Name:	**Teacher**	Class	Reg No.
Date:		4 ___	

Chapter 6: Binomial Theorem
Reference Book: Additional Maths 360 Textbook, Marshall Cavendish

You will learn how to,

- Use the notations $n!$ and $\binom{n}{r}$,
- Expand $(1 + b)^n$ and $(a + b)^n$ to any number of terms, where n is a positive integer,
- Identify and find a particular term in the expansion of $(1 + b)^n$ using the result $T_{r+1} = \binom{n}{r} b^r$,
- Use the result $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ to find a particular term in the expansion of $(a + b)^n$,
- Use the general term formula to find the specific terms, coefficients and unknown values.

The factorial *n*!

The notation $n!$ is read as "n factorial". It is defined for a positive integer n as the product of the first n positive integers. In mathematical terms,

> For a positive integer n,
> $$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Recap…

Recall that $(a+b)^2 = a^2 + 2ab + b^2$.

Hence, when $a = 1$, we have $(1+b)^2 = 1 + 2b + b^2$.

In pairs, expand the following and arrange the terms in ascending powers of b.

(a) $(1+b)^3$

$$(1+b)(1+b)^2 = (1+b)(1+2b+b^2)$$
$$= 1 + 2b + b^2 + b + 2b^2 + b^3$$
$$= 1 + 3b + 3b^2 + b^3$$

(b) $(1+b)^4$

$$(1+b)^4 = (1+b)^2(1+b)^2 = (1+2b+b^2)(1+2b+b^2)$$

or $(1+b)^4 = (1+b)(1+b)^3 = (1+b)(1+3b+3b^2+b^3)$

$$(1+b)^4 = (1+b)(1+b)^3$$
$$= (1+b)(1+3b+3b^2+b^3)$$
$$= 1(1+3b+3b^2+b^3) + b(1+3b+3b^2+b^3)$$
$$= 1 + 3b + 3b^2 + b^3 + b + 3b^2 + 3b^3 + b^4$$
$$= 1 + 4b + 6b^2 + 4b^3 + b^4$$

(c) $(1+b)^5$

$$\begin{aligned}(1+b)^5 &= (1+b)^2(1+b)^3\\ &= (1+2b+b^2)(1+3b+3b^2+b^3)\\ &= 1(1+3b+3b^2+b^3)+2b(1+3b+3b^2+b^3)+b^2(1+3b+3b^2+b^3)\\ &= 1+3b+3b^2+b^3+2b+6b^2+6b^3+2b^4+b^2+3b^3+3b^4+b^5\\ &= 1+5b+10b^2+10b^3+5b^4+b^5\end{aligned}$$

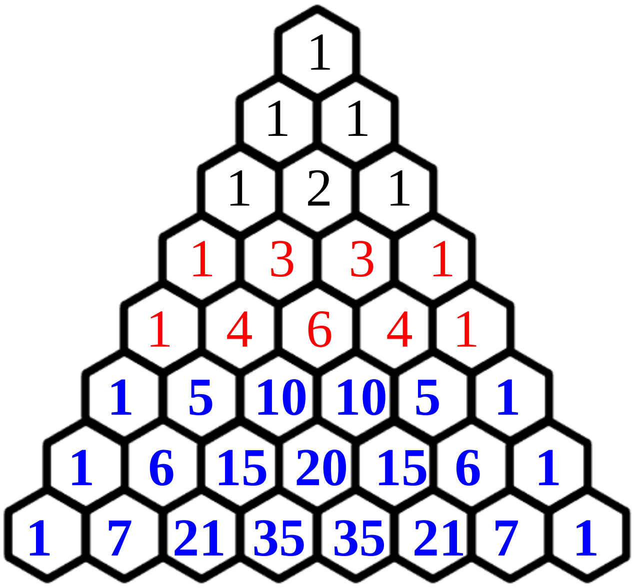
(d) $(1+b)^6$

$$\begin{aligned}(1+b)^6 &= (1+b)(1+b)^5\\ &= (1+b)(1+5b+10b^2+10b^3+5b^4+b^5)\\ &= 1(1+5b+10b^2+10b^3+5b^4+b^5)+b(1+5b+10b^2+10b^3+5b^4+b^5)\\ &= 1+5b+10b^2+10b^3+5b^4+b^5+b+5b^2+10b^3+10b^4+5b^5+b^6\\ &= 1+6b+15b^2+20b^3+15b^4+6b^5+b^6\end{aligned}$$

Is there a faster way to expand $(1+b)^n$ for all positive integer n?

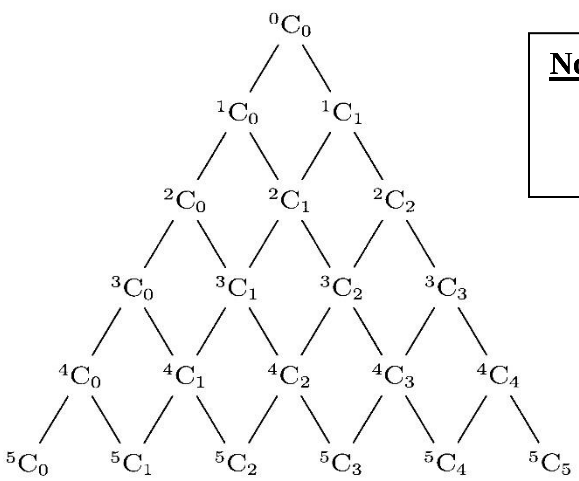
Pascal Triangle

Complete the following array by adding up the adjacent numbers in the preceding row above.



This is known as the **Pascal Triangle**, named after the mathematician, Blaise Pascal, though the ancient Chinese were known to use it before he was born. One of its many interesting properties is that each row in the array corresponds to the **coefficients** in a binomial expansion. These coefficients are also known as **binomial coefficients**.

The **Pascal Triangle** still does not give us a quick way to expand $(1+b)^n$ as it takes time to construct the triangle, especially for large values of *n*. Therefore, we introduce the notation, $\binom{n}{r}$ for the **Pascal Triangle**. $\binom{n}{r}$ is used to find the number of ways of choosing *r* items from a group of *n* different items. When *n* and *r* are both non-negative, with the exception of $\binom{0}{0}$, $\binom{n}{r}$ can also be written as ${}^{n}\mathrm{C}_{r}$. The first few rows of the **Pascal Triangle** looks like:



<u>**Note that**</u>

${}^{n}\mathrm{C}_{0}=1$ and ${}^{n}\mathrm{C}_{n}=1$.

Evaluating $\binom{n}{r}$

(a) $\binom{n}{r} = {}^{n}\mathrm{C}_r$

$$= \frac{n(n-1)(n-2)...(n-r+1)}{r!}$$

$$= \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-1)...(2)(1)}$$

$$= \frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{r-2} \times ... \times \frac{n-r+2}{2} \times \frac{n-r+1}{1}$$

e.g. $\quad {}^{6}\mathrm{C}_4 = \frac{6\times5\times4\times3}{4\times3\times2\times1} = 15$

(b) ${}^{n}\mathrm{C}_r = \frac{n!}{r!(n-r)!}$, where $r! = r\times(r-1)\times(r-2)\times...\times2\times1.$

Is $\quad {}^{n}\mathrm{C}_r = {}^{n}\mathrm{C}_{n-r}$? $\quad$ Y / N

e.g. $\quad {}^{7}\mathrm{C}_2 = \frac{7!}{2!5!} = \frac{7\times6\times5\times4\times3\times2\times1}{(2\times1)(5\times4\times3\times2\times1)} = 21$

$r!$ is read as "r **factorial**" and $0!$ is defined as 1.

Note: Your calculator is able to evaluate ${}^{n}\mathrm{C}_r$ using the $\boxed{\text{nCr}}$ button.
For example, to evaluate ${}^{6}\mathrm{C}_4$, press 6 $\boxed{\text{nCr}}$ 4 to get 15.

<u>Example 1:</u>

Evaluate $\binom{5}{3}$ without the use of calculator.

$$\binom{5}{3} = \frac{5\times4\times3}{3\times2\times1}$$
$$= 10$$

Example 2:

Given that $\binom{n}{3} = 120$. Find the value of *n*.

$$\binom{n}{3} = 120$$

$$\frac{n(n-1)(n-2)}{3(2)(1)} = 120$$

$$n\left(n^2 - 3n + 2\right) = 120(6)$$

$$n^3 - 3n^2 + 2n - 720 = 0$$

Let $\mathrm{f}(n) = n^3 - 3n^2 + 2n - 720$.

$$\mathrm{f}(10) = (10)^3 - 3(10)^2 + 2(10) - 720$$
$$= 0$$

By Factor Theorem, $(n-10)$ is a factor of $\mathrm{f}(n)$.

$$n^3 - 3n^2 + 2n - 720 = (n-10)\left(n^2 + pn + 72\right)$$

By comparing the coefficients of n:

$$2 = 72 - 10p$$
$$10p = 70$$
$$p = 7$$

$$(n-10)\left(n^2 + 7n + 72\right) = 0$$

$n = 10$ or $n^2 + 7n + 72 = 0$

$$n = \frac{-7 \pm \sqrt{7^2 - 4(1)(72)}}{2(1)}$$

$$n = \frac{-7 \pm \sqrt{-239}}{2} \quad \text{(NA)}$$

$\therefore \quad n = 10$

Example 3:

Expand $(1+x)^7$ and arrange the terms in ascending powers of *x*. (refer to Pascal's Triangle)

$$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$$

Expansion of expressions of the form $(1+b)^n$ for a positive integer *n*

Notice that in the expansion of $(1+b)^n$, which is

$$\boxed{\begin{aligned}(1+b)^n &= \binom{n}{0}b^0+\binom{n}{1}b^1+\binom{n}{2}b^2+\binom{n}{3}b^3+\ldots+\binom{n}{r}b^r+\ldots+\binom{n}{n-1}b^{n-1}+\binom{n}{n}b^n \\ &= 1+\binom{n}{1}b^1+\binom{n}{2}b^2+\binom{n}{3}b^3+\ldots+\binom{n}{r}b^r+\ldots+\binom{n}{n-1}b^{n-1}+b^n\end{aligned}},$$

each term consists of two components (or factors), i.e.

- the $\binom{n}{r}$ component and
- the b^r component.

Hence if we can figure out what each component of a term is, we could find that term without having to expand the entire expression.

In general, for a positive integer *n*

$$\boxed{\text{the term containing } b^r \text{ is } \binom{n}{r}b^r.}$$

The term containing b^r is also the $(r+1)^{\text{th}}$ term in the expansion. i.e. $\boxed{T_{r+1}=\binom{n}{r}b^r.}$

Question: How many terms are there in the binomial expansion of $(1+b)^n$? **Ans:** $\underline{n+1}$

<u>Example 4:</u>

Expand $(1+2x)^3$.

$$\begin{aligned}(1+2x)^3 &= 1+\binom{3}{1}(2x)+\binom{3}{2}(2x)^2+(2x)^3 \\ &= 1+3(2x)+3(4x^2)+8x^3 \\ &= 1+6x+12x^2+8x^3\end{aligned}$$

Example 5:

Without expanding $(1-3x)^8$, state

(i) the term containing x^4,

(ii) the coefficient of x^7.

(i) $T_{r+1} = \binom{8}{r}(-3x)^r = \binom{8}{r}(-3)^r x^r$

When $r = 4$, $T_5 = \binom{8}{4}(-3)^4 x^4$

$T_5 = 5670x^4$

The required term is $5670x^4$.

(ii) When $r = 7$, $T_8 = \binom{8}{7}(-3)^7 x^7$

$T_8 = -17\,496x^7$

The coefficient of x^7 is $-17\,496$.

Class Practice 1:

1 Use the Pascal's Triangle to expand $(1-2x)^4$.

$$\begin{aligned}(1-2x)^4 &= 1+4(-2x)+6(-2x)^2+4(-2x)^3+(-2x)^4 \\ &= 1-8x+24x^2-32x^3+16x^4\end{aligned}$$

2 Use the Pascal's Triangle to find the first four terms, in ascending powers of x, in the expansion of $(1+2x)^9$.

$$\begin{aligned}(1+2x)^9 &= 1+9(2x)+36(2x)^2+84(2x)^3+\ \ldots \\ &= 1+18x+144x^2+672x^3+\ \ldots\end{aligned}$$

3 Use the Binomial Theorem to find the first four terms, in ascending powers of x, in the expansion of $(1-2x^2)^7$.

$$\begin{aligned}(1-2x^2)^7 &= 1+\binom{7}{1}(-2x^2)+\binom{7}{2}(-2x^2)^2+\binom{7}{3}(-2x^2)^3+\ \ldots \\ &= 1+7(-2x^2)+21(4x^4)+35(-8x^6)+\ \ldots \\ &= 1-14x^2+84x^4-280x^6+\ \ldots\end{aligned}$$

4 **(i)** Find the first four terms in the expansion of $(2-x)\left(1+\frac{1}{2}x\right)^8$ in ascending powers of x.

(ii) Hence estimate the value of 1.9×1.05^8.

(i) $$(2-x)\left(1+\frac{1}{2}x\right)^8=(2-x)\left[1+8\left(\frac{1}{2}x\right)+\binom{8}{2}\left(\frac{1}{2}x\right)^2+\binom{8}{3}\left(\frac{1}{2}x\right)^3+\ldots\right]$$
$$=(2-x)\left(1+4x+7x^2+7x^3+\ldots\right)$$
$$=2\left(1+4x+7x^2+7x^3+\ldots\right)-x\left(1+4x+7x^2+7x^3+\ldots\right)$$
$$=2+8x+14x^2+14x^3-x-4x^2-7x^3+\ldots$$
$$=2+7x+10x^2+7x^3+\ldots$$

(ii) Let $2-x=1.9$
$x=0.1$

When $x=0.1$, $$1.9\times1.05^8=(2-0.1)\left[1+\frac{1}{2}(0.1)\right]^8$$
$$=2+7(0.1)+10(0.1)^2+7(0.1)^3+\ldots$$
$$\approx 2.81$$

5 **(i)** Find the first two terms in the expansion of $(1+x)^{10\,000}$ in ascending powers of x.

(ii) Hence determine which number is larger, $1.1^{10\,000}$ or 1 000.

(i) $$(1+x)^{10\,000}=1+\binom{10\,000}{1}x+\ldots$$
$$=1+10\,000x+\ldots$$

(ii) Let $1+x=1.1$
$x=0.1$

When $x=0.1$, $$1.1^{10\,000}=(1+0.1)^{10\,000}$$
$$=1+10\,000(0.1)+\ldots$$
$$\approx 1001$$

Hence, $1.1^{10\,000}$ is larger.

6 **(i)** Find the first three terms in the expansion of $(1+4x)^6$ and of $(1-2x)^{14}$ in ascending powers of x.

(ii) Hence find the first three terms in the expansion of $(1+4x)^6(1-2x)^{14}$ in ascending powers of x.

(i) $$\begin{aligned}(1+4x)^6 &= 1+\binom{6}{1}(4x)+\binom{6}{2}(4x)^2+\ldots\\ &= 1+6(4x)+15(16x^2)+\ldots\\ &= 1+24x+240x^2+\ldots\end{aligned}$$

$$\begin{aligned}(1-2x)^{14} &= 1+\binom{14}{1}(-2x)+\binom{14}{2}(-2x)^2+\ldots\\ &= 1+14(-2x)+91(4x^2)+\ldots\\ &= 1-28x+364x^2+\ldots\end{aligned}$$

(ii) $$\begin{aligned}&(1+4x)^6(1-2x)^{14}\\ &= (1+24x+240x^2+\ldots)(1-28x+364x^2+\ldots)\\ &= 1(1-28x+364x^2)+24x(1-28x)+240x^2(1)+\ldots\\ &= 1-28x+364x^2+24x-672x^2+240x^2+\ldots\\ &= 1-4x-68x^2+\ldots\end{aligned}$$

7 **(i)** Find the first four terms in the expansion of $(1-x)^6$ and $(1+2x)^6$ in ascending powers of *x*.

(ii) Hence, expand $\left(1+x-2x^2\right)^6$ up to the term in x^3.

(i) $$(1-x)^6 = 1+\binom{6}{1}(-x)+\binom{6}{2}(-x)^2+\binom{6}{3}(-x)^3+\ldots$$
$$= 1-6x+15x^2-20x^3+\ldots$$

$$\left(1+2x\right)^6 = 1+\binom{6}{1}(2x)+\binom{6}{2}(2x)^2+\binom{6}{3}(2x)^3+\ldots$$
$$= 1+6(2x)+15\left(4x^2\right)+20\left(8x^3\right)+\ldots$$
$$= 1+12x+60x^2+160x^3+\ldots$$

(ii) $$\left(1+x-2x^2\right)^6$$
$$= \left[(1-x)(1+2x)\right]^6$$
$$= (1-x)^6(1+2x)^6$$
$$= \left(1-6x+15x^2-20x^3+\ldots\right)\left(1+12x+60x^2+160x^3+\ldots\right)$$
$$= 1\left(1+12x+60x^2+160x^3\right)-6x\left(1+12x+60x^2\right)+15x^2(1+12x)-20x^3(1)+\ldots$$
$$= 1+12x+60x^2+160x^3-6x-72x^2-360x^3+15x^2+180x^3-20x^3+\ldots$$
$$= 1+6x+3x^2-40x^3+\ldots$$

8 **Personal Finance.** \$20 000 is deposited in an account that pays interest of r % per annum, compounded quarterly. It will amount to $\$20000\left(1+\frac{r}{400}\right)^{4t}$ after t years.

(i) Given that r = 4, use the expansion of $(1+x)^{20}$ in ascending powers of x as far as the term in x^2 to estimate the amount of money received after 5 years.

(ii) The percentage error between the actual amount \$$A$ and the estimated amount \$$E$ is given by $\frac{A-E}{A}\times 100\%$. Using the value found in part **(i)**, calculate the percentage error, correct to two decimal places.

(iii) Based on your calculation obtained in part **(ii)**, determine whether the estimate in part **(i)** is a good estimate.

(i) Amount of money received after 5 years $= 20\ 000\left(1+\frac{4}{400}\right)^{4\times 5}$

$$= 20\ 000\left(1+\frac{1}{100}\right)^{20}$$

$$= 20\ 000\left[1+\binom{20}{1}\left(\frac{1}{100}\right)+\binom{20}{2}\left(\frac{1}{100}\right)^2+\ldots\right]$$

$$= 20\ 000\left[1+20\left(\frac{1}{100}\right)+190\left(\frac{1}{10\ 000}\right)+\ldots\right]$$

$$\approx \$24\ 380$$

(ii) Percentage error $= \dfrac{20\ 000\left(1+\frac{4}{400}\right)^{20}-24\ 380}{20\ 000\left(1+\frac{4}{400}\right)^{20}}\times 100\ \%$

$$\approx 0.097529\ \%$$

$$\approx 0.10\ \%$$

(iii) Yes, it is a good estimate since the percentage error of 0.10 % is insignificant.

9 When $(1-x)(1+ax)^6$ is expanded as far as the term in x^2, the result is $1+bx^2$.
Find the value of *a* and of *b*.

$$(1-x)(1+ax)^6=(1-x)\left[1+\binom{6}{1}(ax)+\binom{6}{2}(ax)^2+...\right]$$
$$=(1-x)\left(1+6ax+15a^2x^2+...\right)$$
$$=1\left(1+6ax+15a^2x^2+...\right)-x\left(1+6ax+15a^2x^2+...\right)$$
$$=1+6ax+15a^2x^2-x-6ax^2-15a^2x^3+...$$
$$\therefore\ 1+bx^2+...=1+6ax-x+15a^2x^2-6ax^2+...$$

By comparing the coefficients of:

$x:\quad 0=6a-1$

$a=\frac{1}{6}$

$x^2:\quad b=15a^2-6a\quad\cdots(1)$

Sub. $a=\frac{1}{6}$ into (1):

$b=15\left(\frac{1}{6}\right)^2-6\left(\frac{1}{6}\right)$

$b=-\frac{7}{12}$

$\therefore\ a=\frac{1}{6},\quad b=-\frac{7}{12}$

10 **(i)** Show that the coefficient of x^2 in the expansion of $(1-2x)^n$ is $4\binom{n}{2}$.

(ii) If the coefficient of x^2 in the expansion of $(1-2x)^n$ is 24, find the value of n.

(i) $$(1-2x)^n = 1+\binom{n}{1}(-2x)+\binom{n}{2}(-2x)^2+\dots$$
$$= 1-2nx+4\binom{n}{2}x^2+\dots$$

Coefficient of x^2 is $4\binom{n}{2}$. (shown)

(ii) $$4\binom{n}{2}=24$$
$$4\left[\frac{n(n-1)}{2}\right]=24$$
$$n(n-1)=12$$
$$n^2-n-12=0$$
$$(n-4)(n+3)=0$$
$n-4=0$ or $n+3=0$

$n=4$ $\quad$ $n=-3$ (NA)

$\therefore \quad n=4$

Expansion of expressions of the form $(a+b)^n$ for a positive integer n

When the first term is not 1, the expansion of $(a+b)^n$ can be expressed, for a positive integer n, as

$$\boxed{(a+b)^n = a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + b^n.}$$

Therefore the general term, also known as the $(r+1)^{\text{th}}$ term, in this expansion is

$$\boxed{T_{r+1} = \binom{n}{r}a^{n-r}b^r, \text{ when } 0 \le r \le n.}$$

Observations:

- powers of a are decreasing
- powers of b are increasing
- sum of powers of a and b is n
- total number of terms in the expansion is $n+1$

Example 6:

Expand $(2+3x)^3$ in ascending powers of x.

$$\begin{aligned}(2+3x)^3 &= 2^3 + \binom{3}{1}(2)^2(3x) + \binom{3}{2}(2)(3x)^2 + (3x)^3 \\ &= 8 + 3(4)(3x) + 3(2)(9x^2) + 27x^3 \\ &= 8 + 36x + 54x^2 + 27x^3\end{aligned}$$

Alternatively, we can take out a common factor such that the first term in the binomial expansion is 1.

This method, though more troublesome, is used more frequently in higher levels.
Hence, learn to be competent in using both methods.

Example 7:

In the expansion of $\left(2+\frac{x^2}{2}\right)^{11}$, find the following:

(a) the 8th term,
(b) the term in x^8,
(c) the coefficient of x^8.

For $\left(2+\frac{x^2}{2}\right)^{11}$,

$$\begin{aligned} T_{r+1} &= \binom{11}{r}(2)^{11-r}\left(\frac{x^2}{2}\right)^r \\ &= \binom{11}{r}(2)^{11-r}\left(x^2\right)^r(2)^{-r} \\ &= \binom{11}{r}2^{11-2r}x^{2r} \end{aligned}$$

(a) For the 8th term, $r = 7$.

$$\begin{aligned} T_8 &= \binom{11}{7}2^{11-2(7)}x^{2(7)} \\ &= 330(2)^{-3}x^{14} \\ &= 330\left(\frac{1}{8}\right)x^{14} \\ &= \frac{165}{4}x^{14} \end{aligned}$$

(b) For the term in x^8,

$$\begin{aligned} x^{2r} &= x^8 \\ 2r &= 8 \\ r &= 4 \end{aligned}$$

Term in x^8, $T_5 = \binom{11}{4}2^{11-2(4)}x^8$

$$\begin{aligned} &= 330(2)^3x^8 \\ &= 2640x^8 \end{aligned}$$

(c) Coefficient of x^8 is 2640.

Example 8:

(i) Write down the first four terms in the expansion of $\left(2-\frac{x}{2}\right)^7$ in ascending powers of x.

(ii) What value of x would you substitute into the expansion of $\left(2-\frac{x}{2}\right)^7$ in order to estimate the value of $(1.995)^7$? Justify your answer.

Hence, find the value of $(1.995)^7$ correct to 4 decimal places.

(i)
$$\left(2-\frac{x}{2}\right)^7 = 2^7 + \binom{7}{1}(2)^6\left(-\frac{x}{2}\right) + \binom{7}{2}(2)^5\left(-\frac{x}{2}\right)^2 + \binom{7}{3}(2)^4\left(-\frac{x}{2}\right)^3 + \ldots$$
$$= 128 + 7(32)\left(\frac{x}{2}\right) + 21(32)\left(\frac{x^2}{4}\right) + \binom{7}{3}(2)^4\left(-\frac{x}{2}\right)^3 + \ldots$$
$$= 128 - 224x + 168x^2 - 70x^3 + \ldots$$

(ii) Let
$$2-\frac{x}{2} = 1.995$$
$$\frac{x}{2} = 0.005$$
$$x = 0.01$$

Hence, we will substitute $x = 0.01$ into the expansion of $\left(2-\frac{x}{2}\right)^7$ in order to estimate the value of $(1.995)^7$.

$$\therefore \quad (1.995)^7 = \left(2-\frac{0.01}{2}\right)^7$$
$$\approx 128 - 224(0.01) + 168(0.01)^2 - 70(0.01)^3$$
$$\approx 125.7767$$

Question:

What do you notice when you expand binomial expressions of the form $(a-b)^n$?

The signs alternate between positive and negative in the expansion, depending whether $(-b)$ is raised to an even or odd power.

Example 9:

(i) Explain why the term in $\frac{1}{x}$ does not exist in the expansion of $\left(5x-\frac{1}{2x^3}\right)^8$.

(ii) Find the constant term in the expansion of $\left(2+x+x^4\right)\left(5x-\frac{1}{2x^3}\right)^8$.

(i) For $\left(5x-\frac{1}{2x^3}\right)^8$,

$$\begin{aligned} T_{r+1} &= \binom{8}{r}(5x)^{8-r}\left(-\frac{1}{2x^3}\right)^r \\ &= \binom{8}{r}(5)^{8-r}(x)^{8-r}\left(-\frac{1}{2}\right)^r\left(x^{-3}\right)^r \\ &= \binom{8}{r}\left(-\frac{1}{2}\right)^r(5)^{8-r}x^{8-r-3r} \\ &= \binom{8}{r}\left(-\frac{1}{2}\right)^r(5)^{8-r}x^{8-4r} \end{aligned}$$

For the term in $\frac{1}{x}$,

$$\begin{aligned} x^{-1} &= x^{8-4r} \\ -1 &= 8-4r \\ r &= \frac{9}{4} \end{aligned}$$

Since $r=\frac{9}{4}$ and it is not an integer, the term in $\frac{1}{x}$ does not exist in the expansion of $\left(5x-\frac{1}{2x^3}\right)^8$.

(ii) For the constant term,

$$\begin{aligned} x^0 &= x^{8-4r} \\ 0 &= 8-4r \\ r &= 2 \end{aligned}$$

For the term in $\frac{1}{x^4}$,

$$\begin{aligned} x^{-4} &= x^{8-4r} \\ -4 &= 8-4r \\ r &= 3 \end{aligned}$$

$$\begin{aligned} \left(2+x+x^4\right)\left(5x-\frac{1}{2x^3}\right)^8 &= 2\left[\binom{8}{2}\left(-\frac{1}{2}\right)^2(5)^{8-2}x^{8-4(2)}\right]+x^4\left[\binom{8}{3}\left(-\frac{1}{2}\right)^3(5)^{8-3}x^{8-4(3)}\right]+\ldots \\ &= 2(28)\left(\frac{1}{4}\right)(15\ 625)+(56)\left(-\frac{1}{8}\right)(3\ 125)+\ldots \\ &= 196\ 875+\ldots \end{aligned}$$

$\therefore$ The constant term is 196 875.

Example 10:

The first three terms in the expansion of $(1+x)(1-px)^n$, where p and n are constants, are $1-5x+9x^2$. Find the value of p and of n.

[Catholic High Prelim P1 2008]

$$\begin{aligned}(1+x)(1-px)^n &= (1+x)\left[1+\binom{n}{1}(-px)^1+\binom{n}{2}(-px)^2+\ldots\right]\\ &= (1+x)\left[1-npx+\frac{n(n-1)}{2}p^2x^2+\ldots\right]\\ &= 1\left[1-npx+\frac{n(n-1)}{2}p^2x^2+\ldots\right]+x(1-npx+\ldots)\\ &= 1-npx+\frac{n(n-1)}{2}p^2x^2+x-npx^2+\ldots\\ &= 1+(1-np)x+\left[\frac{n(n-1)}{2}p^2-np\right]x^2+\ldots\end{aligned}$$

$$\therefore \quad 1-5x+9x^2 = 1+(1-np)x+\left[\frac{n(n-1)}{2}p^2-np\right]x^2+\ldots$$

By comparing coefficients of:

$x:\quad -5 = 1-np$

$\quad np = 6 \quad \cdots(1)$

$$x^2:\quad 9=\frac{n(n-1)}{2}p^2-np$$

$$(n^2-n)p^2-2np=18$$

$$n^2p^2-np^2-2np=18$$

$$(np)^2-(np)p-2np=18 \quad \cdots(2)$$

Sub. (1) in (2): $\quad 6^2-6p-2(6)=18$

$6p=6$

$p=1$

Sub. $p=1$ into (1): $\quad n=6$

$\therefore \quad p=1, \quad n=6$

Class Practice 2:

1 Write down the first four terms in the expansion of $\left(\frac{1}{2x}-2x^2\right)^8$ in ascending powers of x.

$$\begin{aligned}\left(\frac{1}{2x}-2x^2\right)^8 &= \left(\frac{1}{2x}\right)^8-\binom{8}{1}\left(\frac{1}{2x}\right)^7\left(2x^2\right)+\binom{8}{2}\left(\frac{1}{2x}\right)^6\left(2x^2\right)^2-\binom{8}{3}\left(\frac{1}{2x}\right)^5\left(2x^2\right)^3+\ldots\\ &= \frac{1}{256x^8}-8\left(\frac{1}{128x^7}\right)\left(2x^2\right)+28\left(\frac{1}{64x^6}\right)\left(4x^4\right)-56\left(\frac{1}{32x^5}\right)\left(8x^6\right)+\ldots\\ &= \frac{1}{256x^8}-\frac{1}{8x^5}+\frac{7}{4x^2}-14x+\ldots\end{aligned}$$

2 State the general term in the expansion of $(2x-3)^{10}$.

$$T_{r+1}=\binom{10}{r}(2x)^{10-r}(-3)^r$$

3 By considering the general term $\binom{n}{r}a^{n-r}b^r$ in the binomial expansion of $(a+b)^n$, where n is a positive integer, find the indicated term in each of the following expansions.

(a) $(3x-2)^9$, the 4th term **(b)** $(y-2x)^{10}$, the middle term

(a) For $(3x-2)^9$,

$$T_{r+1}=\binom{9}{r}(3x)^{9-r}(-2)^r$$

For the 4th term, $r=3$.

$$\begin{aligned}T_4 &= \binom{9}{3}(3x)^{9-3}(-2)^3\\ &= 84\left(729x^6\right)(-8)\\ &= -489\,888x^6\end{aligned}$$

(b) The expansion has 10 + 1 = 11 terms.
Middle term is the 6th term, where $r=5$.

For $(y-2x)^{10}$,

$$T_{r+1}=\binom{10}{r}(y)^{10-r}(-2x)^r$$

$$\begin{aligned}T_6 &= \binom{10}{5}(y)^{10-5}(-2x)^5\\ &= 252y^5\left(-32x^5\right)\\ &= -8064y^5x^5\end{aligned}$$

4 **(a)** Find the term independent of x in the expansion of $\left(\frac{1}{x}-x^2\right)^{18}$.

(b) Find the middle term in the expansion of $\left(x+\frac{1}{2x^2}\right)^{12}$ in descending powers of x.

(a) For $\left(\frac{1}{x}-x^2\right)^{18}$,

$$\begin{aligned} T_{r+1} &= \binom{18}{r}\left(\frac{1}{x}\right)^{18-r}\left(-x^2\right)^r \\ &= \binom{18}{r}\left(x^{-1}\right)^{18-r}(-1)^r\left(x^2\right)^r \\ &= \binom{18}{r}(-1)^r x^{-18+r+2r} \\ &= \binom{18}{r}(-1)^r x^{3r-18} \end{aligned}$$

For the term independent of x,

$$\begin{aligned} x^0 &= x^{3r-18} \\ 0 &= 3r-18 \\ 3r &= 18 \\ r &= 6 \end{aligned}$$

$$\begin{aligned} \text{Term independent of } x &= \binom{18}{6}(-1)^6 \\ &= 18\,564 \end{aligned}$$

(b) For $\left(x+\frac{1}{2x^2}\right)^{12}$,

$$T_{r+1} = \binom{12}{r}(x)^{12-r}\left(\frac{1}{2x^2}\right)^r$$

The expansion has 12 + 1 = 13 terms.

Middle term is the 7th term, where $r = 6$.

$$\begin{aligned} \text{Middle term} &= \binom{12}{6}(x)^{12-6}\left(\frac{1}{2x^2}\right)^6 \\ &= 924x^6\left(\frac{1}{64x^{12}}\right) \\ &= \frac{231}{16x^6} \end{aligned}$$

5 **(i)** Find the first three terms in the expansion of $(1-2x)^9$ and of $(2+x)^5$ in ascending powers of *x*.

(ii) Hence expand $(1-2x)^9(2+x)^5$ up to the term in x^2.

(i)
$$(1-2x)^9 = 1+\binom{9}{1}(-2x)+\binom{9}{2}(-2x)^2+\dots$$
$$= 1+9(-2x)+36(4x^2)+\dots$$
$$= 1-18x+144x^2+\dots$$

$$(2+x)^5 = 2^5+\binom{5}{1}(2)^4(x)+\binom{5}{2}(2)^3(x)^2+\dots$$
$$= 32+5(16)x+10(8)x^2+\dots$$
$$= 32+80x+80x^2+\dots$$

(ii)
$$(1-2x)^9(2+x)^5$$
$$= (1-18x+144x^2+\dots)(32+80x+80x^2+\dots)$$
$$= 1(32+80x+80x^2)-18x(32+80x)+144x^2(32)+\dots$$
$$= 32+80x+80x^2-576x-1440x^2+4608x^2+\dots$$
$$= 32-496x+3248x^2+\dots$$

6 **(i)** Write down the first four terms in the expansion of $(2x-1)^6$ in descending powers of *x*.

(ii) Find the coefficient of x^5 in the expansion of $(2x-1)^6(x^2-2x+3)$.

(i)
$$(2x-1)^6 = (2x)^6+\binom{6}{1}(2x)^5(-1)+\binom{6}{2}(2x)^4(-1)^2+\binom{6}{3}(2x)^3(-1)^3+\dots$$
$$= 64x^6-6(32x^5)+15(16x^4)-20(8x^3)+\dots$$
$$= 64x^6-192x^5+240x^4-160x^3+\dots$$

(ii)
$$(2x-1)^6(x^2-2x+3)$$
$$= (64x^6-192x^5+240x^4-160x^3+\dots)(x^2-2x+3)$$
$$= -192x^5(3)+240x^4(-2x)-160x^3(x^2)+\dots$$
$$= -576x^5-480x^5-160x^5+\dots$$
$$= -1216x^5+\dots$$

The coefficient of x^5 is -1216.

7 **(i)** Expand $\left(\frac{1}{2}-2x\right)^5$ in ascending powers of x up to the term in x^3.

(ii) If the coefficient of x^2 in the binomial expansion of $(1+ax+3x^2)\left(\frac{1}{2}-2x\right)^5$ is $\frac{13}{2}$, find the coefficient of x^3.

(i) $$\left(\frac{1}{2}-2x\right)^5=\left(\frac{1}{2}\right)^5+\binom{5}{1}\left(\frac{1}{2}\right)^4(-2x)+\binom{5}{2}\left(\frac{1}{2}\right)^3(-2x)^2+\binom{5}{3}\left(\frac{1}{2}\right)^2(-2x)^3+\ldots$$
$$=\frac{1}{32}+5\left(\frac{1}{16}\right)(-2x)+10\left(\frac{1}{8}\right)(4x^2)+10\left(\frac{1}{4}\right)(-8x^3)+\ldots$$
$$=\frac{1}{32}-\frac{5}{8}x+5x^2-20x^3+\ldots$$

(ii) $$(1+ax+3x^2)\left(\frac{1}{2}-2x\right)^5$$
$$=(1+ax+3x^2)\left(\frac{1}{32}-\frac{5}{8}x+5x^2-20x^3+\ldots\right)$$
$$=1(5x^2-20x^3+\ldots)+ax\left(-\frac{5}{8}x+5x^2+\ldots\right)+3x^2\left(\frac{1}{32}-\frac{5}{8}x+\ldots\right)$$
$$=5x^2-20x^3-\frac{5}{8}ax^2+5ax^3+\frac{3}{32}x^2-\frac{15}{8}x^3+\ldots$$

Coefficient of $x^2=\frac{13}{2}$

$$5-\frac{5}{8}a+\frac{3}{32}=\frac{13}{2}$$
$$-\frac{5}{8}a=\frac{45}{32}$$
$$a=-\frac{9}{4}$$

When $a=-\frac{9}{4}$, Coefficient of $x^3=-20+5a-\frac{15}{8}$
$$=-20+5\left(-\frac{9}{4}\right)-\frac{15}{8}$$
$$=-20-\frac{45}{4}-\frac{15}{8}$$
$$=-\frac{265}{8}$$

8 In the expansion of $\left(x^3-\dfrac{2}{x^2}\right)^{10}$, find

(i) the term in x^{10}, **(ii)** the coefficient of $\dfrac{1}{x^5}$, **(iii)** the constant term.

For $\left(x^3-\dfrac{2}{x^2}\right)^{10}$,

$$\begin{aligned}T_{r+1}&=\binom{10}{r}\left(x^3\right)^{10-r}\left(-\frac{2}{x^2}\right)^r\\&=\binom{10}{r}\left(x^{30-3r}\right)\left(-2x^{-2}\right)^r\\&=\binom{10}{r}\left(x^{30-3r}\right)(-2)^r\left(x^{-2r}\right)\\&=\binom{10}{r}(-2)^r x^{30-3r-2r}\\&=\binom{10}{r}(-2)^r x^{30-5r}\end{aligned}$$

(i) For the term in x^{10},

$$\begin{aligned}x^{10}&=x^{30-5r}\\10&=30-5r\\5r&=20\\r&=4\end{aligned}$$

$$\begin{aligned}\text{Term in } x^{10}&=\binom{10}{4}(-2)^4 x^{30-5(4)}\\&=210(16)x^{10}\\&=3360x^{10}\end{aligned}$$

(ii) For the term in x^{-5},

$$\begin{aligned}x^{-5}&=x^{30-5r}\\-5&=30-5r\\5r&=35\\r&=7\end{aligned}$$

$$\begin{aligned}\text{Coefficient of } \frac{1}{x^5}&=\binom{10}{7}(-2)^7\\&=120(-128)\\&=-15\,360\end{aligned}$$

(iii) For the constant term,

$$\begin{aligned}x^{0}&=x^{30-5r}\\0&=30-5r\\5r&=30\\r&=6\end{aligned}$$

$$\begin{aligned}\text{Constant term}&=\binom{10}{6}(-2)^6\\&=210(64)\\&=13\,440\end{aligned}$$

9 **(i)** Find the first four terms in the expansion of $(2+p)^5$ in ascending powers of p.

(ii) Hence find the expansion of $(2+x-2x^2)^5$ in ascending powers of x up to the term in x^3.

(i) $$(2+p)^5 = 2^5 + \binom{5}{1}(2)^4(p) + \binom{5}{2}(2)^3(p)^2 + \binom{5}{3}(2)^2(p)^3 + \ldots$$
$$= 32 + 5(16)p + 10(8)p^2 + 10(4)p^3 + \ldots$$
$$= 32 + 80p + 80p^2 + 40p^3 + \ldots$$

(ii) Let $p = x - 2x^2$.
$$(2+x-2x^2)^5$$
$$= \left[2+(x-2x^2)\right]^5$$
$$= 32 + 80(x-2x^2) + 80(x-2x^2)^2 + 40(x-2x^2)^3 + \ldots$$
$$= 32 + 80x - 160x^2 + 80\left[x^2 - 2(x)(2x^2) + (2x^2)^2\right] +$$
$$40\left[x^3 + 3x^2(-2x^2) + 3x(-2x^2)^2 + (-2x^2)^3\right] + \ldots$$
$$= 32 + 80x - 160x^2 + 80x^2 - 320x^3 + 40x^3 + \ldots$$
$$= 32 + 80x - 80x^2 - 280x^3 + \ldots$$

A4 **(i)** Expand $(1+p)^4$ in ascending powers of p.

(ii) Using the result found in part **(i)**, find the expansion of $(1+x+x^2)^4$ in ascending powers of x up to the term in x^3.

(iii) Suggest a value of x in the expansion in part **(ii)** to evaluate $(1.11)^4$.

(i)
$$\begin{aligned}(1+p)^4 &= 1+\binom{4}{1}p+\binom{4}{2}p^2+\binom{4}{3}p^3+p^4\\ &= 1+4p+6p^2+4p^3+p^4\end{aligned}$$

(ii)
$$\begin{aligned}(1+x+x^2)^4 &= \left[1+(x+x^2)\right]^4\\ &= 1+4(x+x^2)+6(x+x^2)^2+4(x+x^2)^3+(x+x^2)^4\\ &= 1+4x+4x^2+6\left[x^2+2x(x^2)+(x^2)^2\right]+4(x^3+...)+...\\ &= 1+4x+4x^2+6x^2+12x^3+4x^3+...\\ &= 1+4x+10x^2+16x^3+...\end{aligned}$$

(iii) Let $1+x+x^2=1.11$
$$\begin{aligned}x+x^2&=0.11\\ x+x^2&=0.1+0.01\\ x&=0.1\end{aligned}$$

$$\begin{aligned}1.11^4 &= (1+0.1+0.1^2)^4\\ &= 1+4(0.1)+10(0.1)^2+16(0.1)^3+...\\ &\approx 1.52\end{aligned}$$

B4 In the expansion of $(1+ax)^6(2+bx)^5$ in ascending powers of x, the coefficients of x and x^2 are -112 and 80 respectively. Find the integer value of a and of b.

$(1+ax)^6(2+bx)^5$

$=\left[1+\binom{6}{1}(ax)+\binom{6}{2}(ax)^2+\ldots\right]\left[2^5+\binom{5}{1}(2)^4(bx)+\binom{5}{2}(2)^3(bx)^2+\ldots\right]$

$=(1+6ax+15a^2x^2+\ldots)(32+80bx+80b^2x^2+\ldots)$

$=1(32+80bx+80b^2x^2+\ldots)+6ax(32+80bx+80b^2x^2+\ldots)+15a^2x^2(32+\ldots)$

$=80bx+80b^2x^2+192ax+480abx^2+480a^2x^2+\ldots$

Coefficient of $x=-112$

$80b+192a=-112$

$5b+12a=-7$

$a=\dfrac{-7-5b}{12}$ ... (1)

Coefficient of $x^2=80$

$80b^2+480ab+480a^2=80$

$b^2+6ab+6a^2=1$ ... (2)

Sub. (1) into (2):

$b^2+6b\left(\dfrac{-7-5b}{12}\right)+6\left(\dfrac{-7-5b}{12}\right)^2=1$

$b^2+b\left(\dfrac{-7-5b}{2}\right)+6\left[\dfrac{49+2(7)(5b)+(5b)^2}{144}\right]=1$

$b^2+b\left(\dfrac{-7-5b}{2}\right)+\left(\dfrac{49+70b+25b^2}{24}\right)=1$

$24b^2+12b(-7-5b)+49+70b+25b^2=24$

$24b^2-84b-60b^2+49+70b+25b^2=24$

$-11b^2-14b+49=24$

$11b^2+14b-25=0$

$(11b+25)(b-1)=0$

$11b+25=0$ or $b-1=0$

$b=-\dfrac{25}{11}$ (NA) $\qquad b=1$

When $b=1$, $\quad a=\dfrac{-7-5(1)}{12}$

$a=-1$

$\therefore\ a=-1,\ b=1$