Section A: Pure Mathematics [40 marks]

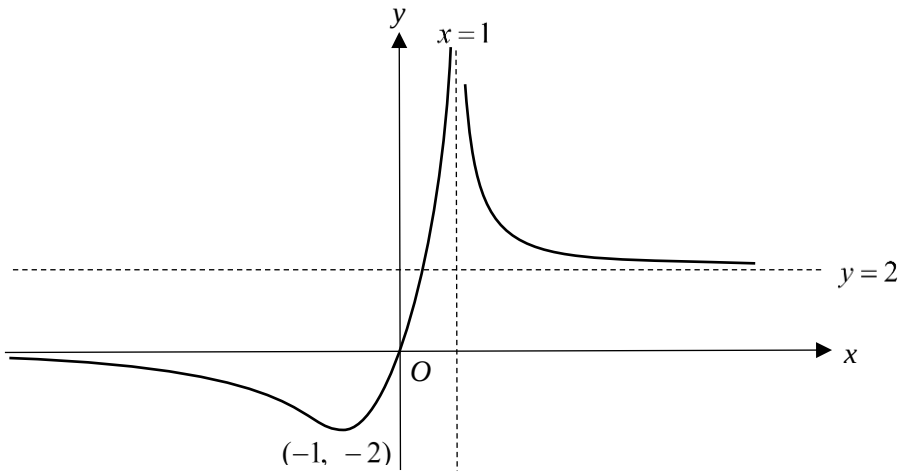
1 It is given that $\mathrm{f}(x)=\sqrt{4-3x}$.

(a) Find the binomial expansion for $\mathrm{f}(x)$, up to and including the term in x^2. Give the coefficients as exact fractions in their simplest form, and state the range of values of x for which this expansion is valid. [4]

(b) By taking $x=\frac{1}{4}$, show that $\sqrt{13}\approx\frac{1847}{512}$. [3]

(c) We may also substitute $x=\frac{1}{13}$ to obtain an approximation of $\sqrt{13}$. Without further calculation, explain why this leads to a better approximation of $\sqrt{13}$ than the value shown in **(b)**. [1]

2 The graph of $y=\mathrm{f}(x)$ is given below. It has one vertical asymptote at $x=1$ and two horizontal asymptotes $y=0$ and $y=2$. The graph passes through the origin O and has a turning point at $(-1,\ -2)$.



(a) On separate diagrams, sketch the following graphs, indicating the coordinates of the points where the graphs cross the axes, the turning points, and the equations of any asymptotes if possible.

(i) $y=\frac{1}{\mathrm{f}(x)}$ [3]

(ii) $y=\mathrm{f}'(x)$ [3]

(iii) $y=\mathrm{f}(|x-1|)$ [2]

(b) State the range of values of a such that $\mathrm{f}(|x-1|)=a$ has only positive root(s). [1]

3 The sum S_n of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by

$$S_n = \ln\left(\frac{\mathrm{e}^n}{3^{n^2}}\right).$$

(a) Show that $u_n = 1+(1-2n)\ln 3$. [2]

(b) Hence, show that the sequence is an arithmetic progression. [2]

(c) Find the sum of the first ten odd-numbered terms. [2]

(d) A geometric sequence $\mathrm{e}^{u_1}, \mathrm{e}^{u_2}, \mathrm{e}^{u_3}, \ldots$ has a common ratio, r. Find the value of r. [2]

(e) Find the least value of n for which the sum of the first n terms of this geometric sequence is within 10^{-8} of its sum to infinity. [3]

4 **(a)** One of the roots of the equation $z^4 + pz^3 + 5z^2 + qz - 26 = 0$, where p and q are real, is $1-2\sqrt{3}\mathrm{i}$. Find the other roots of the equation, and the values of p and q. [5]

Do not use a calculator in answering part (b).

(b) The complex number w is given by $w = \dfrac{\mathrm{i}^3}{\left(-\sqrt{3}+\mathrm{i}\right)^4}$.

(i) Find the exact value of the modulus and argument of w. [4]

(ii) Find the smallest positive integer n such that $\dfrac{\mathrm{i}w^n}{w^*}$ is purely imaginary. [3]

Section B: Probability and Statistics [60 marks]

5 In this question you should state the parameters of any normal distribution you use.

The masses in grams of oranges have the distribution N(150, 14^2) and the masses in grams of kiwis have the distribution N(70, 8^2).

(a) Find the probability that the mass of a randomly chosen orange is less than 180 grams. [1]

6 oranges and 4 kiwis are randomly selected and packed into a randomly chosen empty basket to make a fruit basket. The masses of the empty baskets in grams have the distribution N(750, 168).

(b) Find the probability that a randomly chosen fruit basket is within 25 grams of its mean. [3]

(c) Sketch the distribution for masses of fruit baskets between 1710 grams and 2150 grams. [2]

(d) Three fruit baskets were randomly chosen. Find the probability that exactly one fruit basket weighs more than 2000 grams and exactly one fruit basket weighs less than 1900 grams. [2]

6 A company manufactures screen protectors for handphones. On average, 17% of the screen protectors are cracked. The screen protectors are packaged into boxes of 20. It should be assumed that the number of cracked screen protectors in a box of 20 screen protectors follows a binomial distribution.

(a) Show that the probability of having no more than 3 cracked screen protectors in a randomly chosen box is 0.55041, correct to 5 significant figures. [1]

(b) Find the most probable number of cracked screen protectors in a randomly chosen box. [1]

Every month, the company exports a large shipment of n cartons of screen protectors. In each carton, there are 8 boxes of screen protectors. Every carton is assumed to be filled completely.

(c) Using a suitable approximation, find the least value of n such that in a randomly chosen month, there is at least a 99% chance that the average number of boxes per carton having no more than 3 cracked screen protectors is at most 5. [4]

To determine whether cartons are ready for local or overseas sales, cartons are randomly selected for inspection. In a carton,

- if there are at least 4 boxes that contain no more than 3 cracked screen protectors each, the carton is ready for local sales;
- if there are at least 6 boxes that contain no more than 3 cracked screen protectors each, the carton is ready for overseas sales.

(d) A carton that is not ready for overseas sales is randomly selected. Find the probability that the carton is ready for local sales. [3]

7 In a carnival lucky dip game, a game master places n consolation tickets, m blank tickets, and one golden ticket into a box. A contestant taking part in the game would pay \$1 to draw two tickets from the box. They would then be awarded \$1 for each consolation ticket drawn, and \$10 for the golden ticket if it is drawn. Nothing is awarded for the blank tickets drawn. Let \$$W$ represent the total amount awarded to a contestant after one game.

(a) Show that $\mathrm{P}(W=1)=\dfrac{2nm}{(n+m+1)(n+m)}$, and determine the probability distribution of W. [4]

(b) By considering $\mathrm{E}(W)$, show that if the game master expects to make a profit from the lucky dip game, then $m-n>19$. [3]

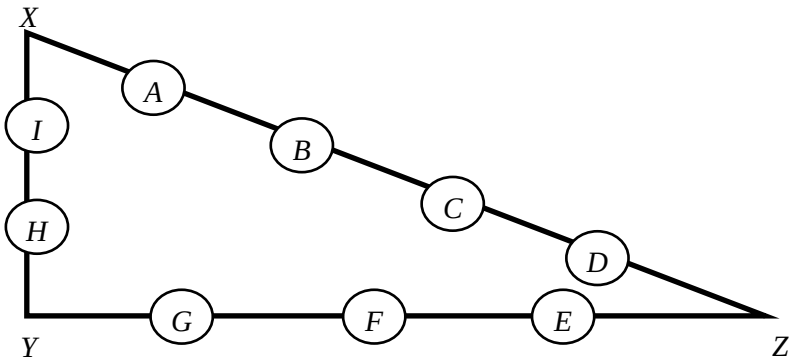
The game master then decides to run the game with 10 consolation tickets and 40 blank tickets. He is also issued with carnival lucky draw tickets to give away, and gives each contestant Y lucky draw tickets after the game, where $Y=|W-4|$.

(c) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [2]

8 In an art installation, three points on the floor, *X*, *Y* and *Z*, were connected using a rope to form a triangle *XYZ*. 9 bulbs, labelled *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*, are then to be placed along the rope to improve the aesthetic appearance.

(a) The 9 bulbs are first distributed into 3 groups, namely Group *XY*, Group *YZ* and Group *ZX*. Find the number of ways the bulbs can be distributed if each group consists of at least 2 bulbs. [2]

In one particular distribution from part **(a)**, the bulbs from Group *XY*, Group *YZ* and Group *ZX* are placed on the sides *XY*, *YZ* and *ZX* of the triangle *XYZ* respectively. The bulbs are arranged as shown in the diagram below.



(b) By connecting the bulbs in this arrangement using additional rope, we can form a smaller triangle within the triangle *XYZ*. For example, the bulbs *B*, *I*, *G* form a smaller triangle when connected. How many different smaller triangles can be formed? [2]

(c) Each bulb can be programmed to produce either red, blue or green light when it is switched on. Find the number of ways the colours can be programmed so that all the 3 colours are used. [3]

In another art installation, 9 bulbs, labelled *A*, *A*, *B*, *B*, *C*, *C*, *D*, *E*, *F*, are laid on the floor in a circular arrangement.

(d) Find the number of possible arrangements if the 2 *A*'s are separated and the 2 *B*'s are separated. [3]

9 Orange is a company that produces laptops and tablets. The company has a patented design for their ultra-slim laptop cooling fans, which operate at an optimal speed of 3100 RPM, such that they maximise the cooling effectiveness while minimising the noise level.

In a routine quality check by the company's internal surveyors, a random sample of 13 laptops were tested and their cooling fan speeds are recorded in the table below.

3202	3033	3103	3013	3023	3032	3145	3054	3154	3083	3099	3101	3102

(a) Calculate the unbiased estimates of the population mean and population variance of the fan speeds. [2]

Based on historical data, it is known that the fan speeds have a standard deviation of 30 RPM.

(b) Stating any necessary assumptions, test at the 10% level of significance whether the population mean fan speed is different from the optimal speed of 3100 RPM. [6]

(c) State the meaning of the *p*-value in this context. [1]

(d) State, to the nearest percent, the smallest level of significance at which the null hypothesis will be rejected using the given set of data. [1]

A random sample of 50 tablets is tested and the standby-time, in hours, is measured. It is known that the standby-time of the tablets is normally distributed with a standard deviation of 3 hours. Furthermore, Orange markets their tablets as having a standby-time of 36 hours.

(e) Determine the range of values for the sample mean standby-time, in hours, such that the null hypothesis will be rejected in a test at the 5% level of significance against an alternative hypothesis that the mean standby-time is less than 36 hours. [2]

10 A group of researchers want to study the relationship between the birth lengths of newborns and the heights of these newborns when they are 16 years old. They record the birth lengths of 10 newborns, l centimetres, and their respective heights, h centimetres, when they are 16 years old.

Birth length, l cm	40.2	41.1	41.8	42.4	45.6	46.0	46.5	48.5	50.2	51.8
Height at 16 years old, h cm	160.5	160.8	161.1	161.9	165.2	176.7	166.5	171.9	178.4	187.6

It turns out that one of the values of h was incorrectly recorded.

(a) Sketch a scatter diagram for the data and circle the point that was incorrectly recorded. [2]

For parts **(b)**, **(c),** and **(d)** of this question, you should **exclude** this incorrectly recorded point.

(b) Use your scatter diagram to explain whether the relationship between l and h is likely to be well-modelled by an equation of the form $h = a + bl$, where a and b are constants. [1]

(c) By calculating the relevant product moment correlation coefficients, determine whether the relationship between l and h is modelled better by $h = a + bl$ or $h = a + bl^3$. Explain how you decide which model is better, and state the equation of the regression line in this case. [5]

(d) Explain whether or not the regression line in part **(c)** should be used to estimate the birth length of a 16-year-old whose height is 170 cm. [1]

(e) Use the regression line in part **(c)** to estimate the value of h for the incorrectly recorded point identified in part **(a)** to 1 decimal place. Comment on the reliability of this estimate. [2]

(f) Give a reason in context why, although there is a strong positive correlation between one's birth length and one's height at 16 years old, we cannot say that one's birth length is the cause of one's height at 16 years old. [1]