# 2018 GCE A Level H2 Further Maths 9649 Paper 1 Solutions

**1** It is given that the equation

$$x \tan x - 1 = 0$$

has a root $\alpha$ in the interval $[3, 4]$.

Use the Newton-Raphson method repeatedly, taking $\alpha_1 = 3.5$ as a first approximation to $\alpha$, to determine a sequence of further approximations to this root. Show your working and state the value of each of the iterates calculated correct to 5 decimal places. Your process should terminate when you have found two successive iterates that are equal when rounded to 5 decimal places. [4]

**[Solution]**

Let $\mathrm{f}(x) = x \tan x - 1$

$\mathrm{f}'(x) = x \sec^2 x + \tan x$

Applying Newton-Raphson method, taking $\alpha_1 = 3.5, \alpha_{n+1} = \alpha_n - \dfrac{\alpha_n \tan \alpha_n - 1}{\alpha_n \sec^2 \alpha_n + \tan \alpha_n}$

$\alpha_2 \approx 3.42875$

$\alpha_3 \approx 3.42562$

$\alpha_4 \approx 3.42562$

Key in recurrence relation in GC to generate the approximations

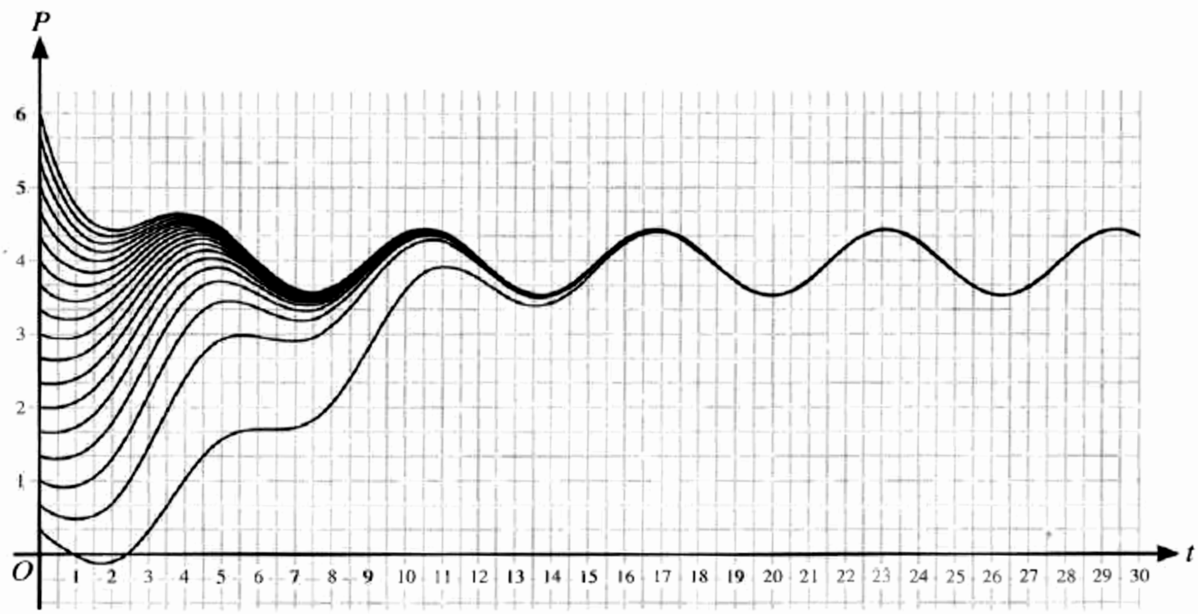
Thus $\alpha = 3.42562$ (correct to 5 d.p.)

**2** The population of deer in a US State is given, in tens of thousands, by $P$ and is modelled by the logistic-type equation

$$\frac{dP}{dt} = \tfrac{1}{2}P(1 - \tfrac{1}{4}P) - \tfrac{1}{2}\cos t,$$

with time $t$ measured in appropriate units. The long-term behaviour of $P$ is cyclical due to seasonal factors such as breeding and hunting.

The diagram below shows several solution curves for the given differential equation, each one arising from a different initial population value $P_0$.




**(i)** Interpret the solution curve shown in the diagram corresponding to an initial deer population of 3300. [1]

**(ii)** Find, to the nearest day, the units in which $t$ is measured, given that the period of the stable population cycle is 365 days. [2]

**(iii)** Write down an approximate equation for $P$ as a function of $t$ during the stable cycle by considering the time $t = 20$ units as a new starting-point. [2]

**[Solution]**

**(i)** When $P_0 = 0.33$, the deer population decreases to zero within 1 unit of time.

**(ii)** A stable population cycle occurs for a period of (26.25 – 20) or 6.25

The units in which $t$ is measured is $\dfrac{365}{6.25} = 58.4 \approx 58$ days

**(iii)** During the stable cycle, $P \approx 4 - \dfrac{1}{2}\cos t$ or $P \approx 4 - \dfrac{1}{2}\cos\left(\dfrac{2\pi}{6.25}t\right)$

**3** Throughout this question, a 'rational approximation' to a real number $r$ refers to any fraction $\frac{p}{q}$, where $p$ and $q$ are integers ($q \neq 0$), which is used in place of $r$ for calculational purposes.

Let $I = \displaystyle\int_0^2 \frac{1}{2+x^2}\,\mathrm{d}x$.

**(i)** Use Simpson's rule with five ordinates to find a rational approximation to $I$. [2]

**(ii)** Calculate, in trigonometric form, the exact value of $I$. [2]

**(iii)** Using $\frac{99}{70}$ as a rational approximation to $\sqrt{2}$, deduce a rational approximation to $\tan^{-1}(\sqrt{2})$. [1]

**(iv)** Use your calculator to show that the error in the rational approximation to $\tan^{-1}(\sqrt{2})$ obtained in part **(iii)** is approximately $\frac{1}{7400}$. [1]

**[Solution]**

**(i)** Let $\mathrm{f}(x) = \dfrac{1}{2+x^2}$

$$I \approx \frac{1}{3}\left(\frac{1}{2}\right)\left[\mathrm{f}(0) + 4\mathrm{f}\left(\frac{1}{2}\right) + 2\mathrm{f}(1) + 4\mathrm{f}\left(\frac{3}{2}\right) + \mathrm{f}(2)\right] = \frac{1}{6}\left[\frac{1}{2} + 4\left(\frac{4}{9}\right) + 2\left(\frac{1}{3}\right) + 4\left(\frac{4}{17}\right) + \frac{1}{6}\right]$$

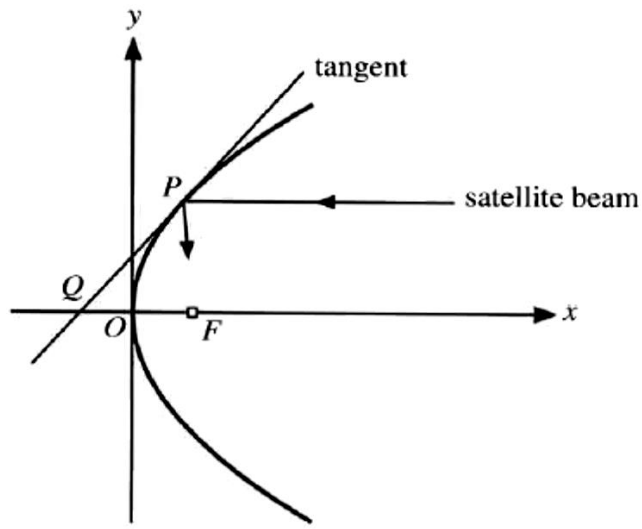
$$= \frac{310}{459}$$

**(ii)** $I = \displaystyle\int_0^2 \frac{1}{2+x^2}\,\mathrm{d}x = \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^2 = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}$

**(iii)** Using **(i)** and **(ii)** results, $\tan^{-1}\sqrt{2} = \sqrt{2}I \approx \dfrac{99}{70} \times \dfrac{310}{459} = \dfrac{341}{357}$

**(iv)** Error $\approx$ $\tan^{-1}\sqrt{2} - \dfrac{341}{357} = 0.00134545 \approx 0.0001351 = \dfrac{1}{7400}$

**4** The diagram shows a cross-sectional view of a parabolic dish, superimposed on a set of coordinate axes so that it has equation $y^2 = 4ax$, where $a$ is a positive constant.




A signal from a satellite, in the form of a beam, is travelling parallel to the $x$-axis and strikes the dish at the point $P\,(ap^2,\ 2ap)$. The tangent to the parabola at $P$ cuts the $x$-axis at $Q$. The focus of the parabola is $F$.

**(i)** State the coordinates of $F$ and determine the equation of the tangent in the form $y = mx + c$. Deduce the coordinates of $Q$. **[4]**

**(ii) (a)** Show that the angle between the beam and the tangent is equal to angle $QPF$. **[4]**

**(b)** Describe the significance of this result in relation to the reception of satellite TV. **[2]**

**[Solution]**

**(i)** Coordinates of $F$ is $(a, 0)$

$$y^2 = 4ax \;\Rightarrow\; 2y\frac{dy}{dx} = 4a \;\Rightarrow\; \frac{dy}{dx} = \frac{2a}{y}$$

At $P$, $\dfrac{dy}{dx} = \dfrac{2a}{2ap} = \dfrac{1}{p}$

Equation of tangent at $P$: $y - 2ap = \dfrac{1}{p}\left(x - ap^2\right)$

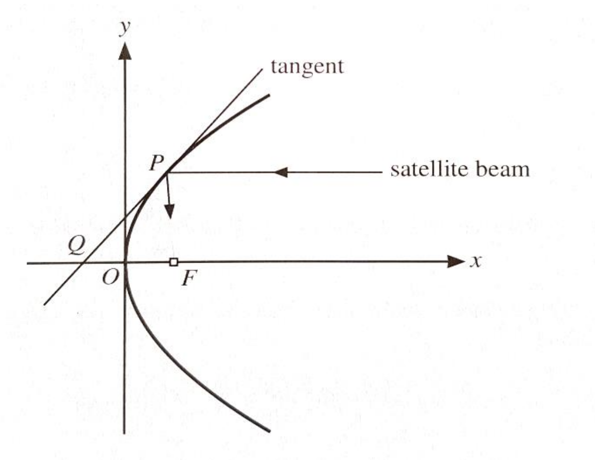
$$y = \frac{1}{p}x + ap$$

When $y = 0$, $x = -ap^2$

Thus the coordinates of $Q$ is $(-ap^2, 0)$

**(ii)** **(a)** Let the angle between the beam and the tangent be $\theta$.

Using the property of corresponding angle, $\angle PQF = \theta$




$QF = ap^2 + a = a(p^2 + 1)$

$PF = \sqrt{(ap^2 - a)^2 + (2ap)^2} = a\sqrt{\left(p^4 - 2p^2 + 1\right) + 4p^2} = a\sqrt{p^4 + 2p^2 + 1}$

$\quad = a\sqrt{\left(p^2 + 1\right)^2} = a(p^2 + 1)$

Since $QF = PF$, $\angle QPF = \angle PQF = \theta$

Thus angle between the beam and the tangent is equal to $\angle QPF$ .

**(b)** All satellite beams are **reflected** at the dish and are **directed to the focus** of the dish.

**5 Do not use a calculator in answering this question.**

A curve is given parametrically by $x = e^t \sin t$, $y = e^t \cos t$ for $0 \leqslant t \leqslant \frac{1}{2}\pi$. When the curve is rotated through $2\pi$ radians about the $x$-axis, a surface of revolution is formed with area $S$. Determine the exact value of $S$. [9]

**[Solution]**

$x = e^t \sin t,\ y = e^t \cos t, \quad 0 \le t \le \dfrac{\pi}{2}.$

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{\left[e^t\left(\cos t + \sin t\right)\right]^2 + \left[e^t\left(\cos t - \sin t\right)\right]^2}$$

$$= \sqrt{e^{2t}\left(\cos^2 t + 2\cos t \sin t + \sin^2 t\right) + e^{2t}\left(\cos^2 t - 2\cos t \sin t + \sin^2 t\right)}$$

$$= e^t\sqrt{2\left(\cos^2 t + \sin^2 t\right)} = \sqrt{2}\, e^t$$

Required surface area $= 2\pi \displaystyle\int_0^{\frac{\pi}{2}} y\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\ dt = 2\pi \int_0^{\frac{\pi}{2}} e^t \cos t\left(\sqrt{2}\, e^t\right)\ dt$

$$= 2\sqrt{2}\pi \int_0^{\frac{\pi}{2}} e^{2t}\left(\cos t\right)\ dt$$

$$\int_0^{\frac{\pi}{2}} e^{2t}\left(\cos t\right)\ dt = \left[\frac{1}{2}e^{2t}\cos t\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{2t}\left(\sin t\right)\ dt$$

$$= -\frac{1}{2} + \frac{1}{2}\left\{\left[\frac{1}{2}e^{2t}\sin t\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{2t}\left(\cos t\right)\ dt\right\}$$

$$= -\frac{1}{2} + \frac{1}{4}e^{\pi} - \frac{1}{4}\int_0^{\frac{\pi}{2}} e^{2t}\left(\cos t\right)\ dt$$

$$\frac{5}{4}\int_0^{\frac{\pi}{2}} e^{2t}\left(\cos t\right)\ dt = \frac{1}{4}(e^{\pi} - 2)$$

$$\int_0^{\frac{\pi}{2}} e^{2t}\left(\cos t\right)\ dt = \frac{1}{5}(e^{\pi} - 2)$$

$\therefore$ Area = $2\sqrt{2}\pi\left(\dfrac{1}{5}\right)\left(e^{\pi} - 2\right)$

**6** **(i)** Write down the five values of $\theta$, $-\frac{1}{2}\pi < \theta \leqslant \frac{1}{2}\pi$, for which $\cot 5\theta = 0$. [2]

**(ii)** Use de Moivre's theorem to prove that $\cot 5\theta = \dfrac{u^5 - 10u^3 + 5u}{5u^4 - 10u^2 + 1}$, where $u = \cot\theta$. [5]

**(iii)** Hence write down, in trigonometric form, the roots of the equation $z^2 - 10z + 5 = 0$. [2]

**(iv)** Deduce the exact value of $\tan^2(\frac{1}{10}\pi) + \tan^2(\frac{3}{10}\pi)$. [3]

**[Solution]**

**(i)** $\cot 5\theta = 0$

$$5\theta = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ldots$$

$$\theta = \pm\frac{\pi}{10},\ \pm\frac{3\pi}{10},\ \frac{\pi}{2} \quad \text{since} \quad -\frac{\pi}{2} < \theta \le \frac{\pi}{2}$$

**(ii)** $\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$ by de Moivre's Theorem

$$= \cos^5\theta + 5\cos^4\theta(\mathrm{i}\sin\theta) + 10\cos^3\theta(\mathrm{i}\sin\theta)^2 + 10\cos^2\theta(\mathrm{i}\sin\theta)^3 + 5\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$$

$$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$$

Comparing the imaginary parts: $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Comparing the imaginary parts: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$$\cot 5\theta = \frac{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta} \times \frac{\frac{1}{\sin^5\theta}}{\frac{1}{\sin^5\theta}}$$

$$= \frac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}$$

$$= \frac{u^5 - 10u^3 + 5u}{5u^4 - 10u^2 + 1} \quad \text{where } u = \cot\theta$$

**(iii)** Let $z = \cot^2\theta = u^2$

$z^2 - 10z + 5 = 0 \Rightarrow u^4 - 10u^2 + 5 = 0 \Rightarrow u^5 - 10u^3 + 5u = 0$

Using (ii), $\cot 5\theta = 0$

Using (i), $z = \cot^2\left(\pm\dfrac{\pi}{10}\right),\ \cot^2\left(\pm\dfrac{3\pi}{10}\right),\ \cot^2\left(\dfrac{\pi}{2}\right)$

$$z = \cot^2\left(\frac{\pi}{10}\right),\ \cot^2\left(\frac{3\pi}{10}\right) \quad \text{since } z \ne 0$$

**(iv)** $z^2 - 10z + 5 = 0 \;\Rightarrow\; z = \dfrac{10 \pm \sqrt{10^2 - 4(1)(5)}}{2} = 5 \pm 2\sqrt{5}$

Since $\cot^2\theta$ is a decreasing function over $\left(0,\ \dfrac{\pi}{2}\right)$,

$\cot^2\left(\dfrac{\pi}{10}\right) = 5 + 2\sqrt{5}$ and $\cot^2\left(\dfrac{3\pi}{10}\right) = 5 - 2\sqrt{5}$

Thus $\tan^2\left(\dfrac{\pi}{10}\right) + \tan^2\left(\dfrac{3\pi}{10}\right) = \dfrac{1}{5+2\sqrt{5}} + \dfrac{1}{5-2\sqrt{5}} = \dfrac{1}{5}\left(5 - 2\sqrt{5} + 5 + 2\sqrt{5}\right) = 2$

**7** An internet grocery-shopping company is set up and, in its first week of business, attracts 100 customers. The company's sales manager models the number of customers that the company has at the end of each following week assuming that a certain proportion $p$ $(0 < p < 1)$ of the previous week's customers will have been retained, and that a constant number, $k$, of new customers will have been attracted in each new week.

Let $C_n$ $(n \geqslant 1)$ denote the number of customers that the company has $n$ weeks after its shopping website was first set up.

**(i)** Write down an expression for $C_{n+1}$ in terms of $C_n$. [1]

**(ii)** Given that the company has 100 customers at the end of its first week, determine $C_n$ in terms of $n$, $p$ and $k$. [5]

The sales manager's business plan requires that the company has 200 customers by the end of its 12th week of trading.

**(iii)** Given that $k = 30$, show that the company needs to retain approximately 87% of its customers, week-by-week. [3]

**(iv)** Given instead that the company finds that it can only retain 75% of its customers, week-by-week, find the least number (assumed constant) of new customers it must attract each week in order to meet, or exceed, its target for week 12. [3]

Assume that the company does indeed retain 75% of its customers, week-by-week, and that it manages to add 100 new customers each week.

**(v)** Use your answer to part **(ii)** to comment on the long-term prospects for the company's success. [2]

**[Solution]**

**(i)** $C_{n+1} = pC_n + k, \quad n \geq 1$

**(ii)** $C_1 = 100$

**<u>Method 1:</u>**

$$
\begin{aligned}
C_2 &= pC_1 + k = 100p + k \\
C_3 &= pC_2 + k = p(100p + k) + k = 100p^2 + k(1+p) \\
C_4 &= pC_3 + k = p\left[100p^2 + k(1+p)\right] + k = 100p^3 + k(1+p+p^2)
\end{aligned}
$$

$\vdots$

$$
\begin{aligned}
C_n &= 100p^{n-1} + k(1 + p + p^2 + \dots + p^{n-2}) \\
&= 100p^{n-1} + k\left(\frac{1-p^{n-1}}{1-p}\right) \\
&= \left(100 - \frac{k}{1-p}\right)p^{n-1} + \frac{k}{1-p}
\end{aligned}
$$

**Method 2:**

Let $C_{n+1}-\alpha = p(C_n-\alpha)$

$pC_n + k - \alpha = pC_n - p\alpha \Rightarrow \alpha(1-p) = k \Rightarrow \alpha = \dfrac{k}{1-p}$

$\dfrac{C_{n+1}-\alpha}{C_n-\alpha} = p$

Let $v_n = C_n - \alpha$. Then $\dfrac{v_n}{v_{n-1}} = p$ is a constant and the sequence $\{v_n\}$ is a geometric progression with first term $v_1 = C_1 - \alpha = 100 - \dfrac{k}{1-p}$ and common ratio $p$.

The general term of the sequence is given by $v_n = \left(100 - \dfrac{k}{1-p}\right)p^{n-1}$ .

$\therefore C_n = \left(100 - \dfrac{k}{1-p}\right)p^{n-1} + \dfrac{k}{1-p}$

**(iii)** $k$ =30

For $c_{12} = 200$, $\left(100 - \dfrac{30}{1-p}\right)p^{11} + \dfrac{30}{1-p} = 200$

From the GC, the graph of $y = \left(100 - \dfrac{30}{1-p}\right)p^{11} + \dfrac{30}{1-p}$ and $y$ = 200 intersect at $p$ = 0.868 ≈ 87%

Hence the company needs to retain approximately 87% of customers, week by week.

**(iv)** If $p = 0.75$

For $c_{12} \geq 200$, $\left(100 - \dfrac{k}{0.25}\right)0.75^{11} + \dfrac{k}{0.25} \geq 200$

$$k[4-4(0.75^{11})] \geq 200 - 100(0.75^{11})$$

$$k \geq 51.1$$

Least $k$ = 52

**(iv)** If $p = 0.75$, $k = 100$

$C_n = \left(100 - \dfrac{100}{0.25}\right)0.75^{n-1} + \dfrac{100}{0.25} = 400 - 300(0.75^{n-1})$

As $n \to \infty$, $0.75^n \to 0$, $\therefore C_n \to 400$

The number of customers will **increase and approach 400 in the long run**. Hence, based on the model, the company will succeed in maintaining a stabilised number of customers at 400, but will not experience any further growth. (To comment on the long term prospect, we need to mention about the growth of the company in the long run based on this model.)

**8** **(i)** Write the $3 \times 3$ system

$$\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 = 0$$
$$\tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 = 1$$
$$\tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 = 1$$

as a matrix equation in the form $\mathbf{Mv} = \mathbf{c}$ and find the solution of this system of equations. [2]

In computer calculations, working is often undertaken in decimal form, with rounding taking place at each stage. Consider the same system of equations when each calculation is undertaken exactly but, when each calculated term is recorded, it is written (correctly rounded) to 3 decimal places and then used in later calculations as if it were exact. The standard row reduction process, leading to an upper echelon form, would thus begin as follows.

$$\textbf{Step 1:}\quad \left(\begin{array}{ccc|c} 0.500 & 0.333 & 0.250 & 0 \\ 0.333 & 0.250 & 0.200 & 1 \\ 0.250 & 0.200 & 0.167 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0.500 & 0.333 & 0.250 & 0 \\ 0 & a & 0.034 & 1 \\ 0 & 0.034 & 0.042 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2' = R_2 - 0.666R_1 \\ R_3' = R_3 - 0.5R_1 \end{array}$$

**(ii)** **(a)** Show that the recorded value of $a$ is 0.028. [1]

**(b)** **Step 2** involves performing further row operations in order to obtain a third row of the form shown below.

$$\rightarrow \left(\begin{array}{ccc|c} 0.500 & 0.333 & 0.250 & 0 \\ 0 & 0.028 & 0.034 & 1 \\ 0 & 0 & b & c \end{array}\right) \quad \begin{array}{l} \\ \\ R_3' = R_3 - kR_2 \end{array}$$

State, in a suitable rational form, the value of the constant $k$ and hence find the recorded values of $b$ and $c$ given by the computer. [3]

**(iii)** **(a)** Use the result of **Step 2** to deduce, in this order, the recorded values of $x_3$, then $x_2$, and then $x_1$, given by the computer. [2]

**(b)** Re-calculate the values of $x_1$, $x_2$ and $x_3$ obtained by a computer which works as above, but this time (correctly) rounding all decimal answers to **4** decimal places instead. [2]

A proposed measure for the accuracy of the recorded values is to calculate

$$\sum_{i=1}^{3} \delta_i^2,$$

where $\delta_i$ ($i = 1, 2, 3$) is the difference between the exact value of $x_i$ and the recorded value found by the computer.

**(iv)** Perform this calculation for the 3 decimal place solution and for the 4 decimal place solution and comment on the results. [3]

**[Solution]**

**(i)** $\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Using GC, $x_1 = -60$, $x_2 = 180$, $x_3 = -120$

**(ii)** **(a)** $a = 0.250 - 0.666(0.333) = 0.028$ (3 d.p.)

**(b)** $k = \dfrac{0.034}{0.028} = \dfrac{17}{14}$

$b = 0.000714 \approx 0.001$ (3 d.p.)

$c = -0.214$ (3 d.p.)

**(iii)** **(a)** $bx_3 = c \;\Rightarrow\; x_3 = \dfrac{-0.214}{0.001} = -214.000$ (3 d.p.)

$0.028x_2 + 0.034x_3 = 1 \;\Rightarrow\; x_2 = 295.571$ (3 d.p.)

$0.500x_1 + 0.333x_2 + 0.250x_3 = 0 \;\Rightarrow\; x_1 = -89.850$ (3 d.p.)

**(b)** $\left(\begin{array}{ccc|c} 0.5000 & 0.3333 & 0.2500 & 0 \\ 0.3333 & .2500 & 0.2000 & 1 \\ 0.2500 & 0.2000 & 0.1667 & 1 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 0.5000 & 0.3333 & 0.2500 & 0 \\ 0 & 0.0278 & 0.0334 & 1 \\ 0 & 0.0334 & 0.0417 & 1 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 0.6666R_1 \\ R_3' = R_3 - 0.5R_1 \end{array}$

$\rightarrow \left(\begin{array}{ccc|c} 0.5000 & 0.3333 & 0.2500 & 0 \\ 0 & 0.0278 & 0.0334 & 1 \\ 0 & 0 & 0.0016 & -0.2014 \end{array}\right) \quad R_3' = R_3 - \dfrac{334}{278}R_2$

$x_1 = -61.8516$, $x_2 = 187.2023$, $x_3 = -125.8750$

**(iv)** For 3 d.p. solution: $E_3 = \sum_{i=1}^{3} \delta_i^2 = 23100$ (3 d.p.)

For 4 d.p. solution: $E_4 = \sum_{i=1}^{3} \delta_i^2 = 89.8$ (3 d.p.)

It is expected that $E_4$ is very much smaller than $E_3$
as rounding off to more decimal places would increase accuracy of calculations.

**9** When modelling the spread of a new, non-fatal, disease within a human population, the S-I-R model splits the population into three distinct groups:

$S$ (for Susceptible) – the number of people who have not yet caught the disease;

$I$ (for Infected) – the number of people who have the disease;

$R$ (for Recovered) – the number of people who have recovered and can no longer catch the disease.

The unit of time used in the model is the average length of time for which a person is infected.

If $N$ is the total size of the population, assumed constant, then we work with the population proportions

$$u = \frac{S}{N}, \quad v = \frac{I}{N} \quad \text{and} \quad w = \frac{R}{N}$$

at any given time.

**(i)** Explain why it may be taken that $u \approx 1$, $v \approx 0$ and $w = 0$ at the outbreak of the disease. [1]

The standard model used in measuring the rate of spread of the disease assumes that $\frac{\mathrm{d}u}{\mathrm{d}t} = -kuv$, where $k$ is a positive constant.

**(ii) (a)** Describe the relevance of each bracketed term in the equation $\frac{\mathrm{d}u}{\mathrm{d}t} = (-k)(uv)$ in terms of the spread of the disease. [2]

**(b)** Justify the assertion that $\frac{\mathrm{d}w}{\mathrm{d}t} \approx v$ and deduce that $\frac{\mathrm{d}v}{\mathrm{d}t} \approx kuv - v$. [2]

**(c)** Describe what happens in each of the cases $ku < 1$ and $ku > 1$. [2]

**(iii)** Given that $v = v_0$ when $t = 0$, use the approximation $u = 1$ to find an expression for $v$ in terms of $t$ that is valid for small $t$. [3]

**(iv)** Consider the case when the population is 1 million, the value of $k$ is 1.2 and the number of infected people at the outbreak of the disease is 100. Use the answer to part **(iii)** to calculate the value of $t$ when 1% of the population is infected with the disease. Comment on the validity of this answer. [3]

**[Solution]**

**(i)** At the outbreak of the disease, we may assume that only a small number of people in a large population have caught the disease, i.e., $I$ is very small compared to $N$ and $R = 0$. Thus $v \approx 0$ and so $w = 0$. Thus $S \approx N$ and so $u \approx 1$.

**(ii)** $\frac{\mathrm{d}u}{\mathrm{d}t} = (-k)(uv), \;\; k > 0$

**(a)** Since $k > 0$ and $uv > 0$, thus the term $(-k)$ means that $\frac{\mathrm{d}u}{\mathrm{d}t} < 0$. Thus as the disease spread, the proportion of people who have not yet caught the disease decreases with time $t$.

The $(uv)$ term being the product of $\frac{S}{N}$ and $\frac{I}{N}$ is a measure of the interaction between the Susceptible and Infected. Thus $\frac{\mathrm{d}u}{\mathrm{d}t}=(-k)(uv)$ means that the proportion of people who have not yet caught the diseases decreases as the interaction between the Susceptible and Infected increases with time *t* until all the Susceptible are infected with the disease.

**(b)** Since the unit of time *t* used in the model is the average length of time for which a person is infected, thus the rate of increase of the population proportion who has recovered from the disease is the same as the proportion who are having the disease at any time *t*. Thus $\frac{\mathrm{d}w}{\mathrm{d}t}\approx v$.

$$S+I+R=N \ \Rightarrow\ u+v+w=1 \ \Rightarrow\ \frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\mathrm{d}w}{\mathrm{d}t}=0$$

Thus $-kuv+\frac{\mathrm{d}v}{\mathrm{d}t}+v\approx 0 \ \Rightarrow\ \frac{\mathrm{d}v}{\mathrm{d}t}\approx kuv-v$

**(c)** When $ku < 1$, $\frac{\mathrm{d}v}{\mathrm{d}t}\approx(ku)v-v<0$ which means that the proportion of people who have the disease decreases over time until no more people are infected. Thus the disease is not spreading in the long term.

When $ku > 1$, $\frac{\mathrm{d}v}{\mathrm{d}t}\approx(ku)v-v>0$ which means that the proportion of people who have the disease increases over time. Thus the disease continue to spread in the long term until all the Susceptible are infected with the disease.

**(iii)** Assuming $u\approx 1$, $\frac{\mathrm{d}v}{\mathrm{d}t}\approx kv-v=(k-1)v$

$$\int\frac{1}{v}\,\mathrm{d}v\approx(k-1)\int\,\mathrm{d}t$$

$\ln v\approx(k-1)t+c$ where $v>0$

$v\approx \mathrm{e}^{(k-1)t+c}$

$v\approx A\mathrm{e}^{(k-1)t}$, $A=\mathrm{e}^{c}$

When $t = 0$, $v = v_0$, $A = v_0$

Thus $v\approx v_0\mathrm{e}^{(k-1)t}$

**(iv)** Given $N = 1\times 10^6$, $k = 1.2$, $v_0 = \frac{100}{1000000}=\frac{1}{10000}$

$$0.01\approx\frac{1}{10000}\mathrm{e}^{0.2t} \ \Rightarrow\ t\approx 23.0$$

The value of $t \approx 23.0$ is not small, thus this answer is not valid since the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} \approx (k-1)v$ is valid for small $t$ only. This also means that the assumption $u = 1$ used in **(iii)** was no longer a valid assumption in this case.

**10** The sequence of integers $\{u_1, u_2, u_3, \ldots, u_n, \ldots\}$ is defined by the second-order recurrence relation

$$u_1 = 1,\ u_2 = m \text{ and } u_{n+2} = 2u_{n+1} - 4u_n \text{ for } n \geqslant 1.$$

**(i)** Prove by induction on $r$ that $u_{3r}$ is a multiple of $(2m - 4)$ for all positive integers $r$. [5]

**(ii) (a)** State in terms of $m$ the value of $u_0$ that is consistent with

$$u_1 = 1,\ u_2 = m \text{ and } u_{n+2} = 2u_{n+1} - 4u_n \text{ for } n \geqslant 0.$$ [1]

**(b)** Solve the given recurrence relation to find the general term, $u_n$. You should leave your answer in a form involving complex numbers. [5]

**(c)** Determine an expression for $u_{3r}$, $r \geqslant 1$, in a form that shows clearly that it is both real and an integer multiple of $(2m - 4)$. [3]

**[Solution]**

**(i)** Let $P_r$ be the statement $u_{3r}$ is a multiple of $(2m - 4)$ for all positive integers $r$.

When $r = 1$, $u_3 = 2u_2 - 4u_1 = 2m - 4$ which is a multiple of $(2m-4)$

Thus $P_1$ is true.

Assume $P_k$ is true for some $k \in \mathbb{Z}^+$ i.e. $u_{3k}$ is a multiple of $(2m - 4)$.

When $r = k + 1$,

$$\begin{aligned} u_{3k+3} &= 2u_{3k+2} - 4u_{3k+1} \\ &= 2\left(2u_{3k+1} - 4u_{3k}\right) - 4u_{3k+1} \\ &= -8u_{3k} \end{aligned}$$

Since $u_{3k}$ is a multiple of $(2m - 4)$, $u_{3(k+1)}$ is also a multiple of $(2m - 4)$.

Since $P_1$ is true and $P_k$ is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, $P_r$ is true for all positive integers $r$.

**(ii)** **(a)** $u_2 = 2u_1 - 4u_0 \Rightarrow u_0 = \dfrac{m-2}{-4} = \dfrac{1}{2} - \dfrac{1}{4}m$

**(b)** Recurrence relation: $u_{n+2} - 2u_{n+1} + 4u_n = 0$

Auxiliary equation: $m^2 - 2m + 4 = 0 \Rightarrow m = \dfrac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}\ \mathrm{i}$

$$= 2\left[\cos\left(\frac{\pi}{3}\right) \pm \mathrm{i} \sin\left(\frac{\pi}{3}\right)\right]$$

Hence, general solution is: $u_n = C(1+\sqrt{3}\ \mathrm{i})^n + D(1-\sqrt{3}\ \mathrm{i})^n$

Since, $u_0 = \frac{1}{2} - \frac{1}{4}m$ and $u_1 = 1$, $A = B = \frac{2-m}{8}$.

$$\therefore u_n = \frac{2-m}{8}\left[(1+\sqrt{3}\ \mathrm{i})^n + (1-\sqrt{3}\ \mathrm{i})^n\right]$$

**(c)** $u_n = 2^n\left[A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right)\right]$, where $A$, $B$ are complex constants.

$$u_0 = \frac{1}{2} - \frac{1}{4}m \Rightarrow A = \frac{1}{2} - \frac{1}{4}m$$

$$u_1 = 1 \Rightarrow 2\left[A\cos\left(\frac{\pi}{3}\right) + B\sin\left(\frac{\pi}{3}\right)\right] = 1$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{4}m\right) + \sqrt{3}B = 1$$

$$\Rightarrow B = \frac{1}{\sqrt{3}}\left(\frac{1}{2} + \frac{1}{4}m\right) = \frac{\sqrt{3}}{3}\left(\frac{1}{2} + \frac{1}{4}m\right)$$

Hence $u_n = 2^n\left[\left(\frac{1}{2} - \frac{1}{4}m\right)\cos\left(\frac{n\pi}{3}\right) + \frac{\sqrt{3}}{3}\left(\frac{1}{2} + \frac{1}{4}m\right)\sin\left(\frac{n\pi}{3}\right)\right], \quad n \geq 0$

$$u_{3r} = 2^{3r}\left[\left(\frac{1}{2} - \frac{1}{4}m\right)\cos(r\pi) + \frac{\sqrt{3}}{3}\left(\frac{1}{2} + \frac{1}{4}m\right)\sin(r\pi)\right], \quad n \geq 0$$

$$= 8^r\left[\left(\frac{2-m}{4}\right)(-1)^r + 0\right]$$

$$= (-8)^{r-1}(2m-4)$$

For $r \geq 1$, $8^{r-1}$ is an integer.

Hence $u_{3r}$ is an integer multiple of $(2m-4)$.