4. Differentiation and its Applications

(I) Tangents and Normals (Direct/Implicit Differentiation)

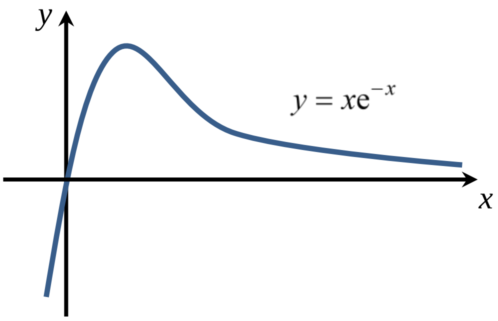
1. RI/2011Prelim/I/4

(a) Differentiate $\dfrac{x}{\sin^{-1}(3x)}$ with respect to x, giving your answer as a single fraction. [3]

(b) Given that $y^{\cos 2x} = x^3$, find the exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \pi$. [4]

2. NYJC/2011Prelim/I/9 (modified)

The diagram shows a sketch of the curve $y = x\mathrm{e}^{-x}$.



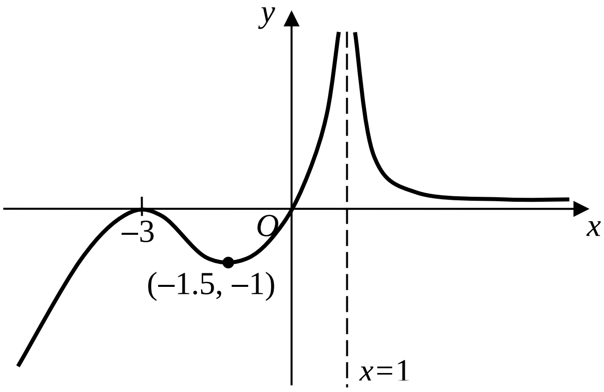
(i) By differentiation, find the range of values of x for which the graph of $y = x\mathrm{e}^{-x}$ is decreasing. [3]

(ii) Determine the range of values of x for which the graph of $y = x\mathrm{e}^{-x}$ is decreasing and concave downwards. [3]

(iii) The tangent to the curve at $P(a, b)$ meets the y-axis at $R(0, h)$. Express h in terms of a, and find the greatest possible value of h. [6]

3. HCI/2009Prelim/I/3

The diagram shows the graph of $y = \mathrm{f}'(x)$. The curve passes through the origin and has turning points at $(-3, 0)$ and $(-1.5, -1)$. The x-axis and $x = 1$ are the two asymptotes of the curve.



(i) Find the range of values of x for which the graph of $y = \mathrm{f}(x)$ is strictly increasing and concave downwards. [1]

(ii) State the x-coordinates of all the stationary points of the graph of $y = \mathrm{f}(x)$ and determine the nature of each point. [2]

(iii) Given that $\mathrm{f}(0) = 1$, sketch the graph of $y = \mathrm{f}(x)$ for $x < 1$. Your sketch should indicate clearly all stationary points, asymptotes and intersections with the axes. [2]

4. ACJC/2009Prelim/I/7

A point (x, y) lies on the curve with equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$, where a is a non-zero constant, $x \neq 0$ and $y \neq 0$.

(i) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y, and explain clearly whether y is an increasing or a decreasing function. [3]

(ii) State, giving a reason, whether there are any stationary points on this curve. [2]

(iii) Find the equation of the tangent to the curve at the point $(2a, 2a)$ and determine if the tangent cuts the curve again. [5]

5. DHS/2009Prelim/I/4

(a) Suppose the following facts are known about the function g and its derivative:

$$\mathrm{g}(5)=1, \quad \mathrm{g}'(5)=-3.$$

If $\mathrm{f}(x)=\tan\left(\mathrm{e}^{\mathrm{g}(x)}\right)$, find the value of $\mathrm{f}'(5)$. [3]

(b) Find, by differentiation, the range of values of x for which the function $y=\frac{x^2}{2x-1}$ increases as x increases. [3]

6. JJC/2015Promo/6

A curve has equation

$$3x^2-4xy+2y^2-2=0.$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x-2y}{2x-2y}$. [3]

(ii) Find the exact coordinates of the points on the curve where the tangent is parallel to the x-axis. [4]

(iii) The normal to the curve at the point P with coordinates (0, 1), meets the curve again at point Q. Find the area of triangle OPQ, where O is the origin. [5]

7. RI/2013Promo/1

A curve C is given by the equation $\sqrt{x}+\sqrt{y}=\sqrt{a}$, for $x>0$, $y>0$, where a is a positive constant.

(i) Show that C has no stationary points. [3]

(ii) What can be said about the tangents to C as $x \to 0$? [1]

8. NJC/2015Promo/8

A curve L has equation

$$(4x-y)^2+16y=48.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [3]

(ii) The tangent to L at the point P is parallel to the x-axis. State the equation of this tangent and the coordinates of P. [2]

(iii) The tangent to L at the point Q is parallel to the y-axis. Find the equation of this tangent and the coordinates of Q. [4]

(iv) The tangents in **(ii)** and **(iii)** intersect at the point R. Find the area of triangle PQR. [1]

(II) Tangents and Normals (Parametric Equations)

9. SRJC/2015Promo/6

The parametric equations of a curve are

$$x=\mathrm{e}^{t}\sin t,\quad y=\mathrm{e}^{-t}\cos t,\quad \text{for } 0<t<\pi$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^{-2t}$. The point P on the curve has parameter p. Find the equation of the normal to the curve at P. [5]

(ii) The normal at P when $p=\frac{\pi}{2}$ meets the x-axis and y-axis at points A and B respectively. Find the area of triangle OAB, where O is the origin. [2]

10. SAJC/2015Promo/8

A curve C is defined by the equations

$$x=t^2,\quad y=2t,\ t\in\mathbb{R}.$$

(i) Find the equation of the normal to C at the point P with parameter p. [3]

(ii) Given that the normal at P where $p=2$ cuts C again at the point Q, prove that the angle QOP is $\frac{\pi}{4}+\tan^{-1}\left(\frac{2}{3}\right)$. [4]

11. YJC/2010Prelim/II/Q1

A curve has parametric equations $x=3(1-t)$, $y=\frac{1}{t^3}$ for $t\neq 0$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and deduce that the curve is an increasing function. [3]

(ii) Find the equation of L_1, the tangent to the curve at the point $\left(3-3t,\frac{1}{t^3}\right)$. Hence, find the coordinates of point P on the curve at which L_1 passes through the origin O. [4]

(iii) The line L_2 is another tangent to the curve which is parallel to L_1. Find the equation of L_2. [3]

(iv) The line L_2 cuts the y-axis at Q. Find the exact area of triangle OPQ. [2]

12. HCI/2011Prelim/II/3

A curve is defined by the parametric equations

$$x=\sqrt{u}, \quad y=\frac{1}{u^2}-2u, \quad \text{where } u>0.$$

(i) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of u. [2]

(ii) The tangent to the curve at $x=1$ meets the x-axis at A and the y-axis at C, while the normal to the curve at $x=1$ meets the x-axis at B and the y-axis at D. Show that $AB=CD$. [5]

(iii) Given that u is increasing at a rate of 0.5 units per second, find the rate at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing when $u=2$. [3]

13. TJC/2015Promo/8

A curve C has parametric equations

$$x=\mathrm{e}^t, \; y=t-\ln t, \qquad \text{where } t>0.$$

(i) Show that there is no tangent to C parallel to the y-axis and find the equation of the tangent to C that is parallel to the x-axis. [5]

(ii) Describe the behaviour of the tangent to C as $t\to 0$. [1]

(iii) Sketch C, showing clearly its asymptote and turning point. [2]

14. MJC/2015Prelim/I/11

A curve C has parametric equations

$$x=2\cos t, \qquad y=\sin t, \qquad \text{for } 0\le t<2\pi.$$

Show that the equations of the tangent and normal to C at the point P with parameter θ are $(\cos\theta)x+(2\sin\theta)y=2$ and $(2\sin\theta)x-(\cos\theta)y=3\sin\theta\cos\theta$ respectively. [5]

(i) Show algebraically that the tangent to C at the point P does not cut the curve C again. [3]

(ii) The normal to C at the point P cuts the x-axis and y-axis at points A and B respectively. The point F is the midpoint of AB. Find a cartesian equation of the curve traced by F as θ varies. Hence give a geometrical description of this curve. [6]

15. HCI/2015Prelim/I/6

A curve C has parametric equations $x = \mathrm{e}^t + \sin t$, $y = \mathrm{e}^t - \cos t$.

(i) Describe the shape of C as $t \to -\infty$. [2]

(ii) Find the Cartesian equation of the normal to C at the point $P\left(\mathrm{e}^\theta + \sin\theta, \mathrm{e}^\theta - \cos\theta\right)$, where $\theta > 0$, giving your answer in the form $y = mx + c$. [3]

The normal to C at P meets the y-axis at the point D, and the curve C meets the positive x-axis at the point E that has integral coordinates.

(iii) Find the coordinates of D and E. [3]

(iv) Give a geometrical description of the path traced by the midpoint of DE as θ varies. [2]

(III) Rate of change and Maximisation/Minimisation problems

16. VJC/2009Prelim/I/1

A man made pond is constructed in the form of a circular cylinder of radius 4.5 m. The ideal depth of water in the pond is 1.2 m. However, after a storm, the depth of water in the pond is 1.9 m. Water is being pumped out of the pond at a rate of 0.8 $\mathrm{m^3 s^{-1}}$. Find

(i) the rate of change of the depth of water, [3]

(ii) the time required for the water to return to its ideal depth. [1]

17. IJC/2010Prelim2/I/1

For any given mass of gas, the volume V cm^3 and pressure p (in suitable units) satisfy the relationship

$$V = \frac{1}{k} p^n,$$

where k and n are constants.

For a particular type of gas, $n = -2.3$. At an instant when volume is 32 cm^3, the pressure is 105 units and the pressure is increasing at a rate of 0.2 unit/s. Calculate the rate of decrease of volume at this instant. [4]

18. NJC/2009Prelim/I/7

(a) In a triangle ABC, $AB = 3$ cm and $AC = 2$ cm. If angle BAC is increasing at a constant rate of 0.1 radians per second, find the rate of increase of the length BC at the instant where $BAC = \dfrac{\pi}{3}$ radians. [4]

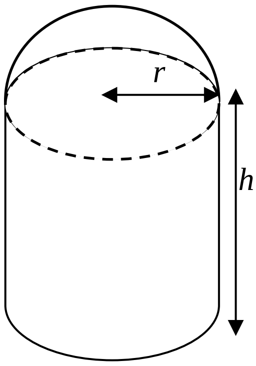
(b) Suppose p is a positive constant. Find the coordinates of the points on the curve

$$x^2 - xy = p^2 + y^2$$

at which the normal is parallel to the x-axis. Leave your answers in terms of p. [4]

19. CJC/2009Prelim/I/7

A candy maker is interested in using containers in the shape shown below to package his candies. The container is made up of an open cylinder of height h cm and radius r cm, with a hollow hemispherical lid of radius r cm.

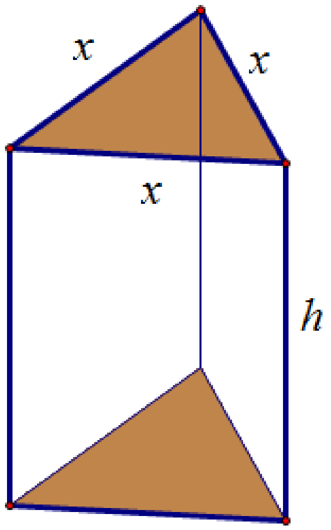


In order to minimise production cost in this difficult time, the candy maker wants to use containers with the least surface area while maintaining the volume of each container at 500 cm^3. If the material used to construct the container cost \$0.015 per cm^2, find, using differentiation, how much a container with minimum surface area costs to the candy maker. Leave your answer to 2 decimal places. [10]

[Volume of sphere, $V=\frac{4}{3}\pi r^3$; Surface area of sphere, $S=4\pi r^2$]

20. SAJC/2010Promo/3

Iron Will, the magician, is constructing a prism with an equilateral triangle base for his latest escape act.



The edges of the prism are made from iron rods, the rectangular faces of the prism are made of glass panels, and the triangular faces of the prism are made of wooden boards.

The volume of the prism is fixed at exactly $2\sqrt{3}$ cubic metres.

Show that $h=\frac{8}{x^2}$, where x is the length of the sides of the triangular base and h is the height of the prism. [2]

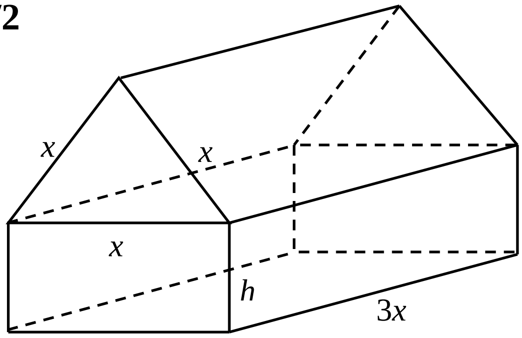
The cost of the iron rods is \$1 per metre, the cost of the wooden boards is $\$2\sqrt{3}$ per sq. metre and the cost of the glass panels is \$2 per sq. metre.

Show that the expression of the total cost C of constructing the prism in terms of x is as follows:

$$C=3x^2+6x+48x^{-1}+24x^{-2}$$ [2]

Using an analytical method, find the minimum cost of constructing this prism. [4]

21. **DHS/2014Prelim/II/2**



An event company builds a tent (as shown in the diagram) which has a uniform cross-sectional area consisting of an equilateral triangle of sides x metres and a rectangle of width x metres and height h metres. The length of the tent is $3x$ metres.

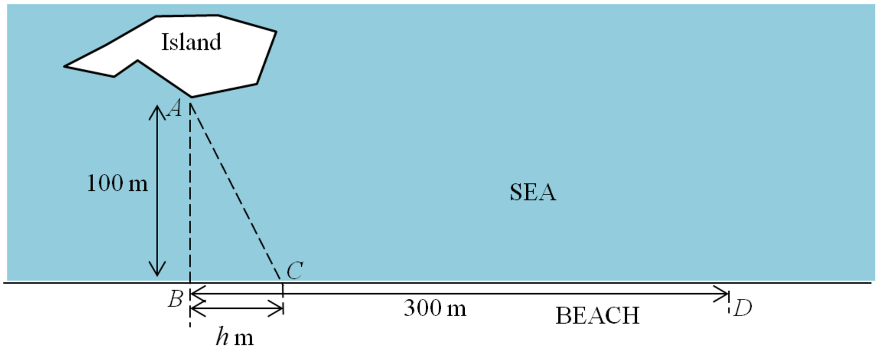
(i) It is given that the tent has a fixed volume of k cubic metres and is fully covered (except the base) with canvas assumed to be of negligible thickness. Show that the area of the canvas used, A, in square metres, is given by $A = 6x^2 - \dfrac{3\sqrt{3}}{2}x^2 + \dfrac{8k}{3x}$.

Use differentiation to find, in terms of k, the value of x which gives a stationary value of A. Determine if A is minimum or maximum for this value of x. [7]

(ii) It is given instead that the tent has a volume of 360 cubic metres and the area of canvas used is 300 square metres. Find the value of x and the value of h. [3]

22. **VJC/2015Promo /7**

A man's running speed on the beach is 4 times as fast as his swimming speed in the sea.



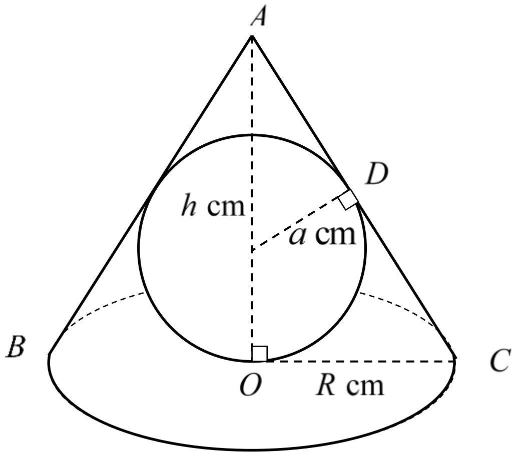
The distances AB, BC and BD are 100 m, h m and 300 m respectively, as shown in the diagram above. The man swims in a straight line from A to C, and then runs from C to D on the beach.

(i) Show that the time taken for him to get from point A to point D through C, is given by $\left(\frac{\sqrt{100^2+h^2}}{v}+\frac{(300-h)}{4v}\right)$s, where v m/s represents his constant swimming speed in the sea. [2]

(ii) Hence, by differentiation, find the value of h that will allow the man to reach D in the shortest amount of time. [4]

23. DHS/2015Promo/9

[It is given that a cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.]

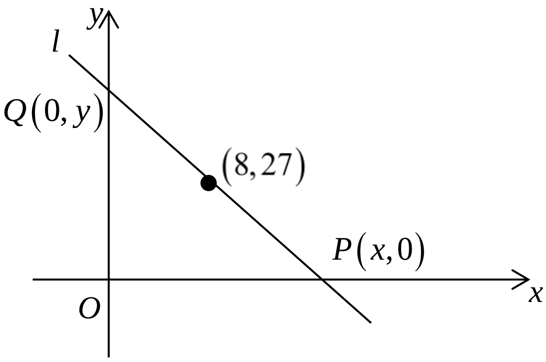


A sphere of constant radius a cm is inscribed in a right circular cone of radius R cm and height h cm, such that the sphere is in contact with the centre of the circular base, O, and the slant surface of the cone.

(i) By considering two different expressions for AC or otherwise, show that $R=\frac{ha}{\sqrt{h^2-2ha}}$. [3]

(ii) As h varies, find the minimum volume of the cone in terms of a. [5]

24. NYJC/2015Promo/5



The line l, with negative gradient, passes through the point with coordinates $(8, 27)$ and meets the x- and y-axes at P and Q respectively. Find an expression in terms of x for the length of PQ.

Given that $x = x_1$ is the value of x which gives the minimum value of length PQ, show that x_1 satisfies the equation $(x-8)^3 - 5832 = 0$. Hence find the exact area of triangle OPQ when PQ is the shortest. [8]

25. NYJC/2013Prelim/I/2

A company manufactures containers in the shape of a right cone. Each container is made from a thin flat sheet of metal in the shape of a sector of a circle, with radius a cm and angle θ radians (see Diagram 1).

The two straight sides of each metal sector are then joined together, without overlap, to form a cone with height h cm and radius r cm (see Diagram 2).

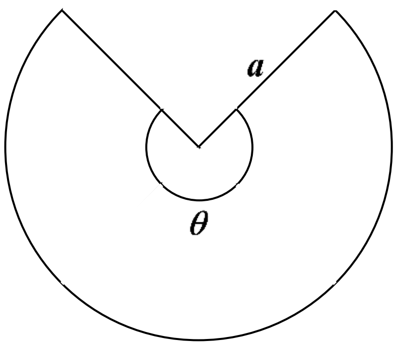


Diagram 1

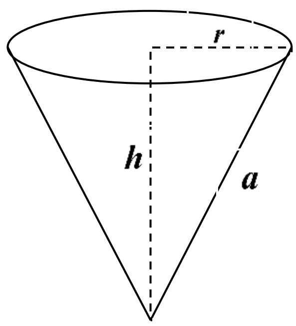


Diagram 2

(i) Express r in terms of a, θ and π. [2]

(ii) Hence, show that the volume, V of a container is given by $V^2 = \dfrac{a^6\theta^4}{576\pi^4}\left(4\pi^2 - \theta^2\right)$. [2]

(iii) Find, in exact form, the maximum volume of the container if the sector of the circle has a radius of 2 cm as θ varies. [4]

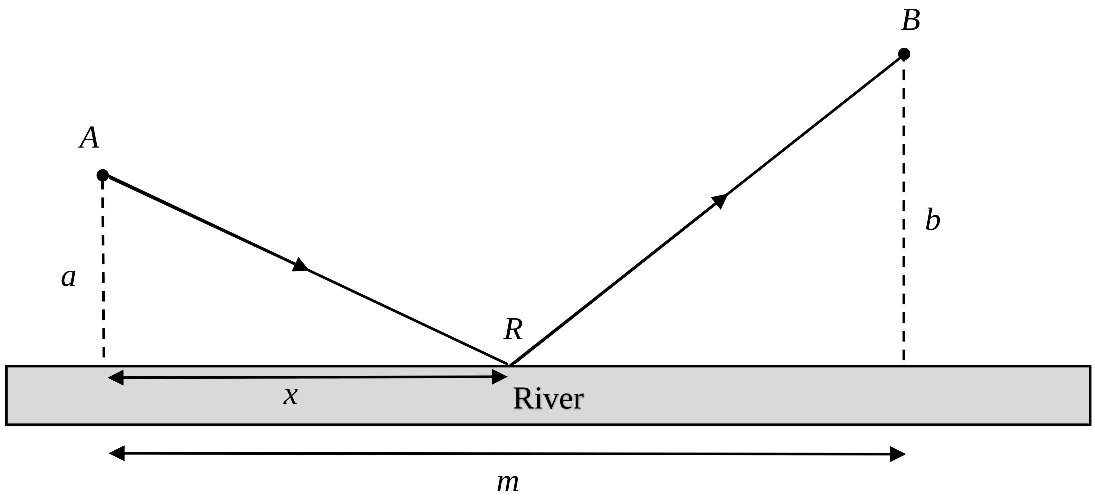
[Note: There is no need to show that the volume is a maximum.]

26. DHS/2020/Y6MCT/10

In a rural area of a developing country, there are two neighbours, Tarzan and Jane, staying at points *A* and *B* respectively near a river. The horizontal distance between *A* and *B* is fixed at *m* units. The shortest distance from *A* and *B* to the river are *a* and *b* units respectively.

A fire occurred at Jane's house and Tarzan wants to help her to put out the fire. He has to fill the buckets with water at the river before reaching Jane's house.

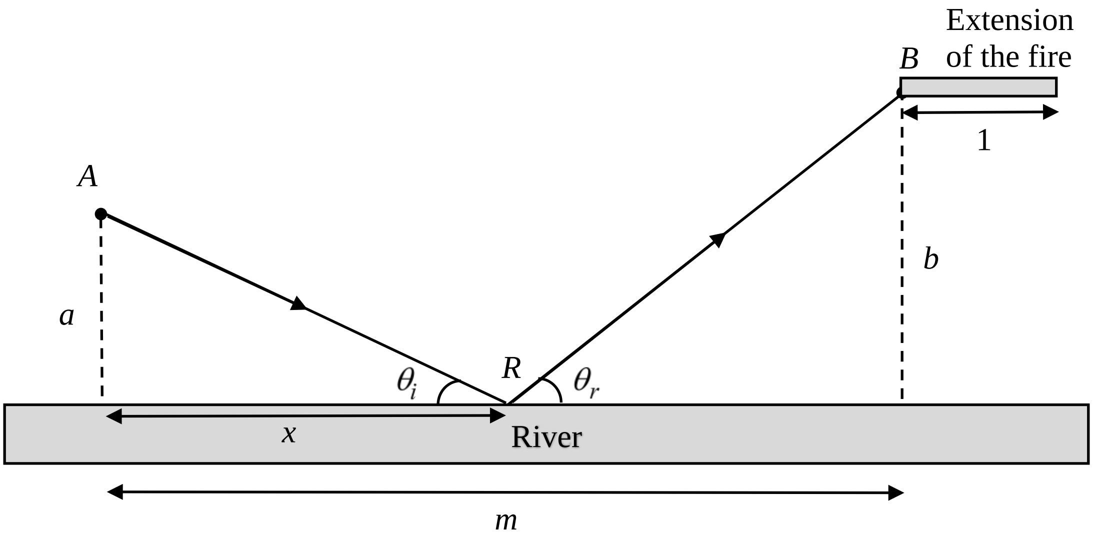
The diagram below shows a possible path taken by Tarzan. Let *x* be the horizontal distance between *A* and *R*, and *L* be the total distance travelled by Tarzan. You may assume that Tarzan runs at a constant speed at all times.



(i) Express *L* in terms of *a*, *b*, *m* and *x*. [1]

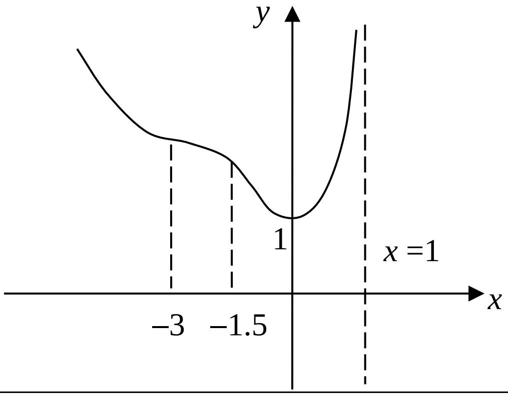
(ii) Using differentiation, show that $x = \dfrac{ma}{a+b}$ is the only value that gives a stationary value of *L*. [6]

(iii) It is known that the stationary value of *L* is the shortest distance that Tarzan can travel to reach Jane's house in the shortest time. θ_i is the angle of the path Tarzan makes when he travels from *A* to the river. θ_r is the angle of the path Tarzan makes when he travels from the river to *B*. Determine whether θ_i is the same as θ_r. [2]



(iv) Due to the wind conditions, the fire at Jane's house is extended 1 unit horizontally from B (see diagram in part **(iii)**). Given that $a = 1,\ b = 2,\ m = 5$ and Tarzan is able to find a lot of his friends at his house to help put out the entire fire, find the range of values of θ_i that his friends can take to reach the fire at the shortest possible time assuming that they run at a constant speed at all times. [3]

Answers

	(I) Tangents and Normals (Direct/Implicit Differentiation)
1	**(a)** $\dfrac{\sqrt{1-9x^2}\,\sin^{-1}(3x)-3x}{\left(\sin^{-1}(3x)\right)^2\sqrt{1-9x^2}}$ **(b)** $3\pi^2$
2	**(i)** $x>1$ **(ii)** $1<x<2$ **(iii)** $h=a^2\mathrm{e}^{-a}$, $h=4\mathrm{e}^{-2}$
3	**(i)** $x>1$ **(ii)** Point of inflexion at $x=-3$, minimum point at $x=0$. **(iii)** 
4	**(i)** y is a decreasing function. **(ii)** No stationary points on curve. **(iii)** $y=-x+4a$; No.
5	**(a)** −9.81 **(b)** $x<0$ or $x>1$
6	**(ii)** $\left(\dfrac{2\sqrt{3}}{3},\sqrt{3}\right)$ and $\left(-\dfrac{2\sqrt{3}}{3},-\sqrt{3}\right)$ **(iii)** $\dfrac{4}{9}$
7	**(ii)** The tangent to C approaches the line $x=0$ (the y-axis).
8	**(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{4(4x-y)}{(4x-y)-8}$ **(ii)** $y=3$, $P\left(\frac{3}{4},3\right)$ **(iii)** $x=\frac{7}{4}$, $Q\left(\frac{7}{4},-1\right)$ **(iv)** 2 unit^2
	(II) Tangents and Normals (Parametric Equations)
9	**(i)** $y=\mathrm{e}^{2p}(x-\mathrm{e}^{p}\sin p)+\mathrm{e}^{-p}\cos p$ **(ii)** $\dfrac{1}{2}\mathrm{e}^{2\pi}$ unit^2
10	**(i)** $y=-px+p^3+2p$
11	**(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{t^4}$ **(ii)** $y=\dfrac{1}{t^4}x-\dfrac{3}{t^4}+\dfrac{4}{t^3}$; $\left(\dfrac{3}{4},\dfrac{64}{27}\right)$ **(iii)** $y=\dfrac{256}{81}x-\dfrac{512}{27}$ **(iv)** $\dfrac{64}{9}$ units^2
12	**(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-4u^{-\frac{5}{2}}\left(1+u^3\right)$ **(iii)** $\dfrac{3}{8\sqrt{2}}$ units per second.

13	**(i)** $y=1$ **(ii)** Tangents to C will tend to the vertical line $x=1$ as $t\to 0$.
14	**(ii)** $\left(\frac{3}{4}\cos\theta, -\frac{3}{2}\sin\theta\right)$; $\frac{x^2}{\left(\frac{3}{4}\right)^2}+\frac{y^2}{\left(\frac{3}{2}\right)^2}=1$ The curve is a vertical ellipse, centre at $(0,0)$, with major axis 3 units and minor axis $\frac{3}{2}$ units.
15	**(i)** A circle of unit radius, with centre at the origin. **(ii)** $y=-\frac{e^{\theta}+\cos\theta}{e^{\theta}+\sin\theta}x+2e^{\theta}$ **(iii)** $D\left(0, 2e^{\theta}\right)$; $E(1,0)$. **(iv)** The path is the half line $x=\frac{1}{2}$ and $y>1$.
	(III) Rate of change and Maximisation/Minimisation problems
16	**(i)** − 0.0126 ms^{-1} **(ii)** 55.7s
17	0.140 cm^3s^{-1}
18	**(a)** 0.196 cm/s **(b)** $\left(-\frac{2p}{\sqrt{5}}, \frac{p}{\sqrt{5}}\right)$ and $\left(\frac{2p}{\sqrt{5}}, -\frac{p}{\sqrt{5}}\right)$
19	\$4.92
20	\$54
21	**(i)** max area when $x=\sqrt[3]{\frac{8k}{9\left(4-\sqrt{3}\right)}}$ **(ii)** $x=3.84,\ h=6.46$
22	**(ii)** $h=\frac{100}{\sqrt{15}}$
23	**(ii)** $\frac{8}{3}\pi a^3$ cm^3
24	$PQ=\sqrt{x^2+\left(\frac{27x}{x-8}\right)^2}$; Area of triangle = 507 $unit^2$
25	**(i)** $r=\frac{a\theta}{2\pi}$ **(iii)** $\frac{16\pi\sqrt{3}}{27}$
26	**(i)** $L=\sqrt{a^2+x^2}+\sqrt{b^2+(m-x)^2}$ **(iii)** Yes **(iv)** $\tan^{-1}\left(\frac{1}{2}\right)\le\theta_i\le\tan^{-1}\left(\frac{3}{5}\right)$