3. Functions

1 PJC/2008Promo/8

The functions f and g are defined by

$$\mathrm{f} : x \to x^2(2-x), \quad x \in \mathbb{R} ,$$

$$\mathrm{g} : x \to 2+\frac{1}{x} \ , \ x \in \mathbb{R}, \ x \neq 0 .$$

(i) Find $\mathrm{g}^{-1}(x)$ and state its domain. [2]

(ii) Determine whether the composite function gf exists. [2]

(iii) Solve the equation fg(x) = 1, give your answer to 3 significance figures. [3]

2 SAJC/2008Promo/8

The function f and g are defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{-x} , \ x \in \mathbb{R}^+$$

$$\mathrm{g} : x \mapsto 3x^2 + 2, \quad x \in \mathbb{R}^-$$

(a) Determine, with reason, whether the inverse for f exists.
If f^{-1} exists, define f^{-1} in a similar form and state its range.
On the same axes, sketch the graphs of f, f^{-1} and ff^{-1}. [5]

(b) Determine, stating reason, whether fg exists. If the function exists, give its domain, rule and range**.** [4]

3 NYJC/2013Promo/7

The function f is defined by

$$\mathrm{f} : x \to x^2 - \frac{1}{x}, \ x \in \mathbb{R}, \ 1 \leq x < 2 .$$

(i) Show, by differentiation, that f is strictly increasing. **[2]**
(ii) State the range of f. **[1]**
(iii) Solve the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, giving your answer to two decimal places. **[2]**

The function g is defined by

$$\mathrm{g}: x \to 1+\sin x, \ x \in \mathbb{R}, \ 0 \leq x < \frac{\pi}{2} .$$

(iv) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. **[3]**
(v) For the composite function which exists, state its range. **[1]**

4 DHS/2009Promo/7

The functions f and g are defined by

$$\mathrm{f} : x \mapsto \tan x + 1, \quad 0 < x < \frac{\pi}{2},$$

$$\mathrm{g} : x \mapsto \frac{1+x}{x}, \qquad x > 0.$$

(i) Find the derivative of $\mathrm{h}(x)$, where $\mathrm{h}(x) = \dfrac{1}{\mathrm{f}(x)}$. Hence show that h is a one-one function. [3]

(ii) Find an expression for $\mathrm{h}^{-1}(x)$. [1]

(iii) Show that the composite function gf exists. Define gf in similar form and state the range of gf. [4]

5 JJC/2012Promo/5

The function f is defined by $\mathrm{f} : x \mapsto \dfrac{x+2}{x-1}$, for $x \in \mathbb{R}$, $x \neq 1$.

(i) Find $\mathrm{f}^2(x)$ and $\mathrm{f}^{2012}(x)$. [3]

The function g is defined by $\mathrm{g} : x \mapsto \cos x$, for $0 < x < 2\pi$.

(ii) Explain why the composite function fg exists. [2]

(iii) Define fg, giving its domain. [2]

(iv) Find the range of fg. [1]

6 JJC/2010Promo/6

An inverse function is defined by $\mathrm{f}^{-1}(x) = \ln(x^2 - 1), \quad x \in \mathbb{R}, x > 1.$

(i) Find f(x) and state the domain of f. [3]

(ii) Explain why ff^{-1} exists and find ff^{-1} in a similar form. [2]

(iii) Sketch the graph of $y = \mathrm{ff}^{-1}(x)$ and state the range of ff^{-1}. [2]

7 HCI/2008Promo/14 [*Part ii removed. Out of syllabus*]

The functions f and g are defined as follows:

$$\mathrm{f} : x \mapsto 3 - 2x - x^2, \ x \in \mathbb{R}, \ x \leq k,$$

$$\mathrm{g} : x \mapsto e^{\sqrt{4-x}}, \ 0 \leq x \leq 4.$$

State the largest value of k such that f^{-1} exists, and find f^{-1} in a similar form. [4]

(i) Show that the composite function gf does not exist. [1]

(iii) Find the set of values of x such that $\mathrm{g}^{-1}\mathrm{g}(x + 1) = \mathrm{g}\,\mathrm{g}^{-1}(x + 1)$. [3]

8 NJC/2010Promo/5

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{|x+1|}, \quad -2 < x < -1,$$

$$\mathrm{g}: x \mapsto x^2 - 4\lambda x, \quad x > 2\lambda, \text{ where } \lambda \text{ is a real constant.}$$

(i) Find $\mathrm{g}^{-1}(x)$ in terms of λ, stating the domain of g^{-1}. [3]

(ii) Determine the set of values of λ for which the composite function gf exist . [2]

(iii) Given that $\lambda = -1$, find the range of gf. [2]

9 NJC/2013Promo/10

The function f is defined as follows:

$$\mathrm{f}: x \mapsto \ln\left(x^2+1\right), \ x \in \mathbb{R}.$$

(i) Without the use of a calculator, find the set of values of x for which the graph of $y = \mathrm{f}(x)$ is concave upwards. **[4]**

(ii) Sketch the graph of $y = \mathrm{f}(x)$. **[2]**

(iii) The function f has an inverse if its domain is restricted to $x \geq k$. State the set of all possible values of k . **[1]**

The function g is defined by $\mathrm{g}: x \mapsto \ln\left(x^2+1\right), \ x \geq 1.$

(iv) Find $\mathrm{g}^{-1}(x)$ and state the exact domain of g^{-1}. **[3]**

10 RI/2013Promo/14

The function f is defined as follows :

$$\mathrm{f}(x) = \begin{cases} 2-x, & 0 \leq x \leq 2, \\ \dfrac{x(2-x)}{4}, & 2 < x \leq 4. \end{cases}$$

(i) Sketch the graph of f and show that the inverse function of f exists. **[3]**

(ii) Sketch the graph of f^{-1} on the same diagram as the graph of f, showing clearly their relationship. State the range of values of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. **[3]**

(iii) Solve $\mathrm{f}^{-1}(x) = 3$. **[2]**

(iv) Find the exact value of $\int_2^3 \mathrm{f}(x)\,\mathrm{d}x$. Hence, or otherwise, find the exact value of $\int_{-\frac{3}{4}}^{2} \mathrm{f}^{-1}(x)\,\mathrm{d}x$. **[4]**

11 DHS/2013MYE/10

The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \begin{cases} -\dfrac{x^2}{4}, & -2 \le x < 0, \\ x^3, & 0 \le x \le 2. \end{cases}$$

$$\mathrm{g}: x \mapsto x, \quad -1 \le x < 0.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$ and show that f^{-1} exists. [3]

(ii) Find f^{-1} in a similar form. [4]

The function h is a restriction of f to $-2 \le x < 0$.

(iii) Show that the composite function hg exists. [1]

(iv) Solve $\mathrm{gh}(x) = \mathrm{hg}(x)$, showing your working clearly. [2]

12 MJC/2015Promo/9

The function f is defined by

$$\mathrm{f}: x \mapsto x^2 - 2x - 1, \quad -1 \le x \le 1.$$

(i) Define f^{-1} in a similar form and sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on a single diagram, showing clearly the relationship between the graphs. [6]

The function g is defined by

$$\mathrm{g}: x \mapsto \begin{cases} 9 - 3x, & 0 \le x < 3, \\ (x-3)^2, & 3 \le x < 6, \end{cases}$$

and that $\mathrm{g}(x) = \mathrm{g}(x+6)$ for all real values of x.

(ii) Sketch the graph of $y = \mathrm{g}(x)$ for $-2 \le x \le 8$. [3]

(iii) Give a reason why the composite function gf exists and hence state its range. [2]

13 PJC/2015Promo/7

Functions f and g are defined by

$$\mathrm{f}: x \mapsto -\frac{2}{x-1}, \ x \in \mathbb{R}, \ x \neq 1,$$
$$\mathrm{g}: x \mapsto 1-2x, \ x \in \mathbb{R}.$$

(i) Only one of the composite functions fg and gf exists. Give a definition, domain and range of the composite that exists, and explain why the other composite does not exist. [4]

A function h is said to be self-inverse if $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$ for all x in the domain of h.

(ii) Show that gf is self-inverse. [3]

14 HCI/2015Promo/5 [*Part iii removed. Out of syllabus*]

The function f and g are defined by

$$\mathrm{f}: x \mapsto 2\cos x, \ x \in [-2\pi, 2\pi],$$
$$\mathrm{g}: x \mapsto \frac{2x-1}{x-1}, \ x \in \mathbb{R}, \ x < 1.$$

(i) Give a reason why f does not have an inverse. [1]

(ii) The function f has an inverse if its domain is restricted to $\frac{\pi}{2} \leq x \leq b$.

Find the greatest value of b. Define f^{-1} in similar form. [3]

15 VJC/2013MYE/12

The functions f is defined by

$$\mathrm{f} : x \mapsto x^2 + 2x - 3, \qquad \text{for} \quad x \le k.$$

(i) Explain why f^{-1} does not exist when $k = 1$. [2]

(ii) State the largest value of k such that f^{-1} exists. [1]

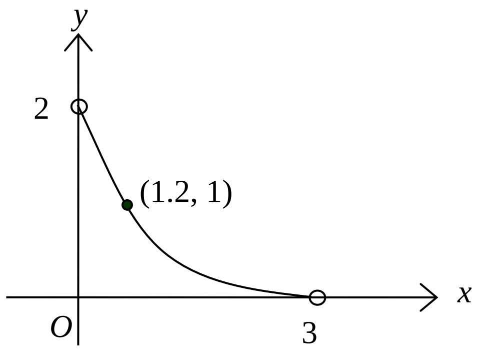
With the value of k found in **part (ii)**,

(iii) define f^{-1} in a similar form, [4]

(iv) sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram, [2]

(v) write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$, and hence find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

The domain of a function g is $\{x \in \mathbb{R} : 0 < x < 3\}$, and its graph passes through the point (1.2, 1). The graph of $y = \mathrm{g}(x)$ is given below.



In the rest of the question, take k to be 1 in the definition of f.

(vi) Give a reason why fg does not exist. [1]

(vii) The function h is defined by

$$\mathrm{h} : x \mapsto \mathrm{g}(x), \qquad \text{for} \quad 1.2 \le x < 3.$$

Find the range of fh, showing your working clearly. [2]

16 IJC/2017 Promo/Q3

It is given that

$$\mathrm{f}(x) = \begin{cases} 8x & , \quad 0 \le x < 1 \\ (3-x)^3 & , \quad 1 \le x \le 3 \end{cases}$$

It is also known that $\mathrm{f}(x) = \mathrm{f}(x+3)$ for all real values of x.

(i) Evaluate $\mathrm{f}(-4) + \mathrm{f}(22)$. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-4 \le x \le 7$. [3]

Answers

1	(i) $g^{-1}: x \to \frac{1}{x-2},\ \ x \neq 2$; (iii) $x = -2.62, -1,$ or -0.382
2	(a) $f^{-1}: x \mapsto -\ln x,\ 0 < x < 1$ ($D_{f^{-1}} = R_f$ =(0,1)) Range of $f^{-1} = \mathbb{R}^+$ (b) fg: $x \mapsto e^{-(3x^2+2)},\ x < 0.$; $R_{fg} = (0, e^{-2})$
3	(ii) $R_f = = \left[0, \frac{7}{2}\right).$ (iii) x = 1.47 (iv) $D_{fg} = D_g = \left[0, \frac{\pi}{2}\right)$ (v) $R_{fg} = \left[0, \frac{7}{2}\right)$
4	$h^{-1}(x) = \tan^{-1}\left(\frac{1}{x} - 1\right)$; $gf: x \mapsto 1 + \frac{1}{\tan x + 1}, 0 < x < \frac{\pi}{2}$, $R_{gf} = (1, 2)$
5	(i) $f^2(x) = x$, $f^{2012}(x) = x$. (ii) Since $R_g \subseteq D_f$, fg exists. (iii) $fg: x \mapsto \frac{\cos x + 2}{\cos x - 1},\ 0 < x < 2\pi$. (iv) $R_{fg} = \left(-\infty, -\frac{1}{2}\right]$
6	(i) $f(x) = \sqrt{e^x + 1}, x \in \mathbb{R}$; (ii) $ff^{-1}(x) = x,\ x > 1$ (iii) $R_{ff^{-1}} = (1, \infty)$
7	$k = -1$; $f^{-1}: x \mapsto -1 - \sqrt{4 - x}, x \in (-\infty, 4]$; (iii) set of values of x = [0, 3]
8	(i) $g^{-1}: x \mapsto 2\lambda + \sqrt{x + 4\lambda^2},\ \ x > -4\lambda^2$ (ii) $\lambda \leq \frac{1}{2}$ (iii) $R_{gf} = (5, \infty)$
9	(i) $-1 < x < 1$ (iii) f^{-1} exist if $x \geq k$, where $k \geq 0$. (iv) $g^{-1}: x \mapsto \sqrt{e^x - 1},\ \ x \in [\ln 2, \infty)$
10	(iii) $-\frac{3}{4}$ (iv) $-\frac{1}{3}; \frac{47}{12}$
11	ii) $f^{-1}: x \mapsto \begin{cases} -\sqrt{-4x}, & -1 \leq x < 0, \\ \sqrt[3]{x}, & 0 \leq x \leq 8. \end{cases}$ (iv) $-1 \leq x < 0$.
12	(i) $f^{-1}: x \mapsto 1 - \sqrt{x + 2}, -2 \leq x \leq 2$ (iii) $R_{gf} = [1, 9]$
13	(i) gf exists; gf(x)=$\frac{x+3}{x-1}$; $D_{gf} = D_f = \mathbb{R} \setminus \{1\}$ $R_{gf} = \mathbb{R} \setminus \{1\}$
14	$b = \pi$; $f^{-1}: x \mapsto \cos^{-1}\frac{x}{2},\ -2 \leq x \leq 0$;
15	(ii) largest k = -1 (iii) $f^{-1}: x \mapsto -1 - \sqrt{x + 4},\ \ x \geq -4$

	(v) $y = \mathrm{f}(x)$, $x = \dfrac{-1-\sqrt{13}}{2}$ (vi) $\mathrm{R_g} = (0, 2)$ and $\mathrm{D_f} = (-\infty, 1]$ $\Rightarrow$ $\mathrm{R_g} \subseteq \mathrm{D_f}$. Thus, fg does not exist. (vii) $\mathrm{R_{fh}} = (-3, 0]$
16	(i) 9