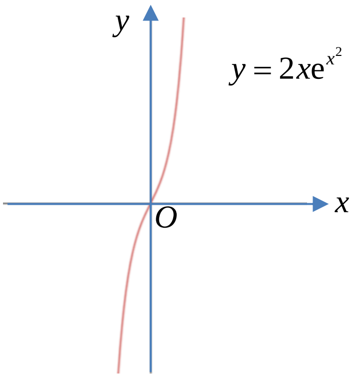
H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Extra Practice Solutions

Q1	2018/MI Promo/1/5(a)
(i)	$\mathrm{f}(x) = x^2 \mathrm{e}^{x^2}$, for $x \in \mathbb{R}$, $\mathrm{f}'(x) = x^2\left(2x\mathrm{e}^{x^2}\right) + 2x\mathrm{e}^{x^2}$ $= 2x\mathrm{e}^{x^2}\left(x^2+1\right)$ For the function to be increasing, $\mathrm{f}'(x) = 2x\mathrm{e}^{x^2}\left(x^2+1\right) > 0$ **<u>Method 1:</u>** By GC,  From the sketch, for $2x\mathrm{e}^{x^2}\left(x^2+1\right) > 0$, $\therefore x > 0$ **<u>Method 2:</u>** Since $x^2+1>0$ and $\mathrm{e}^{x^2} > 0$, for all $x \in \mathbb{R}$, $\therefore x > 0$
(ii)	When $x=1$, $\mathrm{f}'(1) = 2(1)\mathrm{e}(2) = 4\mathrm{e}$, $\mathrm{f}(1) = \mathrm{e}$ Equation of tangent at $x=1$: $y - \mathrm{e} = 4\mathrm{e}(x-1)$ $\therefore y = 4\mathrm{e}x - 3\mathrm{e}$

Q2	**2018/DHS Prelim/2/5(a)**
(i)	$(x+y)^2 = 4\mathrm{e}^{xy}$ $2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 4\mathrm{e}^{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y\mathrm{e}^{xy}-y-x}{y+x-2x\mathrm{e}^{xy}}$
(ii)	When $x = 0$, $(0+y)^2 = 4\mathrm{e}^{(0)y}$ $y = 2 \ (\because y > 0)$ When at (0, 2), $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(2)-2}{2} = 1$ Equation of the tangent to the curve at (0, 2) is $y-2 = 1(x-0)$ $y = x+2$
(iii)	Substitute $y = x + 2$ into $(x+y)^2 = 4\mathrm{e}^{xy}$, $(x+x+2)^2 = 4\mathrm{e}^{x(x+2)}$ $(2x+2)^2 = 4\mathrm{e}^{x^2+2x}$ $(x+1)^2 = \mathrm{e}^{x^2+2x}$ Using G.C., $x = -2$ or $x = 0$ (reject $\because$ it's point A) $\therefore B(-2,0)$

Q3	2018/HCI Prelim/1/6
(i)	$\frac{dx}{dt}=2\cos t$ $\frac{dy}{dt}=-\sin t$ $\frac{dy}{dx}=\frac{-\sin t}{2\cos t}=-\frac{\tan t}{2}$ At P, $t=p$ Equation of tangent at P: $y-1-\cos p=-\frac{\tan p}{2}(x-2\sin p)$ $y=1+\cos p-\frac{\tan p}{2}x+\frac{\sin^2 p}{\cos p}$ $=1+\frac{\cos^2 p+\sin^2 p}{\cos p}-\frac{\tan p}{2}x$ $=1+\sec p-\frac{\tan p}{2}x$ $2y+x\tan p=2(1+\sec p)$ (shown)
(ii)	When $y=0$, $\frac{\tan p}{2}x=1+\sec p$ $x=\frac{2+2\sec p}{\tan p}$ When $x=0$, $y=1+\sec p$
	Method 1: Coordinates of $M=\left(\frac{1+\sec p}{\tan p},\frac{1+\sec p}{2}\right)$ $y=\frac{1+\sec p}{2}\Rightarrow \sec p=2y-1$ $x=\frac{1+\sec p}{\tan p}=\frac{2y}{\tan p}$ Using $1+\tan^2 p=\sec^2 p$, $1+\left(\frac{2y}{x}\right)^2=(2y-1)^2$ $1+\frac{4y^2}{x^2}=4y^2-4y+1$ $y=yx^2-x^2$ $x^2=y(x^2-1)$ $y=\frac{x^2}{x^2-1}$

	Method 2: Coordinates of $M = \left(\cot p + \operatorname{cosec} p, \dfrac{1+\sec p}{2}\right)$ $y = \dfrac{1+\sec p}{2} \Rightarrow \sec p = 2y-1$ $x = \cot p + \operatorname{cosec} p = \dfrac{\cos p + 1}{\sin p}$ Using $\sin^2 p + \cos^2 p = 1$, $\left(\dfrac{\cos p+1}{x}\right)^2 + \left(\dfrac{1}{2y-1}\right)^2 = 1$ $\dfrac{\left(\dfrac{1}{2y-1}+1\right)^2}{x^2} + \dfrac{1}{(2y-1)^2} = 1$ $\dfrac{(2y)^2}{(2y-1)^2 x^2} + \dfrac{1}{(2y-1)^2} = 1$ $4y^2 + x^2 = (4y^2 - 4y + 1)x^2$ $y^2 = y^2x^2 - yx^2$ $x^2 = y(x^2-1)$ $y = \dfrac{x^2}{x^2-1}$

Q4	2018/MJC Promo/1/6
(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=2$, $\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{4}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{2}{t^2}$ Gradient of normal = $\frac{t^2}{2}$ At point M , $t=1$, gradient of normal = $\frac{1}{2}$ Equation of normal: $y-4=\frac{1}{2}(x-3)$ $2y=x+5$
(ii)	At point N : $\left(\frac{4}{t}\right)=\frac{1}{2}(2t+1)+5$ $t^2+3t-4=0$ $\Rightarrow$ $t=1$ $\left(\text{reject} \because \text{ this is point } M\right)$ or $t=-4$ Coordinates of N is $(-7,-1)$
(iii)	Equation of tangent at P : $y-\left(\frac{4}{p}\right)=\frac{-2}{p^2}(x-2p-1)$ At point Q: $y=0 \Rightarrow$ $0-\left(\frac{4}{p}\right)=\frac{-2}{p^2}(x-2p-1)$ $x=4p+1$ At point R: $x=0 \Rightarrow$ $y-\left(\frac{4}{p}\right)=\frac{-2}{p^2}(-2p-1)$ $y=\frac{2(4p+1)}{p^2}$ Area of triangle OQR = $\frac{1}{2}\left(\frac{2(4p+1)}{p^2}\right)(4p+1)$ $=\left(\frac{4p+1}{p}\right)^2$

Q5	**2018/CJC Promo/1/7**

(i)

$$kx^2 + 2xy - 3y^2 = 5$$

$$2kx + \left(2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\right) - 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2kx + 2y}{6y - 2x}$$

$$= \frac{kx + y}{3y - x}\ (\text{shown})$$

(ii)

For tangents parallel to x-axis, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$,

$$kx + y = 0$$

$$y = -kx \quad \text{or} \quad x = -\frac{y}{k}$$

<u>**Method 1:**</u>

Substitute $y = -kx$ into C,

$$kx^2 + 2x(-kx) - 3(-kx)^2 = 5$$

$$x^2\left(-k - 3k^2\right) = 5$$

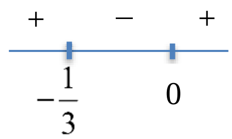
$$x^2 = \frac{-5}{k + 3k^2}$$

Since k is a non-zero constant,

$$k + 3k^2 < 0$$

$$k(1 + 3k) < 0$$

$$-\frac{1}{3} < k < 0$$



<u>**Method 2:**</u>

Substitute $x = -\frac{y}{k}$ into C,

$$k\left(-\frac{y}{k}\right)^2 + 2\left(-\frac{y}{k}\right)y - 3y^2 = 5$$

$$ky^2 - 2ky^2 - 3k^2y^2 = 5k^2$$

$$y^2\left(k + 3k^2\right) = -5k^2$$

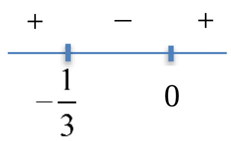
$$y^2 = \frac{-5k^2}{k + 3k^2}$$

$$= \frac{-5k}{1 + 3k}$$

Since k is a non-zero constant,

$$\frac{5k}{1 + 3k} < 0$$

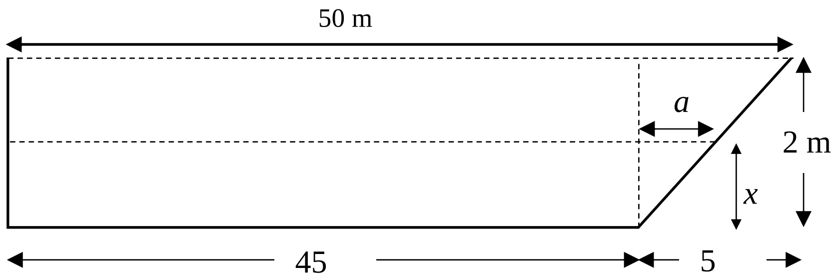
$$-\frac{1}{3} < k < 0$$



<u>**Method 3:**</u> **[Discriminant]**

	Substitute $y=-kx$ into C, $kx^2+2x(-kx)-3(-kx)^2=5$ $x^2(-k-3k^2)=5 \qquad \Rightarrow x^2=\dfrac{-5}{k+3k^2}$ $(k+3k^2)x^2+5=0$ $b^2-4ac\geq 0$ $0-4(k+3k^2)(5)\geq 0$ $k+3k^2\leq 0$ $k(1+3k)\leq 0$ $-\dfrac{1}{3}\leq k\leq 0$ Since $k\neq 0,\quad k\neq -\dfrac{1}{3},\quad -\dfrac{1}{3}<k<0$ **<u>Method 4:</u> [Discriminant]** Substitute $x=-\dfrac{y}{k}$ into C, $k\left(-\dfrac{y}{k}\right)^2+2\left(-\dfrac{y}{k}\right)y-3y^2=5$ $ky^2-2ky^2-3k^2y^2=5k^2$ $y^2(k+3k^2)+5k^2=0 \qquad \Rightarrow y^2=\dfrac{-5k^2}{k+3k^2}$ $b^2-4ac\geq 0$ $0-4(k+3k^2)(5k^2)\geq 0$ $k+3k^2\leq 0$ $k(1+3k)\leq 0$ $-\dfrac{1}{3}\leq k\leq 0$ Since $k\neq 0,\quad k\neq -\dfrac{1}{3},\quad -\dfrac{1}{3}<k<0$
(iii)	$k=13$, $x=1$ and $y=2$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=5$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{13(1)+(2)}{3(2)-(1)}=3$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}=15$ units per second.

Q6	2018/NJC Promo/1/12



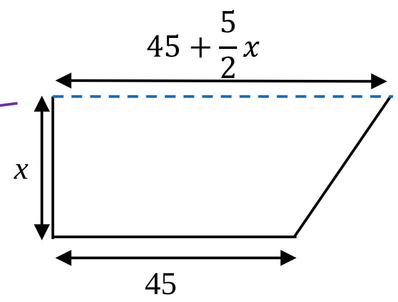
Let V be the volume of water in the pool when the depth of water is x m.

By similar triangles,

$$\frac{a}{5}=\frac{x}{2}\Rightarrow a=\frac{5}{2}x\,.$$

$$\therefore V=\frac{1}{2}\left(45+45+\frac{5}{2}x\right)(x)(20)=900x+25x^2$$

Width of pool is 20m



$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

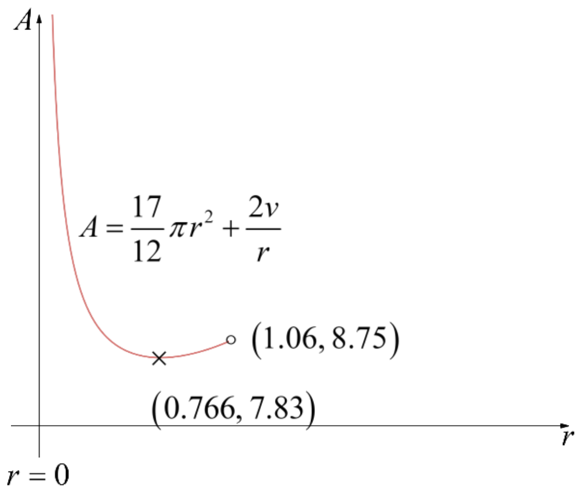
$$=\frac{1}{900x+50x^2}\times 100$$

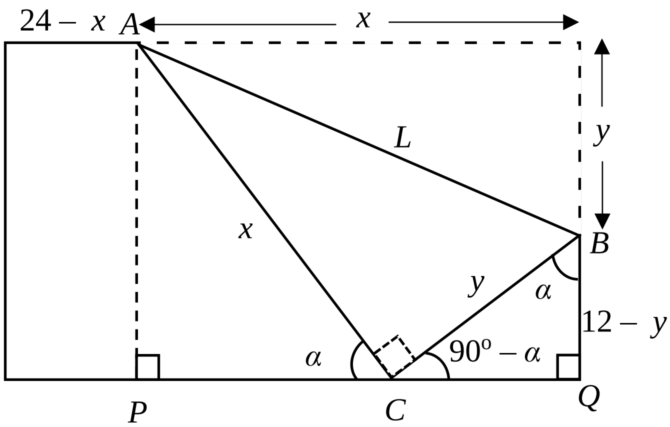
$$=\frac{10}{90+5x}$$

When x = 1.6,

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{10}{90+5(1.6)}=\frac{5}{49}\ \text{m/min}$$

Q7	**2020/SAJC Promo/1/11**
(i)	$v = 2\times\frac{1}{6}\pi\frac{r}{2}\left(3r^2+\frac{1}{4}r^2\right)+\pi r^2 h$ $= \frac{13}{24}\pi r^3 + \pi r^2 h$ $h = \frac{v}{\pi r^2} - \frac{13r}{24}$ $A = 2\pi\left(r^2+\frac{1}{4}r^2\right)+2\pi rh$ $= 2\pi\left(r^2+\frac{1}{4}r^2\right)+2\pi r\left(\frac{v}{\pi r^2}-\frac{13r}{24}\right)$ $= \frac{17}{12}\pi r^2 + \frac{2v}{r}$
(ii)	$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{17}{6}\pi r - \frac{2v}{r^2}$ For stationary values, let $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ $\frac{17}{6}\pi r - \frac{2v}{r^2} = 0$ $17\pi r^3 - 12v = 0$ $r^3 = \frac{12v}{17\pi}$ $r = \sqrt[3]{\frac{12v}{17\pi}}$ Using second derivative test to check for minimum, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = \frac{17}{6}\pi + \frac{4v}{r^3}$ When $r = \sqrt[3]{\frac{12v}{17\pi}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = \frac{17}{6}\pi + \frac{17}{3}\pi = \frac{17}{2}\pi > 0$ Therefore, A is minimum when $r = \sqrt[3]{\frac{12v}{17\pi}}$.

(iii)	$h=\frac{2}{\pi r^2}-\frac{13r}{24}>0$ $\frac{2}{\pi r^2}>\frac{13r}{24}$ $r^3<\frac{48}{13\pi}$ $r<\left(\frac{48}{13\pi}\right)^{\frac{1}{3}}=1.06$ But $r>0$, so $0<r<1.06$.
(iv)	

Q8	**2018/AJC Promo/1/9(b)**
	 $\triangle APC$ and $\triangle CQB$ are similar triangles because: $\angle APC=\angle CQB=90^\circ$ $\angle ACP=\angle CBQ=\alpha$ (3 interior angles are equal) By similar triangles, $\frac{AC}{CB}=\frac{AP}{CQ}$ $\Rightarrow\frac{x}{y}=\frac{12}{CQ}$ $\Rightarrow\frac{x}{y}=\frac{12}{\sqrt{y^2-(12-y)^2}}=\frac{12}{\sqrt{24y-144}}$ $\Rightarrow x=\frac{12y}{2\sqrt{6(y-6)}}=\frac{6y}{\sqrt{6(y-6)}}$

Alternative: $\dfrac{AC}{CB}=\dfrac{PC}{BQ}$

$$\Rightarrow \frac{x}{y}=\frac{\sqrt{x^2-12^2}}{12-y}$$

$$\Rightarrow x^2\left(12-y\right)^2=y^2\left(x^2-12^2\right)$$

$$\Rightarrow x^2\left[y^2-\left(12-y\right)^2\right]=144y^2$$

Rearrange to make x the subject, to get the answer.

Let L be the length of the crease AB formed.

$$L^2=x^2+y^2=\frac{6y^2}{y-6}+y^2$$

$$\Rightarrow L^2=\left[\frac{6y^2+y^2(y-6)}{y-6}\right] \quad \Rightarrow L^2=\left[\frac{y^3}{y-6}\right]$$

Differentiating w.r.t. y,

$$\Rightarrow 2L\frac{\mathrm{d}L}{\mathrm{d}y}=\frac{(y-6)(3y^2)-y^3(1)}{(y-6)^2}$$

$$\Rightarrow 2L\frac{\mathrm{d}L}{\mathrm{d}y}=\frac{2y^3-18y^2}{(y-6)^2}=\frac{2y^2(y-9)}{(y-6)^2}$$

$$\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}y}=\frac{y^2(y-9)}{(y-6)^2}\cdot\frac{1}{L}$$

When $\dfrac{\mathrm{d}L}{\mathrm{d}y}=0 \quad \Rightarrow y=9 \quad (y>0)$

y	9^-	9	9^+
$\dfrac{\mathrm{d}L}{\mathrm{d}y}$	$-$	0	$+$
sketch	\	—	/

$$\therefore L^2=\frac{9^3}{9-3}=243$$

$$\Rightarrow L=\sqrt{243}(\text{since } L>0)=15.6\text{cm}$$

Q9	**2018/MI Promo/1/11**
(i)	The surface area of the rim $=\pi\left(r+\frac{1}{5}\right)^2-\pi r^2$ $=\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}-r^2\right)$ $=\frac{1}{25}\pi\left(10r+1\right)$ (Shown)
(ii)	$V=\pi r^2 h=150\pi$ $\Rightarrow h=\frac{150}{r^2}$ **<u>Method 1:</u>** $=\frac{1}{25}\pi\left(10r+1\right)+\pi r^2+\pi\left(r+\frac{1}{5}\right)^2+2\pi rh+2\pi\left(r+\frac{1}{5}\right)\left(h+\frac{1}{5}\right)$ $=\frac{2}{5}\pi r+\frac{\pi}{25}+\pi r^2+\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}\right)+2\pi r\left(\frac{150}{r^2}\right)+2\pi\left(rh+\frac{1}{5}r+\frac{1}{5}h+\frac{1}{25}\right)$ $=2\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}\right)+2\pi\left(\frac{150}{r}\right)+2\pi\left(r\left(\frac{150}{r^2}\right)+\frac{1}{5}r+\frac{1}{5}\left(\frac{150}{r^2}\right)+\frac{1}{25}\right)$ $=2\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}\right)+2\pi\left(\frac{150}{r}\right)+2\pi\left(\frac{150}{r}+\frac{1}{5}r+\frac{30}{r^2}+\frac{1}{25}\right)$ $=2\pi\left(r^2+\frac{3}{5}r+\frac{2}{25}+\frac{300}{r}+\frac{30}{r^2}\right)$ (Shown) **<u>Method 2:</u>** Total surface area = A $=2\pi\left(r+\frac{1}{5}\right)^2+2\pi rh+2\pi\left(r+\frac{1}{5}\right)\left(h+\frac{1}{5}\right)$ $=2\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}\right)+2\pi r\left(\frac{150}{r^2}\right)+2\pi\left(rh+\frac{1}{5}r+\frac{1}{5}h+\frac{1}{25}\right)$ $=2\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}\right)+2\pi\left(\frac{150}{r}\right)+2\pi\left(r\left(\frac{150}{r^2}\right)+\frac{1}{5}r+\frac{1}{5}\left(\frac{150}{r^2}\right)+\frac{1}{25}\right)$ $=2\pi\left(r^2+\frac{2}{5}r+\frac{1}{25}\right)+2\pi\left(\frac{150}{r}\right)+2\pi\left(\frac{150}{r}+\frac{1}{5}r+\frac{30}{r^2}+\frac{1}{25}\right)$ $=2\pi\left(r^2+\frac{3}{5}r+\frac{2}{25}+\frac{300}{r}+\frac{30}{r^2}\right)$ (Shown)

(iii)

For stationary value of A, $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0$

$$\frac{\mathrm{d}A}{\mathrm{d}r}=2\pi\left[2r+\frac{3}{5}-\frac{300}{r^2}-\frac{60}{r^3}\right]=0$$

$$\Rightarrow 2r+\frac{3}{5}-\frac{300}{r^2}-\frac{60}{r^3}=0$$

By GC, $r=5.2814$ $(\because r>0)$

1st Derivative test:

r	5.2814^+	5.2814	5.2814^-
$\dfrac{\mathrm{d}A}{\mathrm{d}r}$	<0	0	>0
Slope	\	–	/

2nd Derivative test:

When $r=5.2814$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}=\pi\left[2+\dfrac{600}{r^3}+\dfrac{180}{r^4}\right]=39.61>0$

Therefore, A is minimum when $r=5.2814$.

The minimum $A=2\pi\left(r^2+\dfrac{3}{5}r+\dfrac{2}{25}+\dfrac{300}{r}+\dfrac{30}{r^2}\right)=559.33\approx 559$ (3s.f.)

(iv)

Let x be the depth of water in the drinking glass.

Given $\dfrac{\mathrm{d}V}{\mathrm{d}t}=-0.5\text{ cm}^3\text{s}^{-1}$, find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$.

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

Volume of water in the drinking glass, V:

$$V=\pi r^2x\Rightarrow\frac{\mathrm{d}V}{\mathrm{d}x}=\pi r^2\ (r\text{ is constant})$$

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow -0.5=\pi r^2\times\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{-0.5}{\pi r^2}$$

Given $h=2r$.

$$150\pi=\pi r^2h=2\pi r^3$$

$$r=\sqrt[3]{75}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{-0.5}{\pi r^2}=\frac{-0.5}{\pi\left(\sqrt[3]{75}\right)^2}=-0.00895\text{ cms}^{-1}$$

The rate of decrease of the depth of water is 0.00895 cms^{-1}.

Q10	**2016/MJC Promo/1/10**

Let the total time taken for the bird to travel via route *ACDE* be *T* hours.

$$T = \frac{2x}{65} + \frac{y}{90}$$

$$BC = \sqrt{x^2 - 5^2} = \sqrt{x^2 - 25}$$

$$y = k - 2\sqrt{x^2 - 25}$$

$$T = \frac{2}{65}x + \frac{1}{90}\left(k - 2\sqrt{x^2 - 25}\right)$$

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{2}{65} + \frac{1}{90}\left(-\frac{4x}{2\sqrt{x^2 - 25}}\right)$$

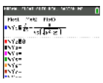
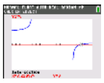
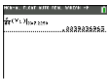
$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{2}{65} - \frac{1}{45}\left(\frac{x}{\sqrt{x^2 - 25}}\right)$$

At stationary point, $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$

$$\frac{2}{65} - \frac{1}{45}\left(\frac{x}{\sqrt{x^2 - 25}}\right) = 0$$

Using GC, $x = 7.2290$ (5s.f)

$x = 7.23$ (3 s.f)

Using GC, at $x = 7.2290$, $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 0.00390$ (3 s.f) > 0

$\therefore$ total time is minimised when $x = 7.23$.

Alternative method to prove minimum:

x	7.2290^-	7.2290	7.2290^+
$\frac{\mathrm{d}T}{\mathrm{d}x}$	\	—	/

$\therefore$ total time is minimised when $x = 7.23$.

	Time taken via route $ACDE$ $=\frac{2}{65}(7.2290)+\frac{1}{90}\left(k-2\sqrt{(7.2290)^2-25}\right)$ $=0.10641+\frac{k}{90}$ Time taken from island A to island B directly $=\frac{k}{65}$ To choose the route $ACDE$ over flying from island A to island E directly, $0.10641+\frac{k}{90}<\frac{k}{65}$ $\frac{k}{234}>0.10641$ $k>24.9$ $\therefore$ minimum value of k is 25 (nearest integer)