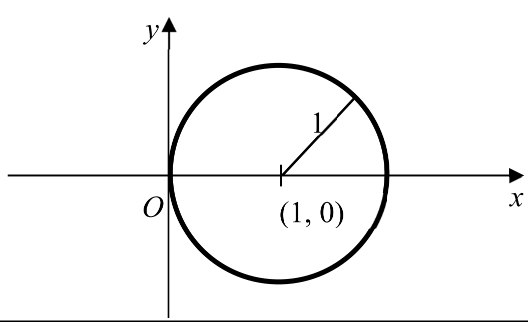
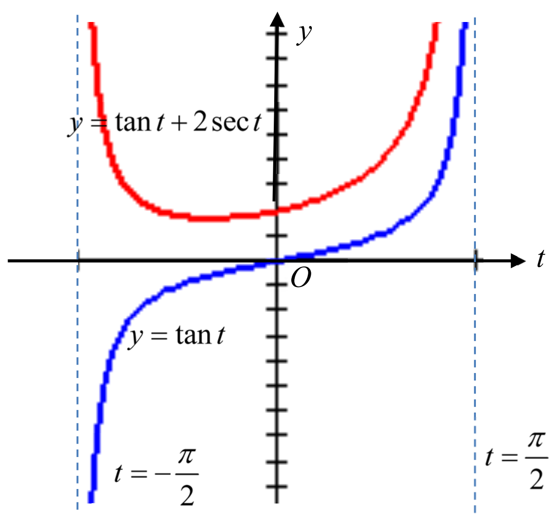
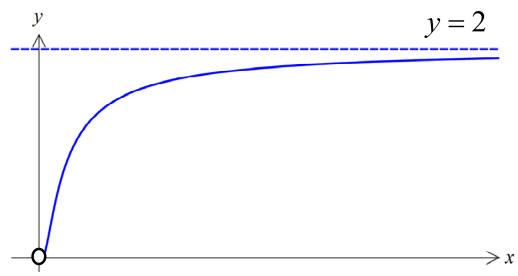
Differential Equations I (Solutions)

Qn	Suggested Solution
1(i)	$2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}=y^2-x^2,\ x\neq 0$ Substitute $y=vx \ \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=x\dfrac{\mathrm{d}v}{\mathrm{d}x}+v$ $2x(vx)(x\dfrac{\mathrm{d}v}{\mathrm{d}x}+v)=(vx)^2-x^2$ $2vx^3\dfrac{\mathrm{d}v}{\mathrm{d}x}+2v^2x^2=v^2x^2-x^2$ $2vx^3\dfrac{\mathrm{d}v}{\mathrm{d}x}=-v^2x^2-x^2$ $x\neq 0$, thus $\dfrac{\mathrm{d}v}{\mathrm{d}x}=-\dfrac{x^2(v^2+1)}{2vx^3}$ Thus, $\dfrac{\mathrm{d}v}{\mathrm{d}x}=-\dfrac{v^2+1}{2vx}$ (shown)
(ii)	$\dfrac{\mathrm{d}v}{\mathrm{d}x}=-\dfrac{v^2+1}{2vx}$ $\displaystyle\int\frac{2v}{v^2+1}\,\mathrm{d}v=-\int\frac{1}{x}\,\mathrm{d}x$ $\ln(v^2+1)=-\ln\lvert x\rvert+c$ $v^2+1=\mathrm{e}^{-\ln\lvert x\rvert+2c}$ $v^2+1=A\left(\dfrac{1}{x}\right),\ A=\pm\,\mathrm{e}^{2c}$ $\left(\dfrac{y}{x}\right)^2+1=A\left(\dfrac{1}{x}\right)$ $y^2+x^2=Ax$, where A is an arbitrary constant.
(iii)	When $x=2$, $y=0$, $A=2$ $y^2+x^2=2x$ $\therefore\ y^2+(x-1)^2=1$ 

Qn	Suggested Solution
2	$\cos t\frac{\mathrm{d}y}{\mathrm{d}t}-y\sin t=\cos t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}+y\cdot\frac{-\sin t}{\cos t}=1$ --- (*) I.F. = $\mathrm{e}^{\int\frac{-\sin t}{\cos t}\,\mathrm{d}t}=\mathrm{e}^{\ln\cos t}=\cos t$ Multiply DE (*) throughout by I.F, $\cos t\frac{\mathrm{d}y}{\mathrm{d}t}-y\sin t=\cos t$ $\frac{\mathrm{d}}{\mathrm{d}t}\left(y\cos t\right)=\cos t$ $y\cos t=\sin t+c$ GS: $y=\tan t+c\sec t$, where c is an arbitrary constant.
(i)	$t=0, y=0$: $c=0$. Thus we sketch $y=\tan t$
(ii)	Sub $t=-\frac{\pi}{6}, \frac{\mathrm{d}y}{\mathrm{d}t}=0$ into DE (*) : $y\left(\frac{1}{\sqrt{3}}\right)=1\Rightarrow y=\sqrt{3}$ $t=-\frac{\pi}{6}, y=\sqrt{3}$: $\sqrt{3}=-\frac{1}{\sqrt{3}}+\frac{2c}{\sqrt{3}}\Rightarrow c=2$. Thus we sketch $y=\tan t+2\sec t$  Question requested for sketch **in a single diagram**.

Qn	Suggested Solution
3(a)	Method 1: Variable separable $\left(1+\cos\theta\right)\dfrac{\mathrm{d}x}{\mathrm{d}\theta}+x=1$ $\left(1+\cos\theta\right)\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=1-x$ $\dfrac{1}{1-x}\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{1}{\left(1+\cos\theta\right)}$ $\displaystyle\int\frac{1}{1-x}\ \mathrm{d}x=\int\frac{1}{2\cos^2\frac{\theta}{2}}\ \mathrm{d}\theta$ $\displaystyle -\ln\lvert 1-x\rvert=\int\frac{1}{2}\sec^2\frac{\theta}{2}\ \mathrm{d}\theta$ $-\ln\lvert 1-x\rvert=\tan\dfrac{\theta}{2}+C$ $\lvert 1-x\rvert=\mathrm{e}^{-\tan\frac{\theta}{2}-C}$ $\left(1-x\right)=A\mathrm{e}^{-\tan\frac{\theta}{2}}$, where $A=\pm\mathrm{e}^{-C}$ is an arbitrary constant $x=1+c\mathrm{e}^{-\tan\frac{\theta}{2}}$, where c is an arbitrary constant. Method 2: Integrating factor $\left(1+\cos\theta\right)\dfrac{\mathrm{d}x}{\mathrm{d}\theta}+x=1$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}+\dfrac{x}{1+\cos\theta}=\dfrac{1}{1+\cos\theta}$ ---------- (*) Integrating factor = $\mathrm{e}^{\int\frac{1}{1+\cos\theta}\ \mathrm{d}\theta}=\mathrm{e}^{\int\frac{1}{2\cos^2\frac{\theta}{2}}\ \mathrm{d}\theta}=\mathrm{e}^{\int\frac{1}{2}\sec^2\frac{\theta}{2}\ \mathrm{d}\theta}=\mathrm{e}^{\tan\frac{\theta}{2}}$ Multiply (*) throughout by I, $\mathrm{e}^{\tan\frac{\theta}{2}}\dfrac{\mathrm{d}x}{\mathrm{d}\theta}+\mathrm{e}^{\tan\frac{\theta}{2}}\dfrac{x}{1+\cos\theta}=\mathrm{e}^{\tan\frac{\theta}{2}}\dfrac{1}{1+\cos\theta}$ $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(x\mathrm{e}^{\tan\frac{\theta}{2}}\right)=\mathrm{e}^{\tan\frac{\theta}{2}}\dfrac{1}{1+\cos\theta}$ $\displaystyle x\mathrm{e}^{\tan\frac{\theta}{2}}=\int\mathrm{e}^{\tan\frac{\theta}{2}}\frac{1}{1+\cos\theta}\ \mathrm{d}\theta$ $x\mathrm{e}^{\tan\frac{\theta}{2}}=\mathrm{e}^{\tan\frac{\theta}{2}}+c$, where c is an arbitrary constant $x=1+c\mathrm{e}^{-\tan\frac{\theta}{2}}$

(b)	$\frac{dy}{dx}+\frac{y}{x}=2y^2\ln x$ Substitute $y=\frac{1}{u} \Rightarrow \frac{dy}{dx}=-\frac{1}{u^2}\frac{du}{dx}$ $-\frac{1}{u^2}\frac{du}{dx}+\frac{1}{ux}=\frac{2}{u^2}\ln x$ $\frac{du}{dx}-\frac{u}{x}=-2\ln x$ ---------- (*) Integrating factor $= e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$ Multiplying (*) throughout by I, $\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}=-\frac{2}{x}\ln x$ $\frac{d}{dx}\left(u\frac{1}{x}\right)=-\frac{2}{x}\ln x$ $\frac{u}{x}=-2\int\frac{1}{x}\ln x\,dx$ $\frac{u}{x}=-2\frac{(\ln x)^2}{2}+C$ $u=-x(\ln x)^2+Cx$ $\frac{1}{y}=-x(\ln x)^2+Cx$ $y=\frac{1}{Cx-x(\ln x)^2}$, where C is an arbitrary constant

Qn	Suggested Solution
4(a)(i)	$x^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = y$ When $y=0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$. Thus, $y=0$ is a solution to the differential equation. When $y \neq 0$, we have $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x^2}$ $\displaystyle\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x^2}\ \mathrm{d}x$ Therefore, $\ln\lvert y\rvert = -\dfrac{1}{x} + C \Rightarrow \lvert y\rvert = \mathrm{e}^{-\frac{1}{x}+C} = A\mathrm{e}^{-\frac{1}{x}}$, where $A = \mathrm{e}^C > 0$. So, $y = B\mathrm{e}^{-\frac{1}{x}}$ where B is an arbitrary real constant.
(ii)	$y \to 2$ as $x \to \infty$. $\Rightarrow 2 = B\mathrm{e}^0$ i.e. $B = 2$ Sketch $y = 2\mathrm{e}^{-\frac{1}{x}}$ for $x > 0$. 
(b)(i)	$\dfrac{\mathrm{d}V}{\mathrm{d}t} + \dfrac{1}{20}V = \dfrac{1}{4}t + 1$. Integrating factor $= \mathrm{e}^{\int \frac{1}{20}\,\mathrm{d}t} = \mathrm{e}^{\frac{1}{20}t}$ $\therefore \dfrac{\mathrm{d}}{\mathrm{d}t}\left(V\mathrm{e}^{\frac{1}{20}t}\right) = \mathrm{e}^{\frac{1}{20}t}\left(\dfrac{1}{4}t+1\right)$ $V\mathrm{e}^{\frac{1}{20}t} = \displaystyle\int \mathrm{e}^{\frac{1}{20}t}\left(\frac{1}{4}t+1\right)\ \mathrm{d}t$ $= 20\mathrm{e}^{\frac{1}{20}t}\left(\dfrac{1}{4}t+1\right) - \displaystyle\int 5\mathrm{e}^{\frac{1}{20}t}\ \mathrm{d}t$ $= 5\mathrm{e}^{\frac{1}{20}t}(t+4) - 100\mathrm{e}^{\frac{1}{20}t} + C$ $V = 5(t+4) - 100 + C\mathrm{e}^{-\frac{1}{20}t}$ $\therefore V = 5t - 80 + C\mathrm{e}^{-\frac{1}{20}t}$, where C is an arbitrary constant. $V = 0$ when $t = 0$: $C = 80$ $\therefore V = 5t - 80 + 80\mathrm{e}^{-\frac{1}{20}t}$ For large values of t, $80\mathrm{e}^{-\frac{1}{20}t} \approx 0$, therefore $V \approx 5t - 80$.
(ii)	Since $V \approx 5t - 80$ which increases indefinitely with time, the tank will overflow in time to come.

5(a)

$$\left(1+x\right)y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{1+x}{x}\right)y = -x^2 + x$$

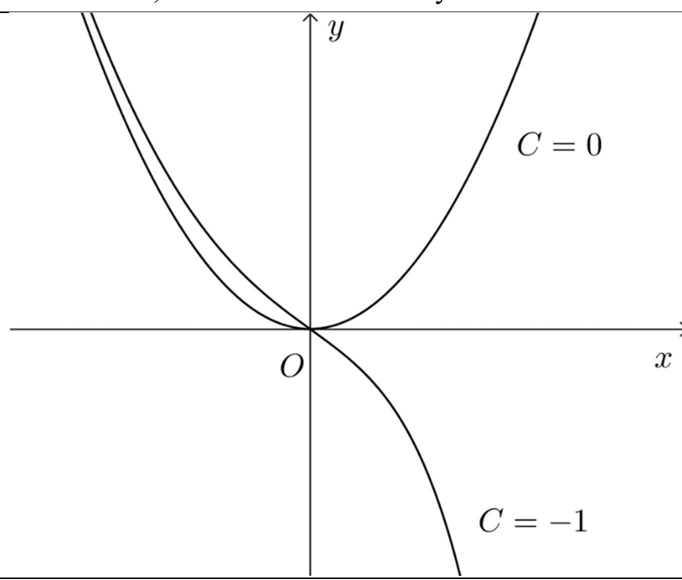
$$\text{Integrating factor} = \mathrm{e}^{\int -\left(\frac{1+x}{x}\right)\mathrm{d}x} = \mathrm{e}^{-\ln x - x} = \frac{1}{x}\mathrm{e}^{-x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\mathrm{e}^{-x}\cdot y\right) = \frac{1}{x}\mathrm{e}^{-x}\left(-x^2 + x\right)$$

$$\begin{aligned}\frac{y}{x}\mathrm{e}^{-x} &= \int \mathrm{e}^{-x}\left(1-x\right)\mathrm{d}x \\ &= -\mathrm{e}^{-x}\left(1-x\right) - \int \mathrm{e}^{-x}\,\mathrm{d}x \\ &= -\mathrm{e}^{-x}\left(1-x\right) + \mathrm{e}^{-x} + C \\ &= x\mathrm{e}^{-x} + C\end{aligned}$$

$y = x^2 + Cx\mathrm{e}^{x}$, where C is an arbitrary constant.

(a)



(b)

Step size, $h = 0.2$,

$t_1 = 0, N_1 = 1$: $u_2 = 1 + (0.2)\left[3(0) - 1^2\right] = 0.8$

$$N_2 = 1 + (0.2)\left[\frac{[3(0) - 1^2] + [3(0.2) - 0.8^2]}{2}\right] = 0.896$$

$t_2 = 0.2, N_2 = 0.896$: $u_3 = 0.896 + (0.2)\left[3(0.2) - 0.896^2\right] = 0.8554368$

$$N_3 = 0.896 + (0.2)\left[\frac{[3(0.2) - 0.896^2] + [3(0.4) - 0.8554368^2]}{2}\right]$$

$$= 0.9225 \text{ (4 d.p.)}$$

6 (i)

$$\mathrm{f}(-0.1) \approx \mathrm{f}(0) + \left(-0.1\right)\mathrm{f}'(0) = 1 - 0.1\left(0 - 1^2\right) = 1.1$$

	$f(-0.2) \approx 1.1 + (-0.1)\left(2(-0.1) - (1.1)^2\right) = 1.241$
(ii)	$\frac{d^2y}{dx^2} = 2 - 2y\frac{dy}{dx} = 2 - 2y\left(2x - y^2\right) = 2 - 4xy + 2y^3$ $\frac{d^2y}{dx^2}$ is positive in the 2nd quadrant because $x < 0$ and $y > 0$. Thus, the approximation is an under-estimate as all the solution curves in the 2nd quadrant are concave up.

7(i)	Rate of X added into cooler = $30 \times 4 = 120$ grams per minute Total amount of mixture at time t = $50 + 4t - 3t = 50 + t$ Concentration of X in mixture at time t = $\frac{\text{amount of } X \text{ in cooler}}{\text{total volume in cooler}} = \frac{x}{50+t}$ Rate of X flowing out of cooler = $\frac{3x}{50+t}$ grams per minute Hence $\frac{dx}{dt} = 120 - \frac{3x}{50+t}$.
(ii)	$\frac{dx}{dt} + \frac{3x}{50+t} = 120$ Integrating factor = $e^{\int \frac{3}{50+t} dt} = e^{3\ln(50+t)} = (50+t)^3$ $\frac{dx}{dt}(50+t)^3 + 3x(50+t)^2 = 120(50+t)^3$ $\frac{d}{dt}\left[x(50+t)^3\right] = 120(50+t)^3$ $x(50+t)^3 = 120 \cdot \frac{(50+t)^4}{4} + C$ $x = 30(50+t) + C(50+t)^{-3}$ When $t = 0$, $x = 0$. Hence $C = -187500000 \Rightarrow x = 30(50+t) - \frac{187500000}{(50+t)^3}$ There are 51 litres in the cooler when $t = 1$. Then $x = 30(51) - \frac{187500000}{(51)^3} = 116.52$ grams Amount of chemical X is <u>117 grams</u> (3 s.f.)
	Let $f(t,\theta) = \frac{d\theta}{dt} = 60e^{0.04t}\left[\cos(0.01\theta) - 1\right]$ $t_1 = 0, \theta_1 = 60$: $u_2 = 60 + 0.5f(0,60) = 54.760$

	$$\theta_2 = 60 + \frac{0.5}{2}\left[\mathrm{f}(0,60) + \mathrm{f}(0.5, 54.760) \right] = 55.142$$ $t_2 = 0.5, \theta_2 = 55.142$: $u_3 = 55.142 + 0.5\mathrm{f}(0.5, 55.142) = 50.606$ $$\theta_3 = 55.142 + \frac{0.5}{2}\left[\mathrm{f}(0.5, 55.142) + \mathrm{f}(1, 50.606) \right] = 50.917$$ Hence after 1 minute, temperature of mixture is 50.9 ºC (1 d.p.).