Knowledge and Inquiry

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<tr><td colspan="2" align="center"><h2><u>Maths: Its Nature, Construction and Validity</u></h2></td></tr>
<tr><td colspan="2"><u>Readings I plan to use</u>
1. Was Mathematics Invented or Discovered: Realism vs Anti-realism
2. Reason_and_arithmetic_Charles_Landesman: Mathematics in philosophy (12)
3. Mathematics_as_an_objective_science_by_Nicolas_D._Goodman: (13)
4. Mathematics-Cultural_product_or_epistemic_exception__by_Susanne_Prediger

<u>Readings Used</u>
1. What is Mathematics
2. The Rise and Role of Mathematics: Mathematics in philosophy
3. The Three Crises in Mathematics: Different Schools of thought
4. The Unreasonable Effectiveness of Mathematics in the Natural Sciences
5. Beliefs Shape Mathematics (3)
6. Is Mathematics Philosophically Analytical (3)
7. Revisiting the 'unreasonable effectiveness' of mathematics (9)</td></tr>
<tr><td colspan="2" align="center">Mathematical Propositions have some distinct characteristics such as...</td></tr>
<tr><td></td><td><u>Nature of Mathematics</u>
Possesses Basic Concepts based on:
● Material/Physical Concepts: E.g Geometry: Point, Line, Plane
● From Ideas: E.g. Negative numbers, calculus, complex numbers
    ○ ≠ Science and Art
● A combination of both: E.g. Instantaneous rate of change

<u>Abstraction</u>
Definition: Process of extracting the underlying essence of a mathematical concept
● Removal of any dependence on real world objects with which it might originally have been connected
● Does not occupy spatial and temporal locations; neither does it engage in causal relationships with other objects.
● Generalization so that it has wider applications or matching among other abstract descriptions of equivalent phenomena

Example: Buying 3 pairs of shoes for $20
● Abstract 3 and 20 and multiply to obtain 60
● Interpret the result to suit the physical situation</td></tr>
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Concrete Real World Concept → *Abstraction* → Mathematical abstract structure

- E.g. Geometry: Calculation of distances and areas in the real world → Properties of shapes
- One abstract concept should be applicable to **all the physical manifestations** of that concept
- Ongoing Process
    - E.g. Geometry: Euclid's Elements + Cartesian Coordinates
- Suggests that Mathematics began as a practical tool that was abstracted to fit more applications

Advantages: Heavy Synergy and Connectedness

- Reveals deep connections between different areas of mathematics
    - Unity of algebra and geometry with cartesian coordinates
    - //Positive Coherentism
- Known results in one area can **suggest conjectures** in a related area
- Techniques and methods from one area can be applied to prove results in a related area

<u>Idealisation</u>

Definition: The assumption that any concept is considered as its ideal

- In applying mathematics, we consider the **ideals of objects** rather than their imperfect physical forms
- E.g. Regarding the Earth as a perfect sphere
- Application of Abstraction to the physical world

<u>Use of Symbols</u>

- E.g. Numbers, infinity sign
    - Numbers and the rules with which they are expressed can vary
    - But they ultimately mean the same thing
    - E.g. 243 vs “两百四十三” (Chinese)
        - Place assumed in English vs Placeholders such as “百” (hundred) are employed in Chinese

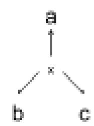
The use of idealisation and abstraction results in foundational truths known as AXIOMS which kickstart mathematical knowledge construction

<u>Axioms</u>

- **Self-evident truths** taken to be established that serve as the **foundation** for mathematical knowledge
    - Also known as postulates
    - //General Epistemology: Foundationalism
        - Difference: **Analytically/Pragmatically** rather than dogmatically derived
            - In contrast with religious knowledge construction where most start from dogmatic foundations about the nature of God to deduce further truths (e.g. God is always right → God’s law is always right),

	mathematical axioms are self-evident as they are analytic (true by the nature of their definition rather than some external truth e.g. All bachelors are unmarried) ○ Deductions are made from Axioms to form an Axiomatic-Deductive System ■ //Intuition-Deduction Thesis in Rationalism ● Example: Euclid's axioms ○ Two things equal to a third are equal to each other ■ x = y, z = y, x = z ○ Equals (even numbers) added to equals make equals ● Self-evident nature seems to imply both realism and anti-realism ○ Anti-Realist Interpretation: Axioms are invented to suit what the concepts are intended to reveal about reality ○ Realist Interpretation: Axioms are known through Platonic Ideals/Aristotelian Intuition ● Sometimes axioms are established whilst the branches of postulates are being added ○ I.e. do not have to dogmatically start from axioms; logic and practical use also advise Criteria for Axioms 1. Consistent: Implications **should not contradict** a. Should not be possible to derive P and ~ P (not-P) from axioms b. Principle of Explosion: One contradictory axiom would lead to further contradictory theorems thus ruining the system c. // Construction method of Proof d. // Coherentism 2. Independent: Should not be possible to derive one axiom from the others 3. Fruitful: Should be possible to derive many theorems from axioms ● Suggests that axioms are defined by utility in consistency ○ *Try, for example, to make a new number system that's like the ordinary one except that it skips some number -- say, 4. It just won't work. Everything will go wrong. You'll have to decide what 2 plus 2 is. If you say that this is 5, then 5 will have to be an even number, and so also must 7 and 9. Then, what's 5 plus 5? Is it 8, or 9, or 10? You'll find that to make the new system at all like arithmetic you'll have to change the properties of all the other numbers. Then, when you're done, you'll find that you have changed only those numbers' names and not their properties at all. Similarly, you could try to make two different numbers be the same -- say, 139 and 145. But then, to make subtraction work, you'll have to make 6 the same as 0 and 4 plus 5 equal to 3. Suddenly, you'll find that the sum of two positive numbers is smaller than either of them--and that scarcely resembles arithmetic at all.*

<u>Sources of Axioms</u>

- Analytically Deduced Axioms: Some concepts have their meaning and properties derived entirely from abstract axioms that prescribe their properties
  - E.g points and straight lines as opposed to triangles
  - A Priori
  - **Cognitively Pragmatic** Axioms: Axioms that are defined as such due to their usefulness in our thinking
    - E.g. Multiplication: a x b x c may be (a x b) x c, (a x c) x b or even 

(a simultaneous application of multiplication to all three numbers
    - However, we accept the first two definitions of multiplication (and its permutations that multiply two numbers at a time) and do not accept the simultaneous application to all three numbers
    - Such a characteristic within the definition of multiplication would demand the need for a new definition that accommodates 4 numbers, 5 numbers, 6 numbers and so forth
    - //Kant on constructivism: Are these axioms defined as so because of the way the world is, or because of the way our mind constructs the world?
- Physical Knowledge Axioms: Some concepts have their meaning and properties derived from an **application in Science**
  - Mathematics serves as the language of Science
    - Density as mass divided by volume
    - Force as the product of mass and acceleration
  - Imposes Arbitrary quantities on natural phenomena
    - Paradox: The 'invented' nature of mathematics vs its strong relevance to the real world
  - //Abstraction: Beneficial for clarity in constructing scientific knowledge in an objective manner

<u>Deduction</u>

- //General Epistemology: Reasoning that moves from the general to the specific
- Suggests that no new propositional knowledge is gained through mathematics
  - At the very most we discover new relations between axioms that provide competence knowledge

Through the use of AXIOMS and DEDUCTIONS, we arrive at mathematical propositions through a process known as...

	<u>Proof</u>
	Definition: The process of determining the **truth or falsehood** of a mathematical statement using only
	1. Axioms: Fundamental Hypotheses
	    a. E.g Euclid's Axioms
	2. Definitions: Fundamental Concepts
	    a. E.g. If e is an odd number e = 2m for some whole numbers n and m, by definition.
	3. Theorems: Previously established Results
	    a. E.g. Pythagoras' Theorem
	4. Logic: Logically correct inference
	    a. //Logicism: Math = Logic
	● Results in theorems
	
	<u>Methods of Proof</u>
	Logical Deduction: Knowledge proved by a sequence of
	● Previously proven propositions
	● Self-evident propositions
	● Propositions that follow by a valid logical argument from earlier propositions
	
	Construction: Proposed objects are explicitly exhibited or constructed
	● Aka Proof by example
	● Not necessary to show construction to prove existence
	
	Mathematical Induction
	1. If a proposition p(n) depending on a natural number n is true for some first number m
	2. And we can show that whenever p(n) is true then so is p(n+1)
	3. Then p(n) is true for all n > m.
	● Guarantees proposition + Guarantees application to larger natural numbers which can all be represented algebraically by p(n+1)
	● Aka complete induction
	● ≠ Science: Guarantees the truth of the conclusion
	
	Contradiction: Proving that a proposition P's negation not-P is contradictory
	● $(\sim p \rightarrow q \ \& \sim q) \rightarrow p$
	● Aka modus tollens
	● Indirect Proof
	
	Prevalence of Logic // Logicism

Therefore, mathematical knowledge construction involves...	
	<u>Mathematical Knowledge Construction/Progress</u> ● Abstraction into Axioms of Symbols ○ Building Models for classes of real phenomena

        - For generalisations to idealised physical phenomena
        - A Priori: Self-evident or cognitively useful axioms
        - A Posteriori: Scientifically useful axioms
    - Proof
        - Proofs are inferred via deduction from axioms resulting in theorems

Example: Proving that an odd number and an even number add together to give an odd number.

Let the odd number be o and the even number be e.

Then o = 2n + 1 and e = 2m for some whole numbers n and m, by definition.

So o + e = 2n + 1 + 2m
= 2m + 2n + 1
= 2(m + n) + 1
= 2p + 1 where p is a whole number

This is of the form 2n + 1 and hence odd.
It's so odd I can't even

Mathematical Knowledge: A Subject S knows P through mathematics if the subject can prove P from a set of axioms

- Axioms are self-evident: Can have immediate knowledge to them and do not need to justify them further

Axiomatic Deductive System

- Mathematics is concerned with the **interrelations** between mathematically 'undefined objects' and the **rules** governing operations within them
    - Mathematics is concerned with the structure and relationships of math objects rather than the nature of math objects themselves
    - Axioms and definitions are not substantial things in themselves

This is in contrast with scientific knowledge construction because...

Maths Construction ≠ Science Construction

Scientific Hypothetico-Deductive Method	Mathematical Axiomatic-Deductive Method
Deduction to a specific case for testing	Deduction to a proposition for proof
Through testing the theory	Through proving the theory
Knowledge gained through Corroboration with repeated tests and peer review	Knowledge gained when we can prove proposition definitively with logical inference from axioms

Falsificationism: Theories are the best explanation given repeated attempts to disprove	Knowledge gained when we can prove proposition definitively with logical inference from axioms
A Posteriori Observations	A Priori Axioms ● Though it may require the ‘experience’ of proving (as with intuitionism), this is not sense experience

With an understanding of the NATURE of MATHEMATICS, we shall now delve deeper into its NATURE and VALIDITY in terms of how it relates to REALITY; differing views on this have led to different schools of thought which fall under...

- Platonism → Logicism → Intuitionism and Formalism

<u>Realism</u>

- Numbers exist **independent** of the mathematician
    - Explains **commonality and uniformity** of mathematics
        - Civilisations being discovered with mathematics
        - Newton and Leibniz independently formulating calculus
        - Discovery of the number zero
- Assertions of mathematics
    - Should be taken literally
    - Have **objective truth-value**
    - Are independent of minds, languages, conventions etc.
        - Though conventions (e.g. Imperial/Metric systems) run in how we express these truths, they are still independent of human minds

<u>Arguments for Realism</u>

Focus on **universal categories** of human thought

- Ideas **naturally ingrained** in our perceptions
- Immediately and inevitably **quantify** the world around us
    - Discovery of other civilisations having maths

Occurrence in Nature

- Nature of Rivers: Pi
    - Irrational
- Plants and Human Lungs: Fractals in Mandelbrot set
    - Can be replicated infinitely such that computers and much less human minds cannot fully comprehend
- Prevalence and regularity + Mathematics could not have been designed in the mind to yield such strong correspondence to nature + **Irrationality/Incomprehensibility** → Independence from mind
    - BUT: Theory-ladeness; Could have been initially based off nature thus yielding such results
        - Reading maths into nature

Godel’s Incompleteness theorem

- A formal system cannot prove its **own consistency** in its **first-order** language (language which uses symbols based in logic)
    - Threat of **circular reasoning** ≠ Formalism
    - Unprovable truths should lie **outside of mathematical systems** in a Platonic World

Independent Mathematicians discovering the same mathematical truths

- Newton and Leibniz developing calculus

Challenges

- To Platonism
    - How can humans know anything about mathematical objects?
        - // Rationalism: Intuition/Innate Knowledge
    - How do we know our assertions of them are true?

Which diverges into..

<u>Platonism</u>

Mathematics involves constructing knowledge of an **ideal mathematical object/concept**

- Maths is the construction method that is the most real/correspondent to reality
- Supports Mathematical Realism: Math exists independently of the human mind in the Platonic World
- Characteristics of Ideal Math objects e.g. the perfect triangle
    - Acausal
    - Eternal
    - Indestructible
    - Out of space-time
        - *And the reach of my brain*
    - A Priori
        - Necessary in nature
- Known through mathematical intuition
- Threat: Unreasonable effectiveness of Mathematics
    - Entities without **spatial and temporal characteristic**s can interact with the physical world with **such applicability**
        - How do we as human access Platonic objects?
        - How do these Platonic objects apply to the real world

Arguments

- Sudden burst of inspiration on challenging problem = Consciousness making contact with platonic world

<u>Logicism</u>

Mathematics can be reduced to **Logic**

- Logical Terminology defines Concepts and objects
- Principles of Logic derive Theorems

- Challenged by Barbershop Paradox

Russell and Whitehead: Tried to reduce mathematics to **set theory**

- Succeeded in showing that all known mathematics could be reduced to set theory and hence axioms from Principia (now refined to formal set theory by Zermelo and Fraenkel, ZF)
- Logicists' definition of Logic: A logical proposition is one that has **complete generality and is true in virtue of its form** > its content
    - E.g. Law of non-contradiction p U ~p holds by its form, regardless of what p is
    - Logical propositions accepted by logicism could lie outside of classical logic
- Threat: Axioms of ZF that could not be reduced to truth by form
    - Axiom of Infinity: There exists infinite sets (sets with infinite elements)
        - Proven by everyday experience with sets: e.g. set of all natural numbers (positive integers and zero)
            - Accepted on the grounds of everyday experience rather than form → Cannot be reduced to logic
            - *Perhaps other grounds for accepting could be discovered?*
    - Axiom of Choice
- Threat: Set theory cannot be reduced to Logic
    - Many incompatible axiomisations
    - Set theory relies on the abstraction of many other objects such as "infinite sets" which cannot be "visualised" by logic
- Advantages
    - Realism: Abstract entities are discovered > constructed → Allows us to accept abstract entities that the mind cannot construct ≠ Intuitionism (e.g. sets of infinite numbers)

Russell's Paradox

- Does a set of sets that do not contain themselves contain itself?
- Challenge to Set theory
- E.g. In a library there are many books, some of which are catalogues of books or even catalogues of catalogues. And a catalogue may well list itself, too. Now consider the catalogue of all catalogues that do not list themselves, does it list itself or not?

REALISM implies...

Postulationism

Knowing some mathematical truth requires that we be able to prove it in its theory from the set of axioms which determines the theory

- The terms used to express our knowledge have their meaning by virtue of and in the context of that theory.
- Refutation: Godel's Incompleteness Theorem

    - The consistency of a formal axiomatic system cannot be proven within the system but only in a ‘larger’ system
        - However in doing so you would be having mathematical knowledge outside of the axiomatic-deductive system; contrary to postulationism
    - If theories must be completely derived from a finite set of axioms, the axioms themselves cannot be proven/refuted
    - *Threat to Coherentism/Foundherentism? Leads to an infinite regress?*

In contrast...

<u>Anti-Realism</u>

Mathematical objects do not exist (as with formalism) or are **dependent** on the mind (// idealism)

- E.g. Nominalism: Linguistic construction
- Are not relative (not subject to bias) with
    - No Truth-Value: As with formalism
    - Idealised truth value: Intuitionism
- No correspondence to reality
- *// Pragmatic Theory of Truth*
- *// Gettier Cases*

<u>Arguments for Anti-Realism</u>

Abstraction: Lack of corresponding reality

- **Unobservable Abstract quantities**: square root of two and imaginary numbers
- Fields such as pure mathematics that studies entirely abstract objects

Subject to changing tastes and preferences of the community

- Theorems fade in importance with newer problems: 9 point theorem
- **Relaxation of axioms** increases mathematical knowledge
    - New fields from relaxing the assumptions of existing fields
        - Rejection of the parallel postulate led to spherical and hyperbolic geometry
    - Suggests that axioms are **not bound to a platonic reality/logic**
    - Two seemingly **contradictory** statements can be mathematically true at the same time
        - Euclidean: Sum of the interior angles of a triangle is 180 degrees
        - Spherical geometry: Sum of interior angles of a triangle is 270 degrees
        - Non-contradiction only holds within axiomatic systems

Which diverges into...

Formalism

Broader definition: Mathematics is a **manipulation of symbols**

- Axiomatic Deductive system was invented (rather than discovered) without correspondence to reality
    - Initial Aim: To formalise an axiomatised theory is to choose an appropriate first order language for i.e. reduce it to:
    - A list of countably many variables
    - Symbols for the connectives from everyday speech
        - E.g. ~ for not
    - The equality sign
    - The "for all quantifier" and "there exist"
        - E.g For all complex numbers there exists a square root
        - E.g there exist irrational numbers
    - Parameters: Undefined terms e.g. point, line and incidence in Euclidean geometry
- Hilbert Program: Aimed to formalise mathematics to a **first-order language** to ensure (i.e. make certain) that mathematics is a consistent system
- Necessity of formalisation: Proves that a theory is free from contradictions (i.e. is consistent)
    - Precise language allows theory to be assessed as a mathematical object
- Aim: Prove that mathematics is free from contradictions
- Mathematical propositions are propositions of the results of rules set on the interaction of symbols
- Focused on the formal techniques (rules) of the symbols
    - Described as a game; the arbitrary manipulation of rules
    - E.g: Douglas Hofstadter's MU puzzle
        - Illustrates the difference between interpretation on the syntactic level (based on the rules within the system) and the semantic level (based on the meanings of the statements i.e. examining how the system works) i.e. reasoning about the system vs reasoning within it
        - Illustrates the inability to express or deduce certain concepts its possesses (MU) within the system // Godel's incompleteness theorem
- Threat: Unreasonable effectiveness of Mathematics
    - Correspondence to the external world is simply coincidence
    - // Gettier Case

Godel's Incompleteness Theorem

1. No consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is **capable of proving all truths** about the **arithmetic of the natural numbers**
    - I.e. No sufficiently expressive mathematical system can be both consistent and complete

	2. For any axiomatic deductive system there will always be statements about the natural numbers that are **true, but that are unprovable** within the system
	
	Arguments
	● Formulated a system that represents statements in a math system with natural numbers
	    ○ E.g "0 = 0" is $2^6 \times 3^5 \times 5^6 = 243{,}000{,}000$
	    ○ In a formal system, it is possible to make a statement that means “this statement is unprovable” (X)
	    ○ X however, is considered to be true by Godel since it cannot be false (proof by contradiction)
	        ■ Therefore consistent systems will not complete (i.e. prove all truths)
	    ○ If you add the truth into the system there would be unprovable truths within the system
	        ■ Therefore complete systems would not be consistent
	    ○ If you add the truth to the system: Can simply bring up ‘“this statement is unprovable” is unprovable’ as another unprovable statement
	
	<u>// Formalism:</u>
	No sentence of a first-order language which can be interpreted as asserting that T is free from contradictions can be proven formally through the said language
	● Formal mathematics cannot prove its own consistency and freedom from contradiction
	
	<u>Threat of Ordinary Language</u>
	In formalisation, one inevitably has to talk about the first-order language as one object, using everyday language
	● Danger of contradictions and other errors slipping in thus threatening our certainty in the consistency of mathematics
	    ○ Hilbert: Suggests that we should employ only finitary reasonings (certain reasons) in the Hilbert program
	        ■ Only use given number of finite objects and functions
	        ■ Functions are well-defined: Can compute their values such that they are unambiguous with only one definition
	        ■ Only state that an object exists alongside proof of how to construct it
	        ■ Never consider the totality of all objects in an infinite collection in communication
	            ● Only say that an argument is true for each and every object in the set taken by itself
	● Analogy of a Chess Game: Meaningless Figures pushed around according to the rules of the game
	    ○ Axioms: Given position of pieces
	    ○ Process of proof: Rules for moving pieces

	○ Theorems: All possible positions that can occur in a game **<u>Formalism: Maths as a Language</u>** Abstract objects entities are merely vocal utterances/written lines (i.e names) ● Have no existence (not even as mental constructions as with intuitionism/conceptualism) ● Knowledge Construction: Mathematics as a Language ○ Study first order language to formalise theories ○ How to form sentences from special sentences (axioms) ○ No sentence can be proven and disproven ○ → Consistent!! ● Grounded in our experience of the world ○ Begins with numbers and operations based on human experience ○ Explains the Unreasonable Effectiveness of Mathematics ■ Some **basis in external reality** → **High applicability** to external reality in postulates ○ E.g. Galileo formulating the gravitational force law as an inverse square law from observations of patterns in the movement of falling objects that implied a formula ■ Observation → Mathematics: Maths abstracts observation into discernible law ● Mystery of how Galileo employed the second derivative to surprising accuracy: Resolved by explanation that physical intuition of acceleration as a property → Use of second derivative (i.e. physical observations guided maths > random chance) ■ Qualitative nature: English lacks the sufficient abstract accuracy to describe the law ● E.g the Object is falling faster at an increasing rate does not define with the same accuracy for further application and relation ○ ∴ Physicists must employ experiments to check their maths ● Analogy: The English Language ○ Able to describe “object A is to the left of object B” ○ Creates a relation that matches our perception yet lacks the problem of Unreasonable Effectiveness? ■ Language clearly arises from utility in the external world whilst mathematics is abstract ■ Languages have less scope to describe one object than mathematics which **can talk about a single event from multiple axiomatic systems** (e.g euclidean geometry on a plane to non-euclidean geometry ● Increases the probability of a relevant axiomatic system ■ **Abstraction to symbols** in mathematics allows for ease of **identifying relevant equations** for physics

	● Identification of 'mass terms' employed across physics: e.g. Term ½ ab^2 : Landau and Lifshitz realised that the total kinetic energy of 2 particles can be reduced to this formula → Can consider 2 particles as one particle

	<u>Intuitionism</u>
	Mathematics is an **idealised mental activity** of carrying out inductive and effective mental constructions **consecutively** (rather than a set of theorems)
	● // Logicism: Reduction of mathematics to basic mental processes (rather than logic)
	● Inductive: Mental constructs must be **built upon previous mental constructs**
	  ○ To construct a natural number one must repeat the mental process for 1 until that number is reached
	    ■ Cannot access the number immediately
	● Effective: Upon the construction of a mental construct it has been constructed entirely as a finished mental construct for our study
	● Constructive: Intuitionistic mathematics involves carrying out inductive and effective constructs over and over
	  ○ E.g. If a real number occurs in an intuitionistic proof, it occurs there as it has been constructed from 1 onwards
	  ○ Cannot be reduced to other sciences (e.g. logic): Mental processes too complex
	    ■ ≠ Logicism: Valid logical processes are constructs and therefore part of mathematics
	● Classical theorems not composed from constructs considered meaningless word combinations
	● Anti-Realist: Claims that mathematics exists in the mind
	● Mathematical propositions are **true when they are proven**
	  ○ Rather than statements that assert truth
	  ○ I.e. the act of adding 1 + 1 = 2 makes it true vs the statement 1 + 1 = 2 has truth-value
	● Rationale: Assuming that mathematics is dependent on the mind (implied by how it was invented) it would not be real until it is thought of, being only abstract unrelated concepts
	  ○ // Access Internalism in General Epistemology
	  ○ // Idealism and Phenomenalism
	  ○ Basic mental processes thought to be intuitive
	
	Paradox: Truth and Falsity are reduced to Provability and Unprovability
	1. P is true + I can prove P is true → P is true
	2. If I can't Prove P is true → P is false or P is not true (i.e. either true or false)?
	  a. The former is implied by Intuitionism; P would **not hold truth value if it can't be proven**

	b. But the latter is more logical; the inability to prove something does not imply that it is false // Fallacy of Ignorance
	3. In proving P is true, I am proving that I cannot prove not P
	    a. P and ~ P are not necessarily contradictory ≠ Law of non-contradiction
	● *Threat to Falsificationism in Science/Reliabilism?*
	● *Reveals the necessity of the 'Truth" condition in addition to the 'Belief' condition in the JTB framework*
	
	Inspiration: Contradiction in Cantor's set theory → Classical Mathematics is faulty
	● Rebuilt axioms from natural numbers based on mental processes
	    ○ Immediate certainty by innate intuition about the definition of the number 1
	    ○ Repetition of the mental process that goes into forming 1 gives 2
	        ■ Based on innate intuition of time
	    ○ Thus humans can construct any **finite** initial segment 1, 2, 3 … n for any natural number n
	        ■ Finite: Limited by mental capabilities
	
	Reconstruction of Natural Numbers
	● Inductive: To construct a natural number one must repeat the mental process for 1 until that number is reached
	    ○ Cannot access the number immediately
	● Effective: Upon the construction of the natural number it has been constructed entirely as a finished mental construct for our study
	● Subsequently developed algebra, set theory, arithmetic etc.
	    ○ Some classical theorems that are not constructed rejected
	● ≠ Logicism: Aimed to develop mathematics with new definition from the ground up rather than relate all current mathematics to logic
	    ○ // Paradigm shifts and incommensurability
	    ○ // Mathematical progress and validity
	
	Reasons for Rejection of Intuitionism by the mathematical community
	1. Loss of many theorems
	    a. Utility: Brouwer fixed point theorem of topology
	        i. Motivation for Rejection?
	2. Constructionist proofs are much longer than classical proofs
	    a. Fundamental theorem of algebra: Half a page → 10 Pages
	    b. Loss of elegance
	    c. Conflict between Maths as the study of relations and Maths down to pure constructs
	        i. Coherentism vs Foundationalism
	3. Contradictions with Classical Maths
	    a. Claim that every real-valued function which is defined for all numbers is continuous
	● CLAIM: Rejection of intuitionism comes from emotional grounds
	    ○ Effect of culture on knowledge

ANTI-REALISM implies...	
	<u>Mathematics by Pragmatism</u> Quine: Mathematics is being tested and corroborated in science (or in everyday life) ● Useful parts of mathematics arise ● Mathematics is Knowledge when connected to the real world ○ Tested and Corroborated

	<u>Mathematics and Physics</u>
	Paradox:
	1. Mathematics turns up in unexpected connections in natural phenomena offering a close and accurate description
	2. Due to its unexpectedness: We cannot know if a theory formulated in maths to describe a natural phenomena is uniquely appropriate
	
	Anti-Realism: Unexpected connections due by design
	● The rules for lesser known theories were built from designed concepts on the rules of known theories
	    ○ Thus unexpected connections are inevitable by deduction
	        ■ Only surprising due to the limits of human thinking
	● Parallels Physics' regularity across time, space and other circumstances
	● with **manageably small set of conditions**
	    ○ Built upon our current scientific knowledge: A result of Theory-Ladeness?
	    ○ Built upon a small set of circumstances such that physicist stumble on math concepts that best describe a phenomenon that mathematicians have already developed independently
	        ■ E.g. Newton's law of gravitation: Built upon the observation that a rock's path on earth // moon's path in sky in being an ellipse
	            ● Theory formulated on single approximate incident
	            ● Employed 2nd Derivative
	        ■ Quantum Mechanics: Laws which did not assume theory of a electrons of Hydrogen atom built upon similarity to a maths concept worked on electrons
	        ■ Assumes realism
	    ○ Quantum Mechanics: Intelligent bets on future properties of the inanimate world
	● Laws of physics already formulated in mathematical language
	    ○ Theory-ladenness, assumed realism and exceptional nature of laws (i.e. laws work under very specific circumstances → The applicability of Mathematics is not 'surprisingly true': It is built on assumptions that cater to its coherence
	● ∴ Mathematics is a mere tool evaluating the implications of already established theories, chosen for their tendency to create elegant arguments

	- HOWEVER: The repeated coincidences between abstract mathematics and natural phenomena render it useful - Law of Gravitation: 4% - Matrix Mechanics → Quantum Mechanics: 1 in 10 million - Complex Spectra - Theory of the Lamb Shift: 1 in 1000 ∴ Mathematics allows us to abstract observations from a limited scope to general laws - Extends the reach of science's falsificationism (i.e allows us to create laws about more concepts that are harder to test so that we can test them and gain scientific knowledge) - Possibility of Union/Conflict of Mathematically implied theories: Macroscopic Theory of Relativity vs Microscopic Quantum Theory - Threat of future theories that disprove current theories based on past disproofs - Bohr's theories on atoms - Ptolemy's epicycles - Free electron theory: Contradicted by observations but faces lack of new theories **<u>Solution: Maths as a Language</u>** Abstract objects entities are merely vocal utterances/written lines (i.e names) - Have no existence (not even as mental constructions as with intuitionism/conceptualism) - Knowledge Construction: Mathematics as a Language - Study first order language to formalise theories - How to form sentences from special sentences (axioms) - No sentence can be proven and disproven → Consistent!! **Grounded in our experience** of the world - Begins with numbers and operations based on human experience - Explains the Unreasonable Effectiveness of Mathematics - Some basis in external reality → High applicability to external reality in postulates - E.g. Galileo formulating the gravitational force law as an inverse square law from observations of patterns in the movement of falling objects that implied a formula - Observation → Mathematics: Maths abstracts observation into discernible law - Qualitative nature: English lacks the sufficient abstract accuracy to describe the law - E.g the Object is falling faster at an increasing rate does not define with the same accuracy for further application and relation - Mystery of how Galileo employed the second derivative to surprising accuracy: Resolved by explanation that physical intuition of acceleration as a property → Use of second

derivative (i.e. physical observations guided maths > random chance)
    - ∴ Physicists must employ experiments to check their maths
- Analogy: The English Language
    - Able to describe “object A is to the left of object B”
    - Creates a relation that matches our perception yet lacks the problem of Unreasonable Effectiveness?
        - Language clearly arises from utility in the external world whilst mathematics is abstract
        - Language have less scope to describe one object than mathematics which can **talk about a single event from multiple axiomatic systems** (e.g euclidean geometry on a plane to non-euclidean geometry
            - Increases the probability of a relevant axiomatic system
        - **Abstraction to symbols** in mathematics allows for ease of identifying relevant equations for physics
            - Identification of ‘mass terms’ employed across physics: e.g. Term $\frac{1}{2} ab^2$: Landau and Lifshitz realised that the total kinetic energy of 2 particles can be reduced to this formula → Can consider 2 particles as one particle

Now that we have considered the NATURE and CONSTRUCTION of math knowledge, we shall now explore if such knowledge is meaningful

<u>View: Maths is Analytic</u>

Mathematics true by the initial axioms/definitions. It is merely deduction from initial postulates - no new knowledge is gained

- Proponent: David Hume
- Proof: Axiomatic-Deductive System

<u>View: Maths is Synthetic</u>

Maths knowledge is true by relation to other things beyond its initial postulates

- Maths progresses through
    - **Extending** previous concepts **beyond their initial definitions**
    - Forming new concepts and applying them to pre-existing maths methods
- E.g. Natural numbers to integers (i.e. conception of negative numbers and zero)
- E.g. Exponentiation: ‘X to the power of N” as “N instances of X all multiplied together” → Rules for negative and zero powers, fractional numbers, complex numbers
- E.g Descartes Analytical Geometry + Gauss on curved surfaces → Non-Euclidean Geometry
- Sensible: Can be incorporated into the axiomatic deductive system
    - Synthetic: Cannot be deduced from axioms

Now that we have considered the NATURE, CONSTRUCTION and VALIDITY of Mathematics, we shall now consider its relation to SOCIETY

Uniqueness of Maths in Culture

- ≠ Arts and Sciences which are invented and culturally influenced
- Formal and abstract > Cultural (Historical, Informal and Social)
    - Objectively accessible and unchanging theory → Independence from the human mind
- Theory: Mathematics was invented by individual mathematicians but gained an autonomous existence
- Human element can only be detected in the Historical development of Mathematics
    - BUT: Mathematisation; where non-mathematical ideas are translated into mathematics?

Historical Evolution

- Greek: Axiomatic Deduction and Axiomatic Geometry
    - Fear of Incommensurable quantities led to fierce adherence to axioms
    - Motivated by the rational understanding of nature
        - Pythagoreans: Diverse phenomena have same mathematical properties → Maths Laws underly nature; physical objects are concrete realisations of abstract concepts
            - “Reduction” of physical phenomena
            - Built astronomical theory out of numerical relationships
        - Atomists: Physical world governed by mathematical laws
        - Plato: Understand World of forms in geometry
            - Maths transcends nature to comprehend the ideal
        - Aristotle: Maths is a description of the natural
        - Euclid’s Axioms
- 16 - 18th Century: Mathematics as a religious quest to understand the world God designed mathematically
    - Influence of the Catholic Church
- French Revolution: Return to precision and rigorous proof
    - Threat: Tension between abstraction and application

Impact on Culture

- Aristotle: Abstracted the law of the excluded middle and the law of contradiction from mathematical deduction

Coherence in Mathematics

- High degree of Coherence: Low human influence due to axiomatic-deductive system
- Collective product without a central authority
- Few Contradictions within theories

- Unexpected Connections
- High Consensus

<u>Absolutism</u>

Our Mathematical Knowledge is certain

- Schools
    - Logicism
    - Formalism
    - Intuitionism: Progressive Absolutism
    - Platonism

<u>Fallibilism</u>

Our Mathematical Knowledge has the possibility of being untrue; it is merely the best possible system we have built

- Arguments:
    - Irreducible axioms that are beliefs that are open to doubt
        - Self-evident?
    - Godel’s incompleteness theorem
- Schools
    - Conventionalism: Reduces mathematics to the lingual and social aspects of society
    - Social Constructivism
        - Grounds
            - Basis in socio-conventional: Linguistic conventions and rules
            - Interpersonal social processes are required to turn an individual's subjective math knowledge into objective math knowledge
            - Objectivity itself is social

<u>Elegance in Mathematics</u>

- Motivation from elegance and beauty
    - // Parsimony
    - E.g. Euler’s formula
- Connections in conventionally beautiful objects: Golden Ratio
- Perfect relation to nature: Fibonacci number, musical ratios

<u>Function of Mathematics</u>

Understanding of Nature

- Plato: Ideal Forms
- Physical
    - Reveals and describes phenomena not readily accessible through the fallible senses

Abstraction of laws to solve practical problems

<u>Value of Mathematics</u>

- Utility: Basis of pragmatic uses

	○ E.g. calculus to calculate rate of change ● Utility: Able to mess with axioms to yield meaningful theorems ○ Assumes anti-realism (especially formalism) ○ New fields from relaxing the assumptions of existing fields ■ Rejection of the parallel postulate led to spherical and hyperbolic geometry ● Utility: Deduction of truth from a set of assumptions ○ E.g Assume that there is only 1 reactant molecule → Able to deduce how concentrations of reactant and product change over time ■ No mathematical basis but high predictive power ● Reliability: Certainty in deductions to necessary truths from axioms ○ Threatened by Godel's Incompleteness Theorem ■ BUT: Fallibilism ■ Not necessarily consistent does not imply necessarily inconsistent, applicability supports ● Reliability: Unreasonable effectiveness ○ Orbit of planets in ellipses // intersection between a plane and a cone ○ Inverse square law in gravitational attraction ○ ∴ By Indispensability argument: Well-established mathematical theories must be true as they are inextricably woven into accepted scientific theories due to their abstract nature that allows for quantitative analysis ■ *BUT: Falsificationism threatens certainty* ○ ∴ Have value through explanations of scientific phenomena

"This is how you are to make it: the length of the ark 300 cubits, its breadth 50 cubits, and its height 30 cubits. Make a roof for the ark, and finish it to a cubit above, and set the door of the ark in its side. Make it with lower, second, and third decks." (Genesis 6:15-16)

<u>Gödel's Incompleteness Theorem summarised:</u>

A (mathematical) system cannot be consistent and complete at the same time, because it relies on some unprovable truths outside the system.

- This is because no statement alone can completely prove itself to be true.
- Layman Eg 1) "I am lying", which is self-contradictory. If it's true, I'm not lying and the statement is false. If it's false, I'm lying and so it's true.
- Layman Eg 2) No one can prove that gravity is consistent all the time, only that it's consistently true every time (inductive reasoning).
- So, no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is **capable of proving all truths** about the **arithmetic of the natural numbers**
    - I.e. No sufficiently expressive mathematical system can be both consistent and complete

How to resolve:

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To-Do

1. Arguments for realism/anti-realism
2. **Russell's Paradox**
3. Axiomatic deductive method
    a. https://www.encyclopediaofmath.org/index.php/Axiomatic_method
4. Schools of Thought: Platonism, Logicism, Intuitionism, Formalism
5. Realism vs Anti-Realism
    a. Idealism
    b. Nominalism
6. Add pictures
7. Hilbert Program