1	(a) Volume $= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} xy \, \mathrm{d}x$ $= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^5 x \, \mathrm{d}x$ $= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^3 x(\sec^2 x - 1) \, \mathrm{d}x$ $= 2\pi \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2 x) \tan^3 x \, \mathrm{d}x - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2 x - 1) \tan x \, \mathrm{d}x \right]$ $= 2\pi \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2 x) \tan^3 x \, \mathrm{d}x - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan x \, \mathrm{d}x + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, \mathrm{d}x \right]$ $= 2\pi \left[\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln\lvert\sec x\rvert \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ $= 2\pi \left[\frac{9}{4} - \frac{3}{2} + \ln 2 - \left(\frac{1}{36} - \frac{1}{6} + \ln \frac{2}{\sqrt{3}} \right) \right]$ $= \pi \left(\frac{16}{9} + \ln 3 \right)$ (b) Let $\mathrm{f}(x) = \frac{\tan^5 x}{x} - \sin x$. Estimate = $\frac{0.5\mathrm{f}(0.8) - 0.8 \cdot \mathrm{f}(0.5)}{\mathrm{f}(0.8) - \mathrm{f}(0.5)} \approx 0.6031511 = 0.603$ (3 s.f.) (c) From GC, $\mathrm{f}'(0.5) = 0.0843$, which is close to zero. The tangent at $x = 0.5$ has a gentle gradient and therefore cuts the *x*-axis far from the actual root required, which is undesirable.
2	(a) Auxiliary equation: $m^2 - 4m + 5 = 0$ $\Rightarrow m = 2 \pm i$ $r = \sqrt{2^2 + 1^2} = \sqrt{5}, \quad \tan\theta = \frac{1}{2}$ $u_n = C(2+i)^n + D(2-i)^n$ $u_n = \left(\sqrt{5}\right)^n (A\cos n\theta + B\sin n\theta)$ $\left.\begin{array}{l} u_0 = 0 \Rightarrow A = 0 \\ u_1 = 1 \Rightarrow 1 = \sqrt{5}B\sin\theta = \sqrt{5}B\left(\frac{1}{\sqrt{5}}\right) \Rightarrow B = 1 \end{array}\right\}$

	$\therefore\ u_n = 5^{\frac{n}{2}} \sin n\left(\tan^{-1}\tfrac{1}{2}\right)$ (b) If $a = 0.1$, then $v_n = (0.1)^{\frac{n}{2}} \times 5^{\frac{n}{2}} \sin n\theta.$ As $n \to \infty, (0.5)^{\frac{n}{2}} \to 0.$ v_n oscillates and converges to 0. If $a = 0.2 = \tfrac{1}{5}$, then $v_n = \left(\tfrac{1}{5}\right)^{\frac{n}{2}} \times 5^{\frac{n}{2}} \sin n\theta = \sin n\theta.$ As $n \to \infty$, so v_n does not converge , is bounded between -1 and 1 and oscillatory.
3	(a) The displacement **decreases** from x_0 and **approaches zero** in the long run. (b) $x_n = x_{n-1} + h(-ax_{n-1}) = (1-ah)x_{n-1}$ $= (1-ah)^2 x_{n-2}$ $\ldots$ $= (1-ah)^n x_0$ (c) Estimates will not decay to zero if $\lvert 1-ah \rvert \geq 1$ $\Rightarrow 1-ah \geq 1$ or $1-ah \leq -1$ $h \leq -\dfrac{2}{a}$ (rej, since $h > 0$) $h \geq \dfrac{2}{a}$ (d) $x_n{}' = x_{n-1}{}' + \dfrac{h}{2}\left[-ax_{n-1}{}' - ax_{n-1}{}'(1-ah)\right]$ $= x_{n-1}{}' - ahx_{n-1}{}' + \dfrac{a^2h^2}{2} x_{n-1}{}'$ $= \left(1 - ah + \dfrac{a^2h^2}{2}\right) x_{n-1}{}'$ $\ldots$ $= \left(1 - ah + \dfrac{a^2h^2}{2}\right)^n x_0$
4	(a) $\left(x+\sqrt{x}\right)^2 = x^2 + 2x\sqrt{x} + x \geq x + x^2$ (since $x\sqrt{x} \geq 0$) Taking square root, $x + \sqrt{x} \geq \sqrt{x + x^2}$ (shown) (b) Let P_n be the proposition: $u_n \geq \sqrt{n!}$ for $n \in \mathbb{Z}$, $n \geq 0$. For n = 1, LHS = $u_0 = 1$, RHS = $\sqrt{0!} = 1$. For n = 2, LHS = $u_1 = 1$, RHS = $\sqrt{1!} = 1$. Since LHS $\geq$ RHS for both base cases, P_0, P_1 are true.

	Assume P_k, P_{k+1} are true for some non-negative integer *k*, i.e. $u_k \geq \sqrt{k!}$ and $u_{k+1} \geq \sqrt{(k+1)!}$, For $n = k+2$, $u_{k+2} = u_{k+1} + (k+1)u_k$ $\geq \sqrt{(k+1)!} + (k+1)\sqrt{k!}$ $= \sqrt{(k+1)k!} + (k+1)\sqrt{k!}$ $= \sqrt{k!}\left(\sqrt{k+1} + (k+1)\right)$ $\geq \sqrt{k!}\sqrt{(k+1)(1+k+1)}$ [using (a)] $= \sqrt{(k+2)!}$ So P_k, P_{k+1} true $\Rightarrow P_{k+2}$ is true. Since P_0, P_1 are true, and P_k, P_{k+1} are true $\Rightarrow P_{k+2}$ is true, by induction P_n is true for all $n \in \mathbb{Z},\ n \geq 0$.
5	(a) Let $\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{M}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\det(\mathbf{M}_1) = \det(\mathbf{M}_2) = 0$, $\mathbf{M}_1, \mathbf{M}_2 \in U$. However, $\mathbf{M}_1 + \mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $\det(\mathbf{M}_1 + \mathbf{M}_2) = 1 \neq 0$. Not closed under addition. Therefore, *A* is not a subspace of *U*. (b) The zero polynomial **0** is in *S*, since $\mathbf{0}(1) = 2 \cdot \mathbf{0}(3) = 0$. Let $f_1, f_2 \in S$. Then $f_1(1) = 2f_1(3)$ and $f_2(1) = 2f_2(3)$. $(f_1 + f_2)(1) = f_1(1) + f_2(1) = 2f_1(3) + 2f_2(3) = 2\left(f_1(3) + f_2(3)\right) = 2(f_1 + f_2)(3)$ For any $\lambda \in \mathbb{R}$, $(\lambda f_1)(1) = \lambda f_1(1) = \lambda \cdot 2f_1(3) = 2(\lambda f_1)(3)$ Closed under addition and scalar multiplication. *S* is a subspace of *V*. $a_0 + a_1 + a_2 + a_3 = 2\left(a_0 + 3a_1 + 9a_2 + 27a_3\right)$ $0 = a_0 + 5a_1 + 17a_2 + 53a_3$ $a_0 + a_1x + a_2x^2 + a_3x^3 = -5a_1 - 17a_2 - 53a_3 + a_1x + a_2x^2 + a_3x^3$ $= a_1(x-5) + a_2(x^2-17) + a_3(x^3-53)$ A basis is $\left\{x-5,\ x^2-17,\ x^3-53\right\}$.

	(c) Matrix = $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Null space = $\left\{ k\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, k \in \mathbb{R} \right\}$ (d) Integrating $7+7x$ twice yields $a+bx+\frac{7}{2}x^2+\frac{7}{6}x^3$. Required set $\left\{ \begin{pmatrix} a \\ b \\ 7/2 \\ 7/6 \end{pmatrix}, \text{ where } a, b \in \mathbb{R} \right\}$.
6	(a)(i) $T_0 = 100\ 000$ is the initial population as given $T_{k+1} = (1-r)T_k$ because a death rate of *r* means that $(1-r)$ of the population is left after each week $0 \le k \le 11$ because the model given is only valid for 12 weeks. (ii) $T_{12} = (1-r)^{12} T_0$ $(1-r)^{12} = \frac{355}{100\,000} \Rightarrow 1-r = \sqrt[12]{0.00355} = 0.62496\ldots \Rightarrow r = 0.375\ (3\text{ sf})$ (b)(i) After 16 weeks, the number of frogs is $0.62496\ldots \times 100\,000 = 54.154\ldots$ So $54.154\ldots \times p \ge 30$ $\Rightarrow p \ge \frac{30}{54.154\ldots} = 0.5539\ldots = 0.554\ (3\text{ sf})$ (ii) There are no external threats to the population, like disease or predators which may reduce the population OR The same weekly death rate continues unchanged. (c) 30 surviving females would produce 75000 eggs, so the population is smaller than it was to start with, so each ‘round’ will result in smaller and smaller population.

7	(a) $e^{i(5\theta)} = \cos 5\theta + i\sin 5\theta$ and $(e^{i\theta})^5 = (\cos\theta + i\sin\theta)^5$ $= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$ Comparing real and imaginary parts, $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ $\tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$ $= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ (shown) (b) Let $\theta = \dfrac{\pi}{7}$ and $t = \tan\dfrac{\pi}{7}$. Then $\tan\dfrac{5\pi}{7} = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$. Since $\tan\dfrac{5\pi}{7} = -\tan\left(\dfrac{2\pi}{7}\right) = -\dfrac{2t}{1-t^2}$ (double angle formula), $-\dfrac{2t}{1-t^2} = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ $2t(1 - 10t^2 + 5t^4) = (t^2 - 1)(5t - 10t^3 + t^5)$ $0 = t^7 - 21t^5 + 35t^3 - 7t$ Since $t = \tan\dfrac{\pi}{7} \neq 0$, we can divide throughout by t and obtain $t^6 - 21t^4 + 35t^2 = 7$. Therefore, $t = \tan\dfrac{\pi}{7}$ is a root of this equation. (c) Any valid fixed-point iteration method with $t_0 = 1$ that yield an approximation 0.482, e.g. $t_{n+1} = \sqrt{\dfrac{21t_n^{\,4} + 7}{t_n^{\,4} + 35}}$ t_1: 0.882, t_2: 0.743, t_3: 0.617, t_4: 0.534, t_5: 0.498, t_6: 0.486, t_7: 0.483, t_8: 0.482, t_9: 0.482 e.g. $t_{n+1} = \sqrt{\dfrac{7}{t^4 - 21t^2 + 35}}$ t_1: 0.683, t_2: 0.524, t_3: 0.488, t_4: 0.483, t_5: 0.482, t_6: 0.482 (d) $\tan 5\theta = 1 \Rightarrow 5\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}, \dfrac{17\pi}{4}$

	$\theta=\frac{\pi}{20},\frac{\pi}{4},\frac{9\pi}{20},\frac{13\pi}{20},\frac{17\pi}{20}$ Set $\tan 5\theta=1$ in **(a)**, $1-10\tan^2\theta+5\tan^4\theta=5\tan\theta-10\tan^3\theta+\tan^5\theta$ $0=\tan^5\theta-5\tan^4\theta-10\tan^3\theta+10\tan^2\theta+5\tan\theta-1$ For the equation $x^5-5x^4-10x^3+10x^2+5x-1=0$, the roots are $\tan\theta$, $\theta=\frac{\pi}{20},\frac{\pi}{4},\frac{9\pi}{20},\frac{13\pi}{20},\frac{17\pi}{20}$. Sum of roots = $-\frac{(-5)}{1}=5$ $\tan\frac{\pi}{20}+\underbrace{\tan\frac{\pi}{4}}_{=1}+\tan\frac{9\pi}{20}+\tan\frac{13\pi}{20}+\tan\frac{17\pi}{20}=5$ $\tan\frac{\pi}{20}+\tan\frac{9\pi}{20}-\tan\frac{7\pi}{20}-\tan\frac{3\pi}{20}=4$ (shown)
8	(a) $\sum_{n=1}^{\infty}\frac{1}{a^{2n-1}}\sin\left(\frac{2\pi(2n-1)t}{T}\right)=\mathrm{Im}\sum_{n=1}^{\infty}\frac{1}{a^{2n-1}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi(2n-1)t}{T}\right)}$ $=\mathrm{Im}\left[\frac{\frac{1}{a}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi t}{T}\right)}}{1-\frac{1}{a^2}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi t}{T}\right)}}\right]$ $=\mathrm{Im}\left[\frac{a\mathrm{e}^{\mathrm{i}\left(\frac{2\pi t}{T}\right)}\left(a^2-\mathrm{e}^{\mathrm{i}\left(-\frac{4\pi t}{T}\right)}\right)}{\left(a^2-\mathrm{e}^{\mathrm{i}\left(\frac{4\pi t}{T}\right)}\right)\left(a^2-\mathrm{e}^{\mathrm{i}\left(-\frac{4\pi t}{T}\right)}\right)}\right]$ $=\mathrm{Im}\left[\frac{a^3\mathrm{e}^{\mathrm{i}\left(\frac{2\pi t}{T}\right)}-a\mathrm{e}^{\mathrm{i}\left(\frac{-2\pi t}{T}\right)}}{a^4-2a^2\cos\left(\frac{4\pi t}{T}\right)+1}\right]$

$$= \frac{a^3 \sin\left(\frac{2\pi t}{T}\right) - a\sin\left(\frac{-2\pi t}{T}\right)}{a^4 - 2a^2\cos\left(\frac{4\pi t}{T}\right) + 1}$$

$$= \frac{a(a^2+1)\sin\frac{2\pi t}{T}}{a^4 - 2a^2\left(1 - 2\sin^2\frac{2\pi t}{T}\right) + 1}$$

$$= \frac{a(a^2+1)\sin\frac{2\pi t}{T}}{(a^2-1)^2 + \left(2a\sin\frac{2\pi t}{T}\right)^2}$$

(b) When $0 < a \le 1$, the coefficients of the sine terms, $\frac{1}{a^{2n-1}}$, are significant for large n values. This means that the remaining terms that are truncated off are significant, resulting in the significant difference in their graphs.

(c) $P = \frac{1}{T}\int_0^T \left[A\mathrm{f}(t)\right]^2 \, \mathrm{d}t = \frac{A^2 a^2 (a^2+1)^2}{T}\int_0^T \left[\frac{\sin\frac{2\pi t}{T}}{(a^2-1)^2 + \left(2a\sin\frac{2\pi t}{T}\right)^2}\right]^2 \mathrm{d}t$

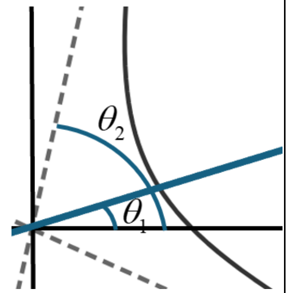
Let $y(t) = \left[\frac{\sin\frac{2\pi t}{T}}{(a^2-1)^2 + \left(2a\sin\frac{2\pi t}{T}\right)^2}\right]^2$.

n	0	1	2	3	4
t_n	0	$\frac{1}{4}T$	$\frac{1}{2}T$	$\frac{3}{4}T$	T
y_n	0	$\frac{1}{[(a^2-1)^2+4a^2]^2}$	0	$\frac{1}{[(a^2-1)^2+4a^2]^2}$	0

Using Simpson's Rule,

	$\int_0^T y\ \mathrm{d}t \approx \frac{1}{3}\left(\frac{T}{4}\right)\left[y_0 + y_4 + 2y_2 + 4(y_1+y_3)\right]$ $= \frac{T}{12}\left[0+0+2(0)+4\left(\frac{2}{[(a^2-1)^2+4a^2]^2}\right)\right]$ $= \frac{T}{12}\left[\frac{8}{[a^4-2a^2+1+4a^2]^2}\right]$ $= \frac{2T}{3(a^2+1)^4}$ For $P=10$, $\frac{A^2a^2(a^2+1)^2}{T}\cdot\frac{2T}{3(a^2+1)^4}=10$ $\frac{2A^2a^2}{3(a^2+1)^2}=10$ $A=\sqrt{\frac{15(a^2+1)^2}{a^2}}$ $=\sqrt{15}\left(a+\frac{1}{a}\right)$
9	(a) Auxiliary equation: $m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1$ CF: $y=(At+B)\mathrm{e}^{-t}$ PI: Try $y=kt^2\mathrm{e}^{-t}+c$ $\left.\begin{array}{l}\frac{\mathrm{d}y}{\mathrm{d}t}=2kt\mathrm{e}^{-t}-kt^2\mathrm{e}^{-t},\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}=2k\mathrm{e}^{-t}-2kt\mathrm{e}^{-t}+kt^2\mathrm{e}^{-t}-2kt\mathrm{e}^{-t}=2k\mathrm{e}^{-t}-4kt\mathrm{e}^{-t}+kt^2\mathrm{e}^{-t}\\ 2k\mathrm{e}^{-t}-4kt\mathrm{e}^{-t}+kt^2\mathrm{e}^{-t}+2\left(2kt\mathrm{e}^{-t}-kt^2\mathrm{e}^{-t}\right)+kt^2\mathrm{e}^{-t}+c=\mathrm{e}^{-t}+1\\ 2k\mathrm{e}^{-t}+c=\mathrm{e}^{-t}+1\end{array}\right\}$ $\Rightarrow k=0.5, c=1 \Rightarrow$ PI $y=0.5t^2\mathrm{e}^{-t}+1$ $y=$ CF + PI $=(At+B)\mathrm{e}^{-t}+0.5t^2\mathrm{e}^{-t}+1$ (b) $t=0, y=1 \Rightarrow B=0$ $\left.\begin{array}{l}y=\left(At+0.5t^2\right)\mathrm{e}^{-t}+1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\left(A+t-At-0.5t^2\right)\mathrm{e}^{-t}\\ t=0, \frac{\mathrm{d}y}{\mathrm{d}t}=9 \Rightarrow A=9\end{array}\right\}$ $\therefore y=\left(0.5t^2+9t\right)\mathrm{e}^{-t}+1$ (c) $\frac{\mathrm{d}y}{\mathrm{d}t}=0 \Rightarrow 0.5t^2+8t-9=0$

	$t > 0 \Rightarrow t = 1.5539$ $\Rightarrow y = 4.50$ (d) $t = 8 \Rightarrow y = \left(0.5\times 8^2 + 9\times 8\right)e^{-8} + 1 = 1.03488\ldots$ This is close to 1 so the model supports the suggestion that the concentration returns to its initial value after 8 hrs.
10	(a) $2r^2\cos^2\theta + 4r^2\sin\theta\cos\theta - r^2\sin^2\theta = 1$ $r^2\left(2\cos^2\theta + 4\sin\theta\cos\theta - \sin^2\theta\right) = 1$ $r^2\left[2\left(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta\right) - 3\sin^2\theta\right] = 1$ $r^2 = \dfrac{1}{2(\cos\theta + \sin\theta)^2 - 3\sin^2\theta}$ $r^2 = \dfrac{\text{cosec}^2\theta}{2(1+\cot\theta)^2 - 3}$ (shown) (b) Area required $= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{4}} r^2 \, d\theta$ $= -\dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{4}} \dfrac{-\text{cosec}^2\theta}{2(1+\cot\theta)^2 - 3} \, d\theta$ $= -\dfrac{1}{2}\left[\dfrac{1}{2\sqrt{6}} \ln\left\lvert \dfrac{\sqrt{2}(1+\cot\theta) - \sqrt{3}}{\sqrt{2}(1+\cot\theta) + \sqrt{3}} \right\rvert\right]_0^{\frac{\pi}{4}}$ $= -\dfrac{1}{4\sqrt{6}}\left[\ln\left\lvert \dfrac{\sqrt{2}(\tan\theta + 1) - \sqrt{3}\tan\theta}{\sqrt{2}(\tan\theta + 1) + \sqrt{3}\tan\theta} \right\rvert\right]_0^{\frac{\pi}{4}}$ $= -\dfrac{1}{4\sqrt{6}}\left[\ln\left(\dfrac{2\sqrt{2} - \sqrt{3}}{2\sqrt{2} + \sqrt{3}}\right)\right]$ $= \dfrac{\sqrt{6}}{24}\ln\left(\dfrac{11 + 4\sqrt{6}}{5}\right)$ (c) Differentiating with respect to x, $4x + 4\left(y + x\dfrac{dy}{dx}\right) - 2y\dfrac{dy}{dx} = 0$ $\dfrac{dy}{dx} = \dfrac{-2(x+y)}{2x - y}$ At vertex (x, kx), $(k)\left(\dfrac{dy}{dx}\right) = -1$

	$$k\left(\frac{-2(x+kx)}{2x-kx}\right)=-1$$ $$\frac{-2k(1+k)}{2-k}=-1$$ $$2k+2k^2=2-k$$ $$2k^2+3k-2=0$$ $k=\frac{1}{2}$ or -2 (reject, since gradient should be positive) (d) Let α be the angle between the transverse axis and the asymptote. Then $\alpha=\theta_2-\theta_1$.  $$\frac{b}{a}=\tan\alpha$$ $$=\tan(\theta_2-\theta_1)$$ $$=\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}$$ $$=\frac{2+\sqrt{6}-\frac{1}{2}}{1+\left(2+\sqrt{6}\right)\frac{1}{2}}$$ $$=\frac{3+2\sqrt{6}}{4+\sqrt{6}}=\sqrt{\frac{3}{2}}$$ Eccentricity, $e=\sqrt{1+\left(\frac{b}{a}\right)^2}=\sqrt{\frac{5}{2}}=\frac{\sqrt{10}}{2}$