CHIJ ST. THERESA’S CONVENT

PRELIMINARY EXAMINATION 2023

SECONDARY 4 EXPRESS / 5 NORMAL (ACADEMIC)

CANDIDATE NAME			
CLASS		INDEX NUMBER	

ADDITIONAL MATHEMATICS 4049/1

Paper 1

29 August 2023

2 hours 15 minutes

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)...(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

1 Find the gradient of the normal to the curve $y = x^{\frac{3}{2}} + 4x$ at $x = 5$, leaving your answers in the form $\left(a\sqrt{5} + b\right)$, where *a* and *b* are constants. [4]

$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} + 4$	**B1 –** Correct Gradient Function
At $x = 5$,	
Gradient of tangent = $\frac{3}{2}\sqrt{5} + 4 = \frac{3\sqrt{5}+8}{2}$	
Hence, Gradient of normal	**M1 –** Using Gradient Function to find gradient of normal
$= -\frac{2}{3\sqrt{5}+8}$	
$= -\frac{2}{3\sqrt{5}+8} \times \frac{3\sqrt{5}-8}{3\sqrt{5}-8}$	**M1 –** Rationalising denominator by multiplying correct conjugate surds
$= \frac{-6\sqrt{5}+16}{-19}$	
$= -\frac{16}{19} + \frac{6}{19}\sqrt{5}$	**A1**

2 Integrate $\frac{5}{3x+1} - \frac{4}{(1-x)^2}$ with respect to *x*. [4]

$\int \frac{5}{3x+1} - \frac{4}{(1-x)^2}\,dx$	**B1 –** $\ln(3x+1)$
$= \int \frac{5}{3x+1} - 4(1-x)^{-2}\,dx$	**B1 –** correct coefficient for ln function
$= \frac{5\ln(3x+1)}{3} - \frac{4(1-x)^{-1}}{(-1)(-1)} + c$	**B1 –** $\frac{1}{1-x}$
$= \frac{5}{3}\ln(3x+1) - \frac{4}{1-x} + c$	**B1 –** correct coefficient for power function and arbitrary constant

3 **(a)** Express $1-16x-16x^2$ in the form $a\left(x+b\right)^2+c$ and hence state the coordinates of the turning point of the curve $y=1-16x-16x^2$. [4]

$1-16x-16x^2$	
$=-16\left(x^2+x\right)+1$	**M1 –** factorising coefficient of x^2
$=-16\left(x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)+1$	
$=-16\left(x+\frac{1}{2}\right)^2+\left(-16\right)\left(-\frac{1}{4}\right)+1$	
$=-16\left(x+\frac{1}{2}\right)^2+5$	**B1, B1 –** correct b and c
Hence, Coordinates of Turning Point $=\left(-\frac{1}{2},5\right)$	**FTA1**

(b) Hence, explain why the curve $y=1-16x-16x^2$ will never intersect $y = 7$ for all values of x. [2]

As $a=-16<0$, Turning point is a **maximum point**	**B1 –** turning point is max point
Since 7 > maximum value of curve = 5, The curve $y=1-16x-16x^2$ will never intersect $y = 7$ for all values of x	**B1 –** 7 is larger than max value of $y = 5$
OR	
$-16\left(x+\frac{1}{2}\right)^2+5=7$	
$\left(x+\frac{1}{2}\right)^2=-\frac{1}{8}$	**M1 –** Showing $\left(x+\frac{1}{2}\right)^2$ is equals to negative
Since $\left(x+\frac{1}{2}\right)^2$ > 0, the curve $y=1-16x-16x^2$ will never intersect $y = 7$ for all values of x	**A1 –** Conclude appropriately

4 The equation of a curve is $y = a\cos bx + c$, where $a > 0$.

(a) Given that the maximum and minimum values of *y* are 1 and – 9 respectively, state the values of *a* and *c*. [2]

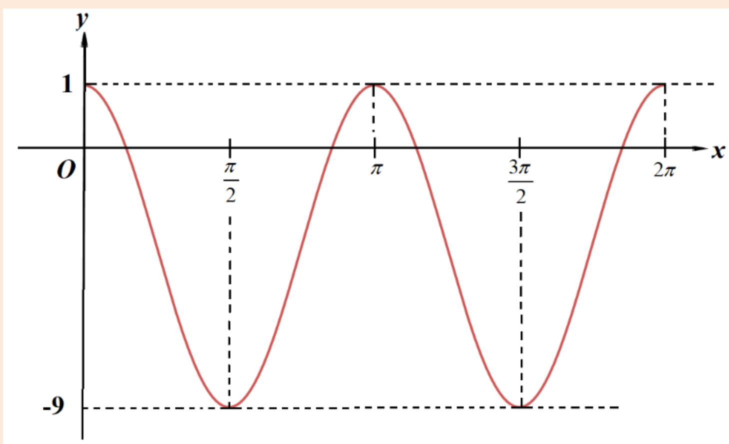
$a = 5, \quad c = -4$ **B1, B1**

(b) Given that the period of *y* is π, find the value of *b*. [1]

b = number of cycles = $\frac{2\pi}{\pi} = 2$

B1

(c) Hence, sketch the curve $y = a\cos bx + c$ for $0 \le x \le 2\pi$ radians, labelling the maximum and minimum points clearly. [3]



B1 – correct shape (cosine)
B1 – 2 complete cycles
B1 – correct max and min points

5 The function $f(x) = ax^3 + 5x^2 + bx + 1$ has a factor $(2x-1)$ and leaves a remainder of 9 when divided by $(x+1)$.

(a) Show that $b = -5$ and find the value of a. [4]

Since $(2x-1)$ is a factor,

$f\left(\frac{1}{2}\right) = 0$ — **M1 –** Correct use of factor theorem to form equation

$a\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 1 = 0$

$a + 4b = -18$

$a = -4b - 18$

Since $f(x)$ leaves a remainder of 9 when divided by $(x+1)$,

$f(-1) = 9$

$a(-1)^3 + 5(-1)^2 + b(-1) + 1 = 9$ — **M1 –** Correct use of remainder theorem to form equation

$-a - b = 3$

$a = -b - 3$

Equating both equations,

$-4b - 18 = -b - 3$ — **M1** – Solving to find b

$3b = -15$

$b = -5 \quad \Rightarrow a = 2$

$\therefore a - b = 2 - (-5) = 7$ — **A1** – Correct value of a.

(b) Given that the quadratic expression $x^2 + px - 1$ is also a factor of $f(x)$, find the value of the constant p. [2]

$2x^3 + 5x^2 - 5x + 1 = (2x-1)(x^2 + px - 1)$

Comparing coefficients of x^2 terms: — **M1 –** comparing x^2 or x terms

$5 = 2p - 1$

$p = 3$ — **A1**

OR

Comparing coefficients of x terms:

$-5 = -2 - p$

$p = 3$

6 Express $\dfrac{3x^3-4x^2+4x-17}{3x^2-x-4}$ in partial fractions. [7]

As fraction is improper, using long division/synthetic division:	
$$\begin{array}{r l} & \quad x \;\; -1 \\ 3x^2-x-4 & \big) \overline{3x^3-4x^2+4x-17} \\ & -\left(3x^3-x^2+4x\right) \\ \hline & \qquad -3x^2+8x-17 \\ & \quad -\left(-3x^2+x\;\;+4\right) \\ \hline & \qquad\qquad 7x-21 \end{array}$$	**M1 –** Correct use of long division or synthetic division to express improper fraction into mixed terms
$$\therefore \frac{3x^3-4x^2+4x-17}{3x^2-x-4}=x-1+\frac{7x-21}{3x^2-x-4}$$	**A1**
$$\frac{7x-21}{3x^2-x-4}=\frac{7x-21}{(x+1)(3x-4)}$$	**M1 –** Correct factorisation of denominator
$$\frac{7x-21}{(x+1)(3x-4)}=\frac{A}{x+1}+\frac{B}{3x-4}$$ $$7x-21=A(3x-4)+B(x+1)$$	**B1** – Correct identification of partial fractions
Subst $x=-1$, $-28=7A$ $A=4$	**M1** – Using appropriate substitution to find *A*
Subst $x=0$, $-21=-4A+B$ $-21=-16+B$ $B=-5$	**M1** – Using appropriate substitution to find *B*
$$\therefore \frac{3x^3-4x^2+4x-17}{3x^2-x-4}=x-1+\frac{4}{x+1}-\frac{5}{3x-4}$$	**A1**

7 The number of bacteria cells, N, in millions, in a circular patch after t hours is given by $N = ke^{2t}$, where k is a constant. The initial number of bacteria cells is 10,000,000.

(a) Explain why $k = 10$. [1]

At $t = 0$, $N = 10$

Hence,
$N = ke^{2t}$
$10 = ke^{0}$
$k = 10$

M1 – substitution of $t = 0$, $N = 10$

(b) Find the rate at which the number of bacteria cells is increasing after 30 minutes. [2]

$N = 10e^{2t}$

$\frac{dN}{dt} = 20e^{2t}$

M1 – Differentiating N w.r.t t

After 30 minutes, $t = \frac{1}{2}$

$\frac{dN}{dt} = 20e^{2\left(\frac{1}{2}\right)} = 20e = 54.366$

$= 54.4$ million/hour

A1

(c) As the number of bacteria cells increases, the area, A units^2, of the circular patch also increases. Given that $4A = N^2$, find the rate of change of A after 5 hours, leaving your answer in exact form. [4]

$4A = N^2 \quad \Rightarrow \quad A = \frac{1}{4}N^2$

$\frac{dA}{dN} = \frac{1}{2}N$

M1 – Differentiating A w.r.t N

When $t = 5$,

$\frac{dA}{dN} = \frac{1}{2}N = \frac{1}{2}\left(10e^{2(5)}\right) = 5e^{10}$

$\frac{dN}{dt} = 20e^{2(5)} = 20e^{10}$

M1 – Finding values of $\frac{dA}{dN}$ and $\frac{dN}{dt}$ when $t = 5$

$\therefore \frac{dA}{dt} = \frac{dA}{dN} \times \frac{dN}{dt}$

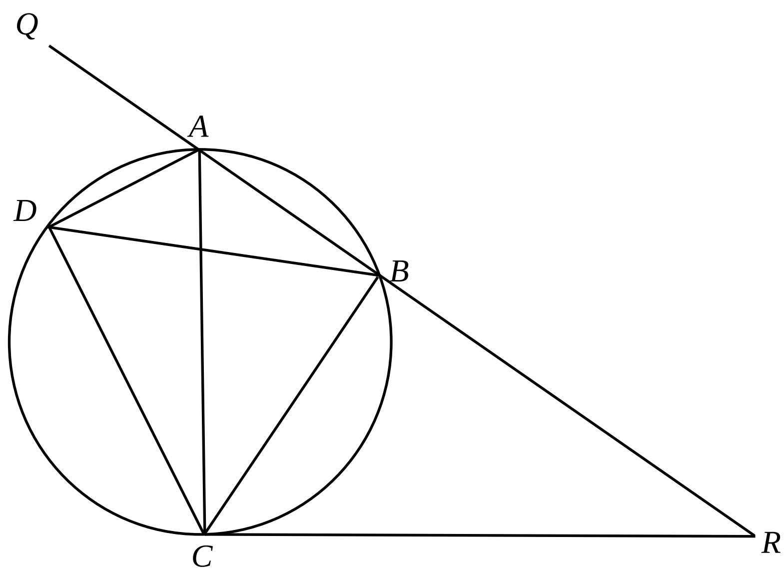
$= 5e^{10} \times 20e^{10}$

$= 100e^{20}$ units^2/hour

M1 – Substituting values into correct connected rates of change equation

A1

8 In the diagram, CR is a tangent to the circle at C, and line QR cuts the circle at points A and B. D lies on the circumference of the circle.



(a) Prove that triangle CBR and triangle ACR are similar. [2]

$\angle BRC = \angle CRA$ (common angle)
$\angle BCR = \angle CAR$ (alt. segment thm) **M1**

$\therefore$ By AA test, triangle CBR is similar to triangle ACR **A1** – AA test

(b) Given that $\angle ADB = \angle BRC$**,** show that AC is the diameter. [4]

Let $\angle BRC = \angle ADB = \angle ACB = x$ (angles in the same segment) **M1**

Let $\angle BAC = \angle BCR = y$ (alt. segment thm or properties from similar triangles) **M1**

Considering triangle ACR,
$x + y + x + y = 180°$ (angle sum of triangle)
$x + y = \angle ACR = 90°$ **M1**

$\therefore$ AC is diameter (rad $\perp$ tangent) **A1**

9 **(a)** Prove the identity $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}=2\sec\theta$. [4]

LHS

$=\dfrac{\cos^2\theta+(1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}$

M1 – Making into common denominator

$=\dfrac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$

M1 – Correct expansion of $(1+\sin\theta)^2$

$=\dfrac{2+2\sin\theta}{(1+\sin\theta)\cos\theta}$

M1 – Using the identity $\sin^2\theta+\cos^2\theta=1$ to simplify

$=\dfrac{2(1+\sin\theta)}{(1+\sin\theta)\cos\theta}$

M1– Factorising and simplifying $(1+\sin\theta)$ to get RHS

$=\dfrac{2}{\cos\theta}$

$=2\sec\theta$

$=RHS \quad (shown)$

(b) Hence solve the equation $\dfrac{\cos\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}+\dfrac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=5$ for $-180°\le\theta\le180°$. [4]

Using (a),

$2\sec\dfrac{\theta}{2}=5 \quad\Rightarrow\quad \dfrac{2}{\cos\frac{\theta}{2}}=5$

$\cos\dfrac{\theta}{2}=\dfrac{2}{5}$

M1 – Finding the correct equivalent equation

Since $-180°\le\theta\le180°$,

$-90°\le\dfrac{\theta}{2}\le90°.$

$Basic\ \angle\alpha=\cos^{-1}\dfrac{2}{5}=66.422°$

M1 – Finding basic angle using $\cos^{-1}$

As $\cos\dfrac{\theta}{2}$ is positive, $\dfrac{\theta}{2}$ is in Quad 1 and 4

$\therefore\ \dfrac{\theta}{2}=66.422°$ or $-66.422°$

$\theta=132.8°$ or $-132.8°$

A1, A1

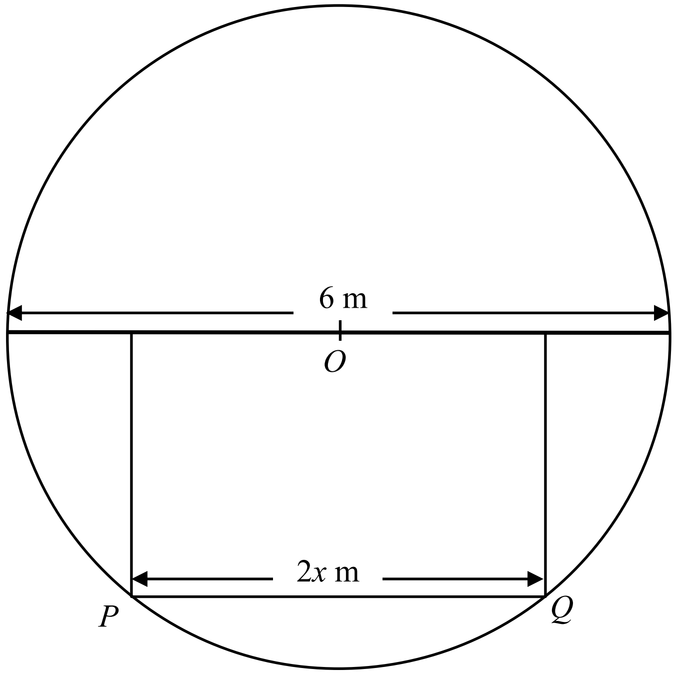
10 **(a)** Given that $\log_8 x+\log_2(1-y)=1$, express y in terms of x. [4]

$\log_8 x+\log_2(1-y)=1$	
$\dfrac{\log_2 x}{\log_2 8}+\log_2(1-y)=1$	**M1 –** Change to common base
$\dfrac{1}{3}\log_2 x+\log_2(1-y)=1$	
$\log_2 x^{\frac{1}{3}}+\log_2(1-y)=1$	**M1 –** Use of Product rule to combine terms into single term
$\log_2\left[x^{\frac{1}{3}}(1-y)\right]=1$	
$x^{\frac{1}{3}}(1-y)=2$	**M1 –** Removing logarithms by using $\log_2 2=1$ or by expressing logarithmic form in index form
$1-y=\dfrac{2}{x^{\frac{1}{3}}}$	
$y=1-\dfrac{2}{x^{\frac{1}{3}}}$ or $y=1-\dfrac{2}{\sqrt[3]{x}}$	**A1**

(b) Solve the equation $\dfrac{5^{2x-3}}{3^x}=2$. [4]

$5^{2x-3}=2(3^x)$	**M1 –** Correct use of Product rule to express $\lg 2(3^x)=\lg 2+\lg 3^x$
$\lg 5^{2x-3}=\lg 2(3^x)$	
$(2x-3)\lg 5=\lg 2+\lg 3^x$	**M1 –** Correct use of Power rule to remove variable from power
$2x\lg 5-3\lg 5=\lg 2+x\lg 3$	
$2x\lg 5-x\lg 3=\lg 2+3\lg 5$	**M1 –** Isolating x terms on one side of the equation
$x=\dfrac{\lg 2+3\lg 5}{2\lg 5-\lg 3}$	
$x=2.6041 \quad \approx 2.60$	**A1**

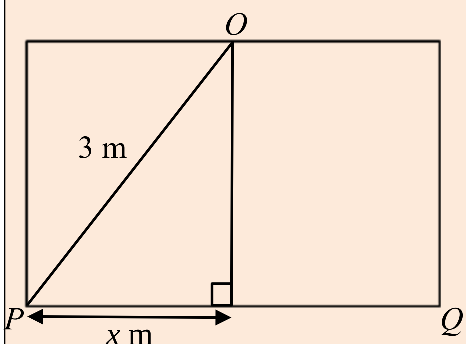
11 The diagram represents the circular cross-section area of a commercial plane, with center O. Cargo is stored in a rectangular container in the lower half of the plane, touching the fuselage at points P and Q. The plane is 6 m wide and the width of the rectangular container is $2x$ m.



(a) Show that the area, A m^2, of the cargo container, is given by $A = 2x\sqrt{9-x^2}$ m^2. [3]

From the diagram, as cross sectional area is circular,
OP = Radius of circle = 3m.

B1 – Correctly identifying OP = 3m



Using Pythagoras theorem,
Height of container $= \sqrt{3^2 - x^2} = \sqrt{9 - x^2}$
Area of container, A = base x height
$= 2x\sqrt{9-x^2}$

M1 – Finding the height of the container using Pythagoras Thm.

AG1

(b) Given that x can vary, find the stationary value of A and determine its nature. [6]

$A = 2x\sqrt{9-x^2}$

$u = 2x \quad , \quad v = \sqrt{9-x^2}$

$\dfrac{du}{dx} = 2 \quad , \quad \dfrac{dv}{dx} = \dfrac{1}{2\sqrt{9-x^2}}(-2x)$

$= \dfrac{-x}{\sqrt{9-x^2}}$

$\dfrac{dA}{dx} = 2\sqrt{9-x^2} + 2x\left(\dfrac{-x}{\sqrt{9-x^2}}\right)$

B1, B1 – Correct differentiation using Product rule

$= \dfrac{2(9-x^2)-2x^2}{\sqrt{9-x^2}}$

$= \dfrac{18-4x^2}{\sqrt{9-x^2}}$

At stationary point,

$\dfrac{dA}{dx} = 0 \quad \Rightarrow \quad \dfrac{18-4x^2}{\sqrt{9-x^2}} = 0 \quad \Rightarrow 18-4x^2 = 0$

M1 – Combining derivative into a single fraction and equating to 0

$x = \sqrt{\dfrac{9}{2}} \quad or \quad -\sqrt{\dfrac{9}{2}}(rej)$

$\therefore A = 2\left(\sqrt{\dfrac{9}{2}}\right)\sqrt{9-\dfrac{9}{2}}$

$A = 2\left(\sqrt{\dfrac{9}{2}}\right)\left(\sqrt{\dfrac{9}{2}}\right) = 9\text{m}^2$

A1

Using the first derivative test,

x	2.1	$\sqrt{\dfrac{9}{2}}$ or 2.121	2.2
$\dfrac{dA}{dx}$	+ ve	0	– ve
tangent	/	—	\

M1 – Using First or Second Derivative test

A is a **maximum** when $x = \sqrt{\dfrac{9}{2}}$

A1

12 **(a)** The curve $y^2 = \frac{2}{3}x + 2$ and the line $3y + 2x = 0$ intersect at the points A and B. Find the coordinates of A and B. [4]

$y^2 = \frac{2}{3}x + 2 ----(1)$

$3y + 2x = 0$

$y = -\frac{2}{3}x -----(2)$

Subst (1) into (2),

$\left(-\frac{2}{3}x\right)^2 = \frac{2}{3}x + 2$

M1 – Substitution

$\frac{4}{9}x^2 = \frac{2}{3}x + 2$

$4x^2 = 6x + 18$

$2x^2 - 3x - 9 = 0$

M1 – Factorisation/ Quadratic Formula

$(2x+3)(x-3) = 0$

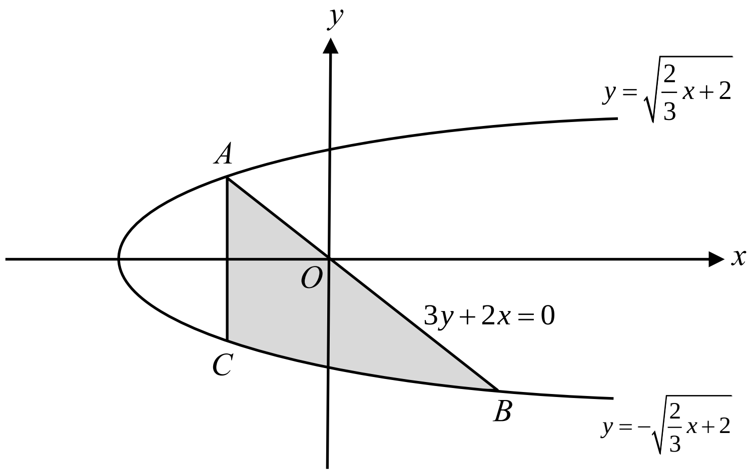
$x = -\frac{3}{2} \quad or \quad x = 3$

$y = 1 \quad or \quad y = -2$

$\therefore A\left(-\frac{3}{2}, 1\right), \quad B(3, -2)$

A1, A1

(b) The curve of $y^2 = \frac{2}{3}x + 2$ is made of two parts; the part above the x-axis is given by $y = \sqrt{\frac{2}{3}x + 2}$ and the curve below the x-axis is given by $y = -\sqrt{\frac{2}{3}x + 2}$. The diagram shows the curve $y^2 = \frac{2}{3}x + 2$ and the line $3y + 2x = 0$.



Find the area of the shaded region bounded by the curve, the line $3y + 2x = 0$, and the vertical line AC. [5]

$$\int_{-\frac{3}{2}}^{3} -\frac{2}{3}x - \left(-\sqrt{\frac{2}{3}x+2}\right) \mathrm{d}x$$

B1 – Correct limits
M1 – Line minus curve

$$= \int_{-\frac{3}{2}}^{3} -\frac{2}{3}x + \left(\frac{2}{3}x+2\right)^{\frac{1}{2}} \mathrm{d}x$$

$$= \left[-\frac{1}{3}x^2 + \frac{\left(\frac{2}{3}x+2\right)^{\frac{3}{2}}}{\frac{3}{2}\left(\frac{2}{3}\right)} \right]_{-\frac{3}{2}}^{3}$$

M1, M1 – for each correct integral

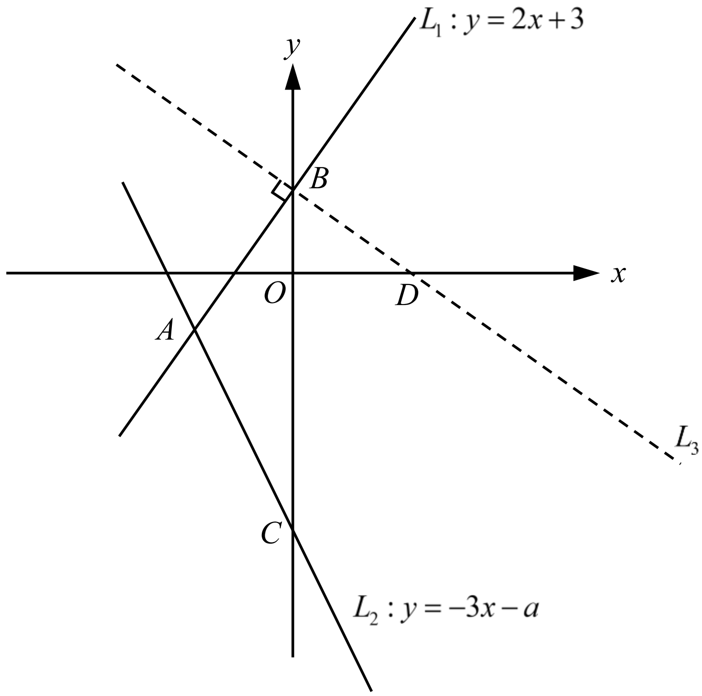
$$= \left[-\frac{1}{3}x^2 + \left(\frac{2}{3}x+2\right)^{\frac{3}{2}} \right]_{-\frac{3}{2}}^{3}$$

$$= [-3+8] - [-\frac{3}{4}+1]$$

$$= 4\frac{3}{4} \quad \text{or} \quad 4.75 \text{ units}^2$$

A1

13 The diagram, which is not drawn to scale, shows the line $L_1 : y = 2x + 3$ and the line $L_2 : y = -3x - a$. Lines L_1 and L_2 intersect at A. Points B and C lie on the y-axis.



(a) Given that the area of triangle ABC is 10 units^2, show that $a = 7$. [3]

Since Lines L_1 and L_2 intersect at A, $2x + 3 = -3x - a$ $x = \dfrac{-3-a}{5} = -\left(\dfrac{3+a}{5}\right)$	**M1 –** Equating L_1 and L_2
Hence, $Area = \dfrac{1}{2}(3+a)\left(\dfrac{3+a}{5}\right)$ $10 = \dfrac{(3+a)^2}{10}$	**M1 –** Finding base and height using the intercepts and coordinates of A
$(3+a)^2 = 100$ $3 + a = 10 \quad or \quad -10(rej)$ $a = 7\ (shown)$	**M1 –** Solving equation to get a = 7

(b) Line L_3 passes through *B* and is perpendicular to L_1. Given that it intersects the *x*-axis at *D*, find the coordinates of *D*. [3]

Gradient of *BD* = $-\frac{1}{2}$	**B1 –** gradient of *BD*
Vert intercept = 3	
Hence,	
$L_3 : y = -\frac{1}{2}x + 3$	**B1 –** equation of *BD*
When $y = 0$, $x = 6$.	
$\therefore D(6,0)$	**B1**

(c) Hence, find the area of quadrilateral *ABCD*. [2]

Area of quadrilateral *ABCD*	
= Area of Triangle *ABC* + Area of Triangle *BDC*	**M1 –** Sum of 2 triangles or shoelace method
$= \frac{1}{2}(10)(2) + \frac{1}{2}(10)(6)$	
$= 40 \text{ units}^2$	**A1**

(d) Is *ABCD* a trapezium? Justify your answer. [2]

$D(6,0), \quad C(0,-7)$	
Gradient of *CD* = $\frac{0-(-7)}{6-0} = \frac{7}{6}$	**M1 –** comparing gradient of *AB* and *CD*
As gradient of *CD* $\neq$ gradient of *AB*	
AB and *CD* are not parallel.	
Hence *ABCD* is not a trapezium.	**A1**

~~~ End of Paper ~~~
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