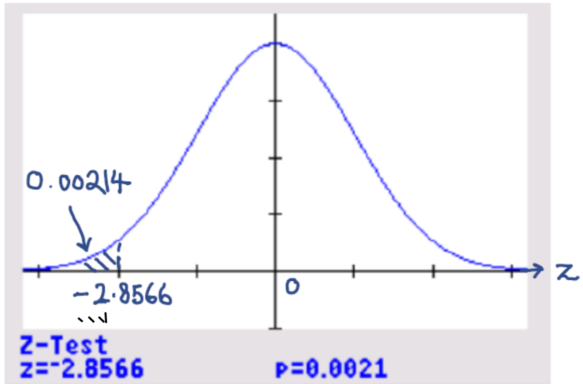
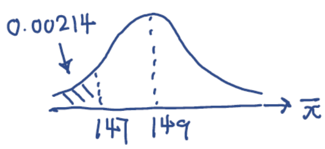
(iv)	$X \sim B(10, 0.68)$ $\mathrm{P}(X > 6) = 1 - \mathrm{P}(X \le 6) = 0.595637$ Let Y be the number of concluded courses out of 8 courses, with more than 6 students achieving distinction $Y \sim B(8, 0.595637)$ $P(Y \le 7 \mid Y \ge 5) = \dfrac{P(5 \le Y \le 7)}{P(Y \ge 5)}$ $= \dfrac{P(Y \le 7) - P(Y \le 4)}{1 - P(Y \le 4)}$ $= \dfrac{0.984156 - 0.416069}{1 - 0.416069}$ $\approx 0.973\ (3sf)$	Many students did not realise the need to find conditional probability for this part and only found $\mathrm{P}(5 \le Y \le 7)$ or $\mathrm{P}(Y \le 7)$ instead. Other common mistake include: $\mathrm{P}(X > 6) = 1 - \mathrm{P}(X \le 5)$ $\mathrm{P}(X \ge 6) = 1 - \mathrm{P}(X \le 6)$

10	Solution [10] C&R P2 Q10	
(a)	 For regression line of *y* on *x*, the <u>values of *x* are considered as accurate</u> and the <u>sum of squares of deviation in the *y*-direction is minimized.</u> For regression line of *x* on *y*, the <u>values of *y* are considered as accurate</u> and the <u>sum of squares of deviation in the *x*-direction is minimized.</u> 	Many students had difficulties in the clear explanation of the difference between the regression line of *y* on *x* and that of *x* on *y*. Common errors include: 1. Mixing up of the vertical and horizontal residue (the deviations e_i as shown in the diagrams) for both the regression lines 2. Not stating fully the meaning of each respective regression line in that the regression line of *y* on *x* considers the **minimum sum of squares of deviation in the *y*-direction** while that of the regression line *x* on *y* considers the **minimum sum of squares of deviation in the *x*-direction** 3. Wrong labelling of *x* an *y* axis for the diagram explaining regression line of *x* on *y* 4. The regression lines are reflection of each other about the *x*-axis / the line $y = x$

(b) (i)	 Not appropriate as the scatter diagram indicates that as x increases, y increases at a decreasing rate.	Some students did not label the range of values of x an y on their scatter diagram. In the explanation of whether a linear model is appropriate, it was not sufficient to just state that the scatter diagram showed a non-linear or curvilinear relation. Students should explain that as x increases, y increases at a decreasing rate.
(ii)	$y=-3.506924046+6.082293948\ln x$ $y=-3.51+6.08\ln x$ When y = 12, $12=-3.506924046+6.082293948\ln x$ $x=12.80095\approx 12.8$ ml	Generally no problem for most students except for some students mistook '$\ln x$' as 'x' in the use of the regression line to solve for predicted value of x.
(iii)	Not reliable as y = 12 is outside the input data range, $3\le y\le 10$.	Some students wrongly quoted that $x=12.8$ is outside the data range of $4\le x\le 10$ instead.
(iv)	No change in the product-moment correlation coefficient. y in cm. y' in mm Then $y=\dfrac{y'}{10}$. $\dfrac{y'}{10}=-3.506924046+6.082293948\ln x$ $y'=-35.1+60.8\ln x$	Most students were able to explain that there was no change to the value of the product-moment correlation coefficient r. Most common error in the derivation of the equation for y' was that students use the wrong substitution of $y=10y'$ or $y=\dfrac{y'}{100}$ instead.

11	Solution [12] Normal Dist P2 Q11	
(i)	Mass of a completed ornament Y = 1.05(0.9X) = 0.945X $Y \sim N(283.5, 357.21)$ $P(290 < Y < 350) = 0.365238 \approx 0.365$	Generally well done.
(ii)	Required prob $= \binom{9}{2} 0.365^2 (1-0.365)^7\, 0.365 = 0.072878121 \approx 0.0729$	Some students found this question difficult. Not everyone was able to identify how to have the third such ornament in the 10th position. Students who solved this, often used Y~B(9,0.365) Prob = P(Y=2)*(0.365)
(iii)	Let B be the mass of a box. $B = \alpha Y \sim N(283.5\alpha, 357.21\alpha^2)$ Let S be the total mass of an ornament and its box. $S = Y + B \sim N(283.5\alpha + 283.5, 357.21\alpha^2 + 357.21)$ $S \sim N(283.5(\alpha+1), 357.21(\alpha^2+1))$ $P(S > 360) = 0.4$  By GC, $\alpha = 0.252$	This question was not well done. Students were often able to find the required distribution, but not able to proceed from there. Some students misinterpreted the meaning of 40% in this context. For some students, a term of $(1+\alpha)^2$ appeared in the variance.
	If $W \sim N(130, 80^2)$, $P(W < 0) \approx 0.0521$. That is, Approx 5.21% of the blocks are of negative masses. Thus, this distribution is not appropriate. <u>Alternatively</u> 99.7% of the population should be within the range of $\mu \pm 3\sigma$ i.e. (-110, 370). However, this range contains negative values which are not possible. Thus the distribution is not appropriate.	This question was very poorly done. Many misinterpreted the question and attempted to explain why the CLT could/could not apply.

12	Solution [12] Hypothesis Testing P2 Q12	
(i)	$\bar{x} = \frac{7644}{52} = 147$ $\Sigma(x-20) = \Sigma x - 20(52) = 6604$ $s^2 = \frac{1}{51}\left[840008 - \frac{6604^2}{52}\right] = \frac{16900}{663} \approx 25.5$	Almost all students were able to find the value of $\bar{x}$ Many students struggled to find s^2. Common errors: $s^2 = \frac{52}{51}\left[\frac{840008}{52}\right]$ $s^2 = \frac{1}{51}\left[840008 - \frac{7644^2}{52}\right]$
(ii)	Test H$_0$: $\mu = 149$ against H$_1$: $\mu < 149$ at 5% level of significance. Test statistic: Under H$_0$, $Z = \frac{\bar{X} - 149}{s/\sqrt{52}} \sim N(0,1)$ approximately $z_{\text{calculated}} = -2.86$ p-value = $P(Z < z_{\text{calculated}}) = 0.00214 < 0.05$, we reject H$_0$ There is sufficient evidence at 5% level of significance that the newly developed filter is more effective.	Presentation was poor for a number of students. In some cases, this cost students marks as they omitted or incorrectly presented important steps. Generally, students were able to correct establish appropriate hypotheses and test statistic(s). Students often lost marks due to incorrect s^2 values from the previous part. A number of students were confused as to what the appropriate conclusion to the test should be. There were many variations on the standard phrasing, not all of which were appropriate. Students should be reminded that if their p-value > 0.5 it is likely they have made a mistake.
(iii)	p-value = $0.00214 = P(Z < -2.86) = P(\bar{X} < 147)$ p-value of 0.00214 refers to a probability of 0.00214 of obtaining a sample mean as extreme as the observed sample mean of 147 value given that the true population mean is 149.	Very poorly done. Common errors for the sketch: • p-value appeared along the horizontal axes as

	 Alternatively *p*-value of 0.00214 means that the least significance level the test concluding that the new filter is more effective is 0.214%. 	opposed to area under the curve. • Students often did not label their sketch with z_{calc} or $\overline{x}$ • Wrong tail was sketched • $\overline{x}$ and μ were swapped Many students were not able to define the p-value. Common errors: • Many of those who did only gave the generic definition without reference to the context in (ii). • Some students confused p-value with level of significance. • Many did not recognize that the p-value was a <u>conditionally</u> probability.
	Let Y denotes the amount of impurities in water of another town. Test $H_0: \mu = 150$ against $H_1: \mu \neq 150$ at 5% level of significance. $s^2 = \frac{100}{99}(29.85)^2$ $s \approx 30$ Test statistic: Under H_0, $Z = \frac{\overline{Y} - 150}{30/\sqrt{100}} \sim N(0,1)$. To reject Ho,	Not well done. Presentation was poor for many students. Many students did not find s^2 for this question, and just took σ = 29.85. Those who did try to find s^2 often forgot to square 29.85. Many students were unable to identify the correct critical region for the test. Some identified only one tail. Others identified two-tail with 5% at each tail. Others improperly used the CENTER option on their GC and identified two-tail with 47.5% at each tail. Others found the values of

	$\lvert z_{cal} \rvert > z_{0.975} = 1.959964$ $\left\lvert \frac{\bar{y}-150}{3} \right\rvert > 1.959964$ $\frac{\bar{y}-150}{3} < -1.959964 \quad or \quad \frac{\bar{y}-150}{3} > 1.959964$ $0 < \bar{y} < 144 \quad or \quad \bar{y} > 156$	$\bar{y}$ outside the critical region, rather than inside. Almost no one identified that $\bar{y}$ needed to be non-negative.