Additional Notes

Topic 10: Oscillations

Content:

- Simple harmonic motion
- Energy in simple harmonic motion
- Damped and forced oscillations, resonance

Learning Outcomes:

Candidates should be able to:

(a) describe simple examples of free oscillations.
(b) investigate the motion of an oscillator using experimental and graphical methods.
(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.
(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.
(e) recognise and use $x = x_o \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.
(f) recognise and use $v = v_o \cos \omega t$ and $v = \pm\omega\sqrt{\left(x_o^{\,2} - x^2\right)}$
(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.
(h) describe the interchange between kinetic energy and potential energy during simple harmonic motion.
(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping in cases such as a car suspension system.
(j) describe practical examples of forced oscillations and resonance.
(k) describe graphically how the amplitude of a forced oscillation changes with driving frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.
(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

Nature of Science

We can gain a deep understanding of periodic motion by analysing the mathematically simplest case of free oscillations, known as simple harmonic motion (SHM). Such sinusoidally varying motion is essentially a projection of uniform circular motion, and provides a mathematical basis upon which to describe more complicated oscillations.

Relating Science and Society

Oscillations play an important role in engineering and nature. The study and control of oscillation is needed to achieve important goals in engineering, e.g. to prevent the collapse of a building due to waves created by an earthquake.

A Introduction

A.1 Periodic Motion

- Any motion that repeats itself in equal time intervals.
  Examples: (a) mass oscillating at end of a spring
  (b) oscillation of a simple pendulum
  (c) planetary motion

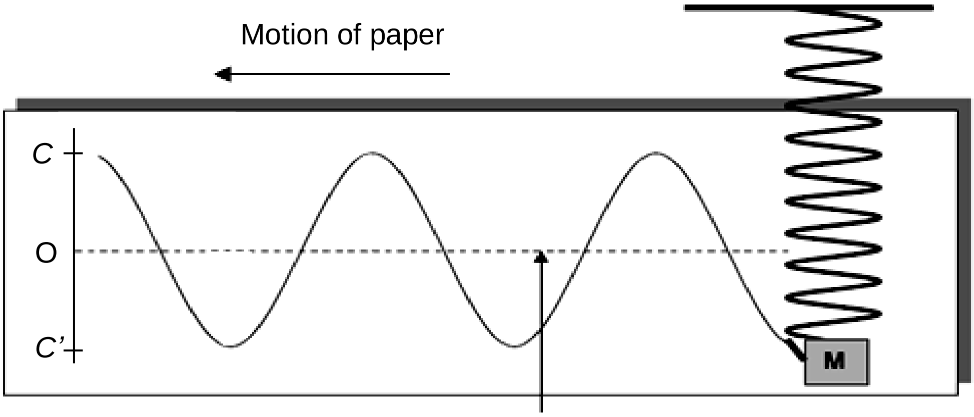
A.2 Oscillatory or Vibratory Motion

- A particle in oscillatory or vibratory motion moves back and forth over the same path.
  Examples: (a) mass oscillating at end of a spring
  (b) oscillation of a simple pendulum
  (c) vibrations of strings in musical instrument

Oscillation	
Free Oscillation	**Forced oscillation** (dealt with later in Section E)
the to-and-fro movement of a body about a fixed point in which no external driving force acts on it.	Occurs when the body is acted upon by an external periodic driving force, causing the body to oscillate at the frequency of the periodic force, rather than the natural frequency of the body.
Example: A child on a swing given an initial push and swings freely.	**Example:** A child on a swing kept in motion by pushing the child at equal intervals of time.

Additional Notes

- An example of oscillatory motion is the spring-mass system.



If we attach a pen to the mass (above diagram) and the mass is set into vertical oscillations, the pen will trace a fine line on the paper placed behind the mass. If the paper is pulled steadily (at a constant speed) at right angles to the oscillations, then the pen ought to trace out a sinusoidal wave on the paper.

The motion of the mass is about an equilibrium point O and limited between *C* and *C'*. The path marked by the pointer on the paper represents the displacement-time plot of the mass. Since the displacement-time plot of the mass having such a motion is *sinusoidal* (can be expressed as sine or cosine functions), this periodic, oscillatory or vibratory motion is commonly referred to as simple **harmonic motion**.

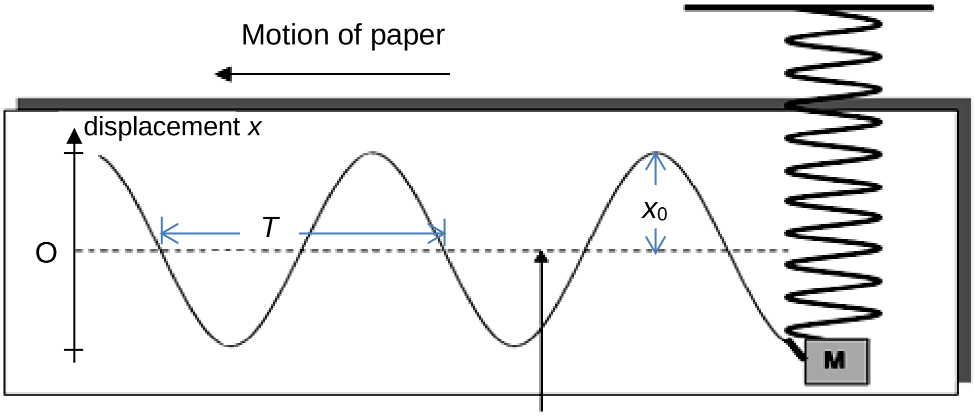
In a mechanical oscillation there is a continual interchange of potential and kinetic energies - during parts of the cycle, potential energy is changed to kinetic energy of the moving body, and during other parts of the cycle the kinetic energy is converted to potential energy.

The motion can continue indefinitely if there is no energy loss, although energy is lost in real situations. (This is dealt with later in Section D.)

- In this topic we will discuss the following quantities of an oscillating body:
  - Its **motion** (i.e. displacement, velocity, acceleration) – note that these quantities are related to each other, as well as related to time.
  - The **forces** acting on it, and in particular, the **net force** (i.e. restoring force) which results in the above motion.
  - The variation of its mechanical energy (i.e. **potential energy** and **kinetic energy**) as a result of its motion.

Restoring force is the net force that acts on the object to bring it back to its equilibrium position.

A.3 Key terms specific to oscillations



- **Equilibrium position O** is the position at which no net force acts on an oscillating body.
- **Displacement *x*** of the object is the linear distance of the oscillating body from its equilibrium position in a specified direction. It is a vector quantity and can be positive or negative.
- **Amplitude** of the oscillation, x_0 is the maximum displacement of an oscillating body from its equilibrium position. It is a scalar quantity, always positive.
- **Period *T*** of the motion is the time taken to complete one oscillation.
- **Frequency *f*** is the number of complete oscillations per unit time (Unit: **Hertz** (Hz)). For a system in free oscillation, this frequency is known as the natural frequency.

$$\boxed{f = \frac{1}{T}}$$

There is a difference in the meaning of frequency (f) and angular frequency (ω).

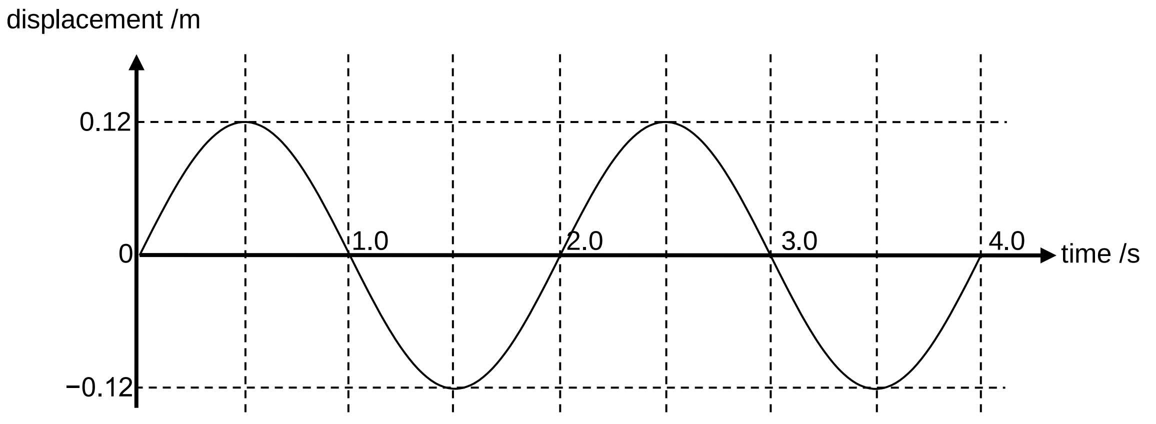
Although the symbols (ω) used are the same. Angular frequency is different from angular velocity.

- **Angular frequency** ω is a constant of a given oscillator and is related to its natural frequency f by $\omega = 2\pi f$.

Additional Notes

Check Your Understanding 1

The displacement of the mass described on page 8-3 varies with time as shown.



Determine for this oscillation, its
(a) amplitude, **(b)** period and **(c)** frequency.

Worked Example 1

A bungee jumper undergoes an oscillation between a height of 3.0 m and 8.0 m above sea level. It takes 2.0 s for him to move from the lowest point to the highest point.

Determine the **(a)** amplitude and **(b)** period of oscillation.

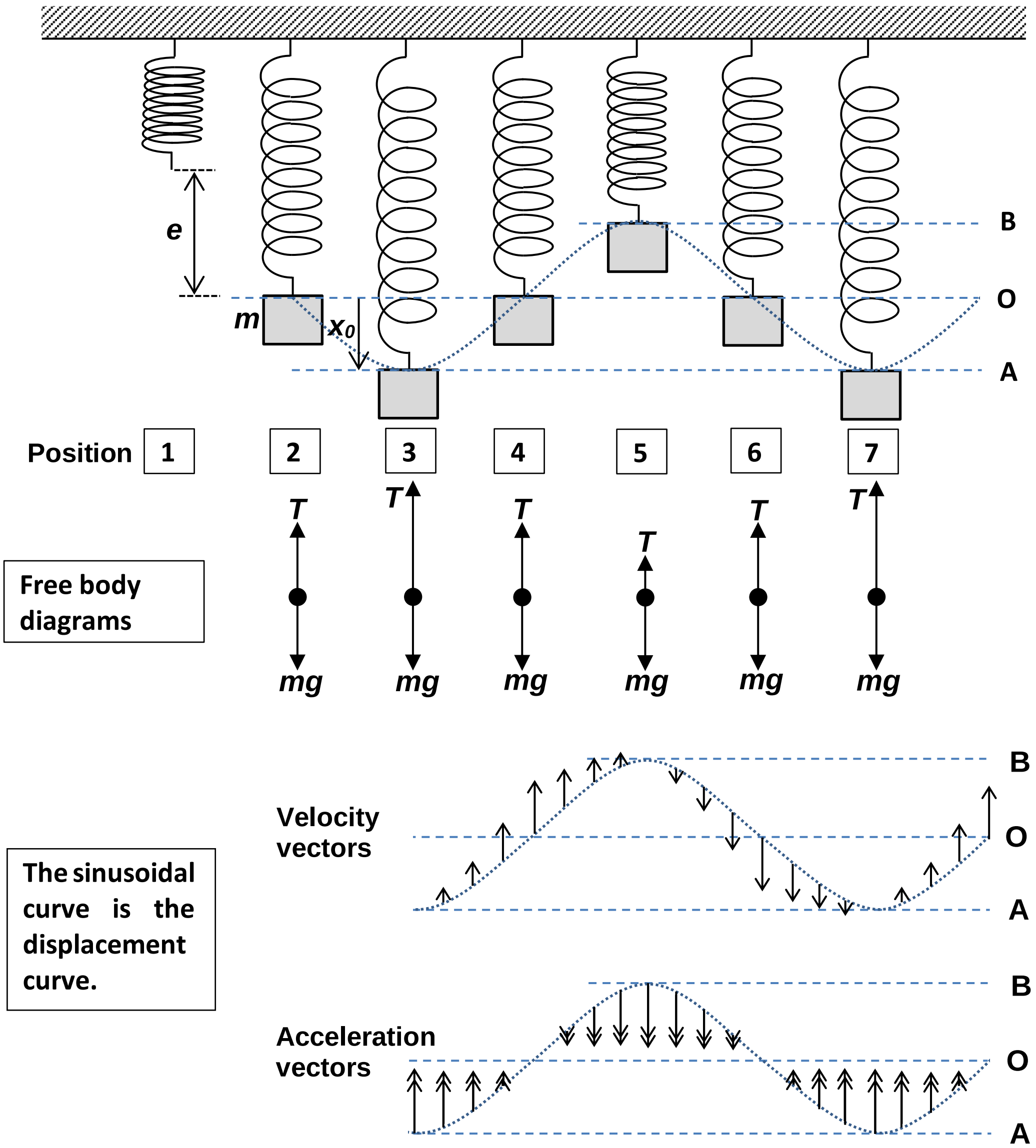
Solution:
(a) amplitude = ½ (distance between the highest and lowest points) = ½ (5.0)
= 2.5 m

(b) period = 2 x (time taken to travel between the highest and lowest points)
= 2 x 2.0 = 4.0 s

B Simple Harmonic Motion

B.1 Example of simple harmonic motion: Vertical Spring-Mass Oscillation

A mass *m* attached to the end of a long spring (with force constant *k*) can be made to oscillate in a straight line about O, between points A and B.



Compare the difference in the length of the arrows that represents velocity and acceleration.

Note the positions of zero velocity, maximum velocity, zero acceleration and maximum acceleration.

At Position 1

Spring is unstretched.

At Position 2

Mass *m* is attached to the spring and slowly lowered till **equilibrium position** where net force on it, $\boldsymbol{F}_{net}$ **= 0.** At this point, the spring has extended by *e*.

i.e. Upward tension = Downward force due to weight of suspended mass

$$\Rightarrow k\,e = m\,g$$

Additional Notes

At Position 3

Pull the mass m down a further distance x_0, further stretching the spring
→ Upward tension is now greater than downward force due to weight of mass
→ **Net upward force** on mass, $F_{net} = k(e + x_0) - mg$ → $F_{net} = kx_0$

Do not memorise F_{net}. Show derivation of F_{net} in your working.

From Position 3 to Position 4:

- When mass m is released at position 3, the restoring force in spring pulls m up and the mass accelerates towards its equilibrium position O with increasing (upward) velocity.
- When the mass is *between* positions 3 and 4 (i.e. displacement x), then the net upward force on mass will be $F_{net} = k(e + x) - mg$ → $F_{net} = kx$
- The **net force decreases** as mass m approaches O (since the spring is stretched less) and so **velocity increases at a decreasing rate**.

x_0 is the **amplitude** of the oscillation whereas x is the **displacement** of the mass from equilibrium position (pg. 4)

At Position 4

At equilibrium position O, the **net force on mass *m* is zero** but it is now moving with **maximum upward velocity**.

From Position 4 to Position 5:

- Because of momentum, mass m overshoots the equilibrium position and continues to move upward. Upward tension decreases while downward force due to weight of mass remains constant, giving a **net downward force**. Thus mass m slows down.
- The **net downward force increases** as mass m approaches B and so **velocity decreases at an increasing rate**.

Position 5

Mass m comes to a **momentary rest** at B (amplitude position)
→ **Net downward force** on mass, $F_{net} = mg - k(e - x_0)$ → $F_{net} = kx_0$

From Position 5 to Position 6:

- The **net downward force** pushes m down and the mass accelerates (downward) towards its equilibrium position O with increasing velocity (downward).
- The net downward force decreases as mass m approaches O (since the spring becomes more stretched) and so velocity (downward) increases at a decreasing rate.

Position 6

At equilibrium position O, the net force on mass m is zero but it is now moving with maximum downward velocity.

From Position 6 to Position 7:

- Because of momentum, mass m continues to move downward past the equilibrium position. Tension in the spring increases while weight of mass remains constant, giving an upward net force. Thus mass m slows down.
- The **net upward force increases** as mass m approaches A and so **velocity decreases at an increasing rate**.

Position 7

Mass m comes to a momentary rest at A (amplitude position)
→ **Net upward force** on mass, $F_{net} = k(e + x_0) - mg$ → $F_{net} = kx_0$
and the motion repeats itself.

Additional Notes on *e* and x_0

In this example, the amplitude of the oscillation (i.e. x_0) is less than e. Hence, at position 5, the spring is still stretched and there will be tension. If the amplitude of oscillation is equal to e instead, the tension at the highest position will be zero.

Consider **Displacement**, **Velocity** and **Acceleration**

When *m* is *below* O	***m* is slowing down from O to A**	***m* is momentarily stationary at A**	***m* is speeding up from A to O**
Direction of Displacement	Downward (always measured from equilibrium position)		
Direction of Velocity	Downward	Zero	Upward
Direction of Acceleration	Upward (**always directed towards equilibrium position**)		

When *m* is *above* O	***m* is slowing down from O to B**	***m* is momentarily stationary at B**	***m* is speeding up from B to O**
Direction of Displacement	Upward (always measured from equilibrium position)		
Direction of Velocity	Upward	Zero	Downward
Direction of Acceleration	Downward (**always directed towards equilibrium position**)		

Consider the **Variation of Acceleration with displacement**

- Magnitude of net force ($F_{net} = k\,x$) increases proportionally with displacement but always acts towards the equilibrium position (i.e. O).

- **Acceleration** must therefore behave likewise, i.e. **increases proportionally with displacement** but being directed to equilibrium position O whatever the displacement. (i.e. if *m* is below O, the displacement is downward and the acceleration upward, but if *m* is above O, the displacement is upward and the acceleration is downward.) **Acceleration and displacement always have opposite signs in an oscillation.**

- Such an oscillation in which the acceleration of the body is proportional to its displacement but in opposite direction is called a simple harmonic motion. (s.h.m.)

Acceleration is ***not constant*** in SHM thus we **cannot** use kinematics equations in SHM.

B.2 Definition of Simple Harmonic Motion

The **defining equation for s.h.m.** is

$$\boxed{a = -\,\omega^2 x}$$

where a = acceleration of the object undergoing SHM
ω = angular frequency of oscillation (this will be constant for each system)
x = displacement of the object (measured from the equilibrium position)

"Simple" in SHM refers to when the sinusoidal motion has a single frequency and a constant amplitude.

- When a body undergoes simple harmonic motion (s.h.m.), its acceleration is:

    1. proportional to its displacement from the equilibrium position AND

    2. always directed towards that point (or in opposite direction to its displacement).

Additional Notes

Example 1

The diagrams below show the spring-mass system (with mass m and spring constant k) at its equilibrium and amplitude positions respectively. The horizontal surface is frictionless and the spring obeys Hooke's Law. Using Newton's Second Law, show that the horizontal spring-mass system undergoes s.h.m. with $\omega = \sqrt{\frac{k}{m}}$.

Fig. a

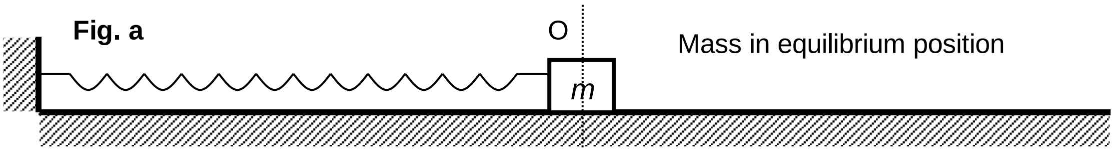


Fig. b

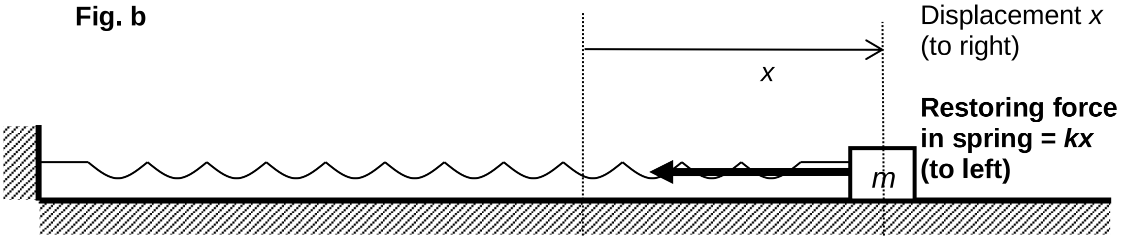
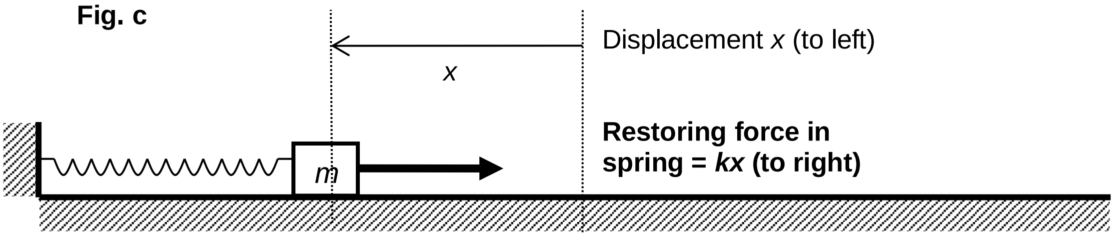


Fig. c



Recall:
In Forces, we learnt that
$F_{external} = kx$
$F_{internal} = -kx$

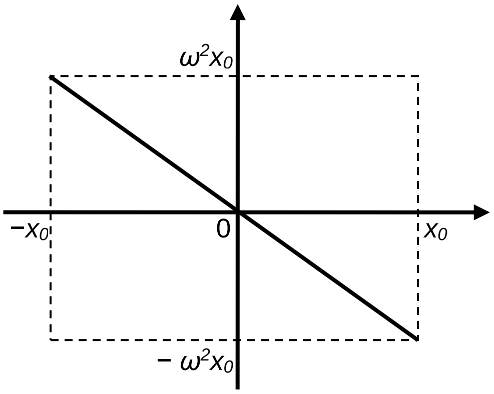
The negative sign indicates that the internal / spring force $F_{internal}$ always acts in the direction opposite to the direction in which the system is displaced.

In this horizontal spring-mass system, $F_{internal}$ is the restoring / net force acting on the system when the mass is released.

B.3 Variation of Acceleration with Displacement

- Straight line passing through the origin (gradient = $-\omega^2$): Acceleration is proportional to displacement.
- Negative gradient: Acceleration and displacement in opposite directions.
- The maximum value of a occurs at $\pm x_0$, where the magnitude is $\omega^2 x_0$.

Maximum acceleration is usually quoted as a positive value.



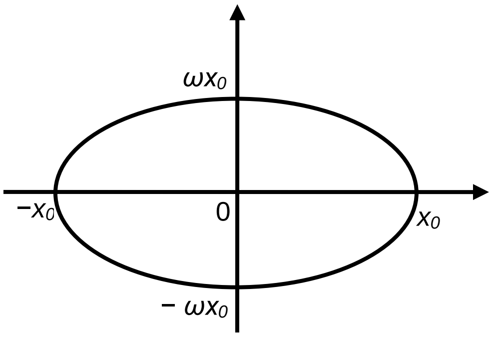
B.4 Variation of Velocity with Displacement

- Integrating $a = -\omega^2 x$ gives an equation that relates the velocity v of an object in simple harmonic motion to its displacement x :

$$\boxed{v = \pm\omega\sqrt{\left(x_0^2 - x^2\right)}}$$

- Graph is an ellipse
- The maximum value of v occurs at $x = 0$ (i.e equilibrium position), where it has a value **$v_0 = \omega x_0$**

When drawing an ellipse, do ensure that it is symmetrical on both sides.



Worked Example 2

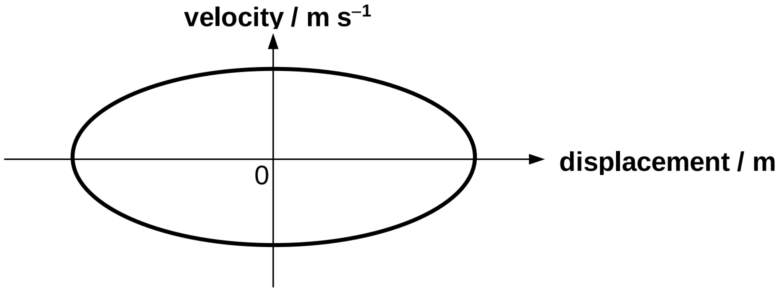
A particle performs s.h.m. of amplitude 2.0×10^{-3} m and period 0.10 s. Determine its maximum speed.

$v_o = \omega x_o$
$= (2\pi / T)\, x_o$
$= (2\pi / 0.10)\,(2.0 \times 10^{-3})$
$= 0.13$ m s^{-1}

Additional Notes

Example 2

The variation with displacement *x* of velocity *v* of the body in Check Your Understanding 1 (CYU 1) is shown in the graph below. (The amplitude of oscillation is 0.12 m and its period is 2.0 s)



(a) Label on the graph the amplitude of the oscillation.

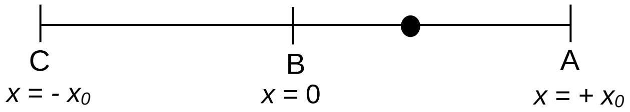
(b) Using the values from CYU 1, determine the angular frequency of the oscillation. [3.14 rad s^{-1}]

(c) Hence determine the maximum speed of the body, and label the value on the graph. [0.377 m s^{-1}]

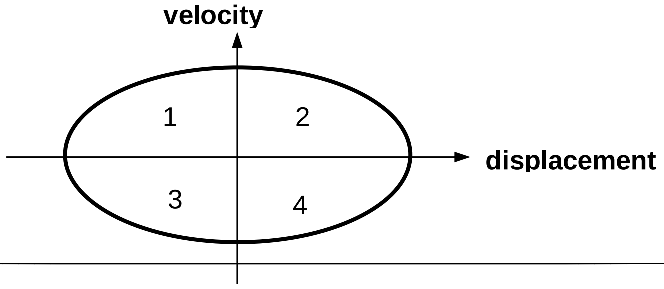
(d) Write down the equation which describes the graph, and use the equation to calculate the speed of the body at x = 0.08 m. [0.281 m s^{-1}]

Check Your Understanding 2

A particle is oscillating in simple harmonic motion between points C and A, the equilibrium position being at point B.



Which quadrant of the velocity-displacement graph represents the motion of the particle travelling from point A to point B?



B.5 Variation of Displacement with Time

- One solution for the defining s.h.m. equation is

$$\boxed{x = x_0 \sin \omega t}$$

  where x_o is the amplitude of the oscillation, t the time and ω the angular frequency.

- If the oscillating particle is at its ***equilibrium*** *position* when timing starts (initial condition, $t = 0$), then $\boldsymbol{x = x_o \sin \omega t}$.

- If the oscillating particle is at its ***amplitude*** *position* when timing starts (initial condition, $t = 0$), then $\boldsymbol{x = x_o \cos \omega t}$.

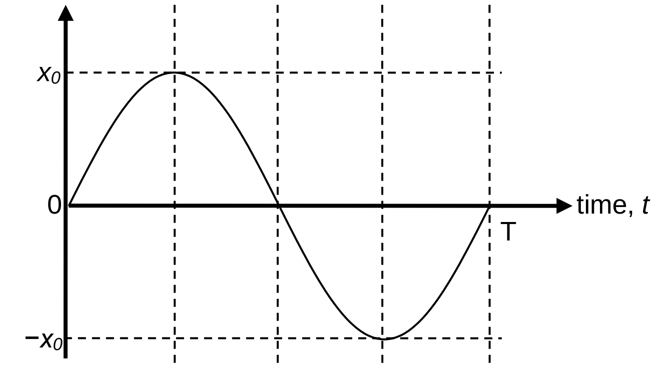
Additional Notes

- The equations for the variation of velocity and acceleration with time can be obtained by differentiating the equation relating displacement to time. The following table summarises this for the case of $x = x_0 \sin\omega t$. (i.e. when the particle is at its ***equilibrium*** *position* when $t = 0$.)

The form of the equation varies according to the **initial condition**.

Displacement-time graph

$$x = x_0 \sin\omega t$$



If x = x_0 cos ωt. instead, then the velocity-time and acceleration-time graphs will change accordingly.

Velocity-time graph

v = gradient of x-t graph

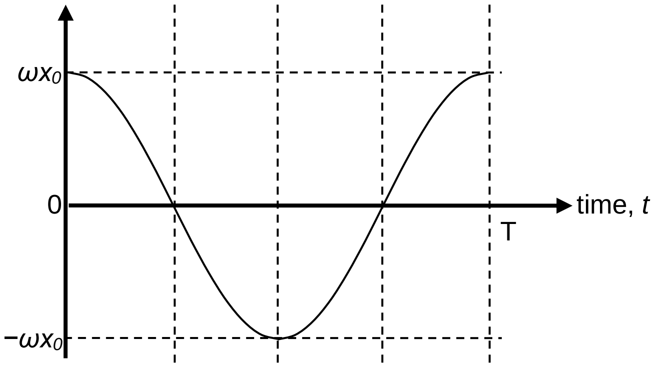
$$v = \frac{dx}{dt}$$

$$= \frac{d(x_0 \sin\omega t)}{dt}$$

$$= \omega x_0 \cos\omega t$$

$$v = v_0 \cos\omega t$$

$$(v_0 = \omega x_0)$$



Acceleration-time graph

$$a = -\omega^2 x$$

$$= -\omega^2 x_0 \sin\omega t$$

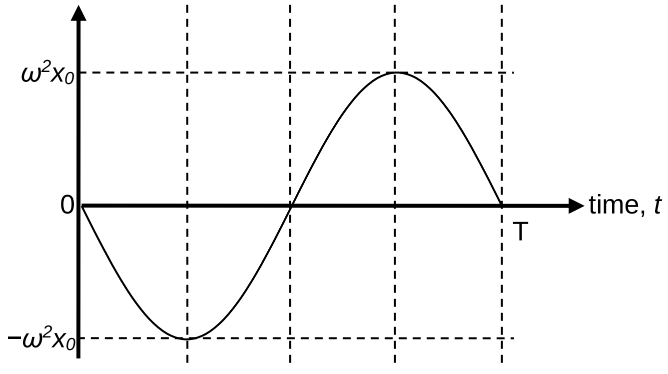
or

a = gradient of v-t graph

$$a = \frac{dv}{dt}$$

$$= \frac{d(\omega x_0 \cos\omega t)}{dt}$$

$$= -\omega^2 x_0 \sin\omega t$$



Worked Example 3

The displacement of a mass vibrating with s.h.m. is given by $x = 4.0 \sin (20\pi\, t)$, where x is in cm and t is in s. Determine the **(a)** amplitude x_o, **(b)** period T and **(c)** velocity v at t = 0 s.

(a) $x_o = 4.0$ cm
(b) since $\omega = 2\pi / T = 20\pi \Rightarrow T = 0.10$ s
(c) $v = 80\pi \cos (20\pi\, t) \Rightarrow$ At $t = 0$, $v = 80\pi = 251$ cm s^{-1}

Example 3

By considering the graph in CYU 1, form an equation to relate the displacement x (in metres) to the time t.

Using the equation, calculate

(a) the displacement of the body 0.25 s after the oscillation has begun. [0.085 m]

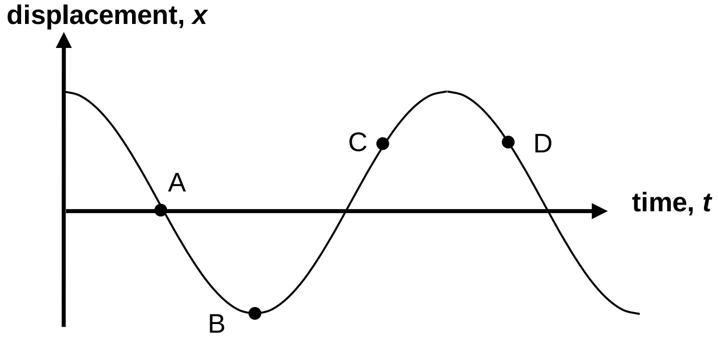
(b) the time at which the body initially passes the 0.080 m mark. [0.23 s]

(c) for one oscillation, the duration which the body's displacement is greater than the 0.080 m mark. [0.54 s]

Note: From CYU 1, the amplitude of oscillation is 0.12 m and its period is 2.0 s.

Check Your Understanding 3

The diagram shows a displacement-time graph of a body performing simple harmonic motion. At which instant is the body travelling and accelerating in the same direction?



Additional Notes

C Energy in Simple Harmonic Motion

In an oscillation there is a constant interchange of energy between the kinetic and potential energies. If there are no resistive forces, its total energy is constant.

C.1 Variation of Kinetic Energy and Potential Energy with Displacement

- At displacement x from the equilibrium position, the total energy is the sum of the kinetic energy and the potential energy.

$$\boxed{\text{Total Energy} = \text{KE} + \text{PE}}$$

Kinetic Energy

- The velocity of a particle of mass m at a distance x from its centre of oscillation O is

$$v = \pm\omega\sqrt{\left(x_0^2 - x^2\right)}$$

- Kinetic energy at displacement x

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left(x_0^{\,2} - x^2\right)$$

- The **maximum kinetic energy occurs at $x = 0$** where the speed of the particle is the greatest ($v = v_0 = \omega x_0$).

$$\text{KE}_{max} = \frac{1}{2}mv_0^{\,2} = \frac{1}{2}m\omega^2 x_0^{\,2}$$

- If the potential energy is taken to be zero at equilibrium position (e.g. horizontal spring-mass system), then the maximum kinetic energy must also be the total energy.

Total Energy
$$= \text{KE}_{max} = \frac{1}{2}m\omega^2 x_0^{\,2}$$

Do not memorise the equations. Understand how the equations are obtained from basic equations.

Most importantly, relate these equations to the physical situation – take note where maximum and minimum KE and PE occur.

Total energy of system is different from the total energy of oscillation (To be addressed in tutorial).

Potential Energy (applicable for PE =0 at equilibrium position)

- We can also find an expression for the variation of potential energy with displacement using the equations above.

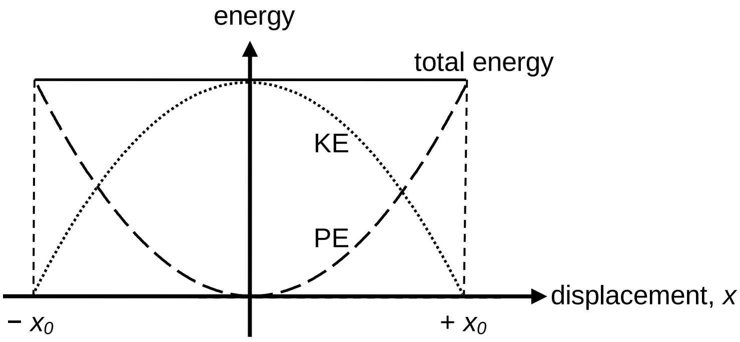
$$\begin{aligned}\text{PE} &= \text{Total Energy} - \text{KE}\\ &= \frac{1}{2}m\omega^2 x_0^{\,2} - \frac{1}{2}m\omega^2\left(x_0^{\,2} - x^2\right)\\ &= \frac{1}{2}m\omega^2 x^2\end{aligned}$$

- **Maximum potential energy is at $x = x_0$** (where the body is at rest momentarily, and kinetic energy is zero)

Total Energy
$$= \text{PE}_{max} = \frac{1}{2}m\omega^2 x_0^{\,2}$$

For vertical spring-mass system (pg 6), a) potential energy refers to the sum of GPE and EPE; b) PE may NOT be zero at equilibrium position (due to EPE). Hence, equations for KE_{max} and PE_{max} may be different from those listed in the table.

C.2 Graphs of Energy against displacement for a system in s.h.m:



In summary, for cases when <u>PE = 0 at equilibrium position,</u>

	Equations	
Kinetic Energy	$KE = \frac{1}{2}m\omega^2\left(x_0^{\ 2} - x^2\right)$	$KE_{max} = \frac{1}{2}m\omega^2 x_0^{\ 2}$
Potential Energy	$PE = \frac{1}{2}m\omega^2 x^2$	$PE_{max} = \frac{1}{2}m\omega^2 x_0^{\ 2}$

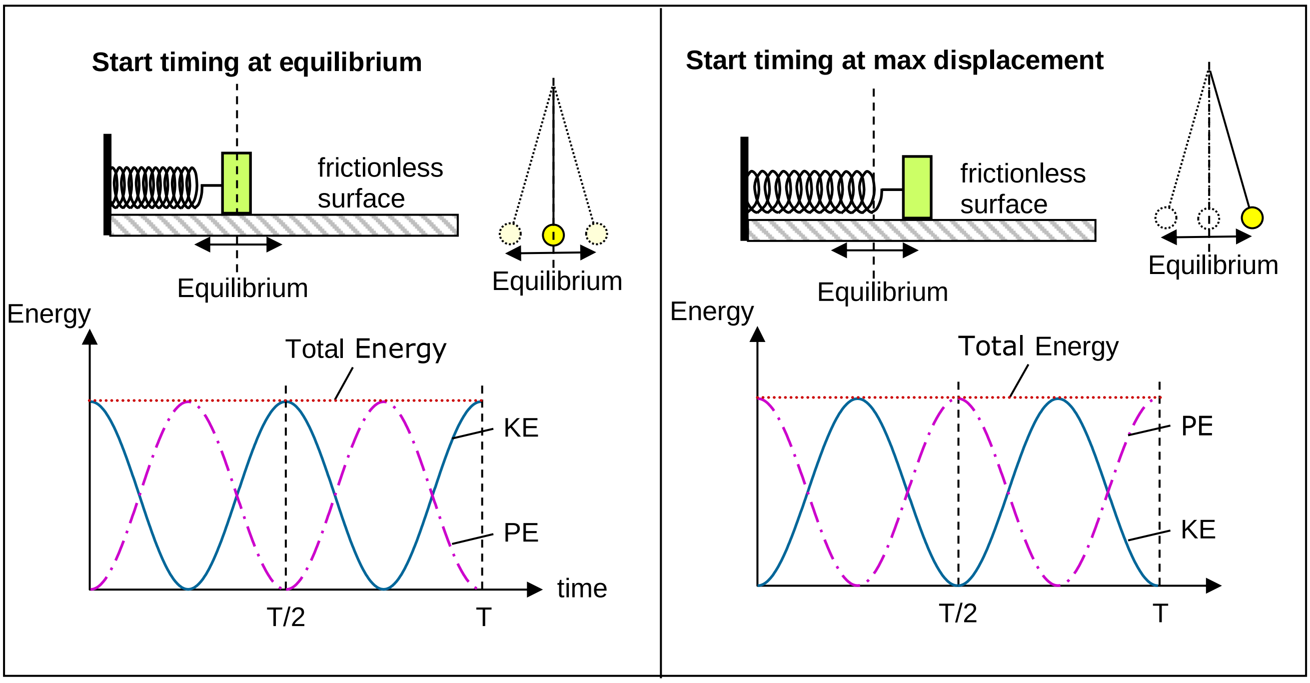
Additional Notes

C.3 Variation of Kinetic Energy and Potential Energy with Time

To appreciate how the kinetic energy and potential energy of a system vary with time, it is best to picture an object undergoing simple harmonic motion. Consider either a horizontal spring-mass system or a pendulum.

- At the equilibrium position, the kinetic energy of the object is at a maximum while the potential energy is zero.
- At the amplitude positions, the kinetic energy of the object is zero while the potential energy of the object is at a maximum.
- Throughout the motion, as long as there is no dissipative force present, the total energy remains constant. (Total Energy = KE_{max} = PE_{max})

As such, depending on where the initial position of the object is at time t = 0, the energy variations with time will change accordingly as shown below.



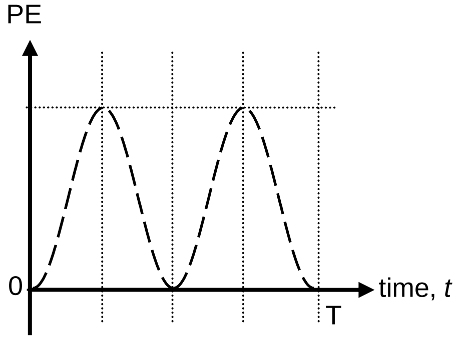
Additional Notes

A mathematical treatment of the case where the system is <u>initially at equilibrium position</u> is shown below.

Do not "blindly" memorise these graphs.

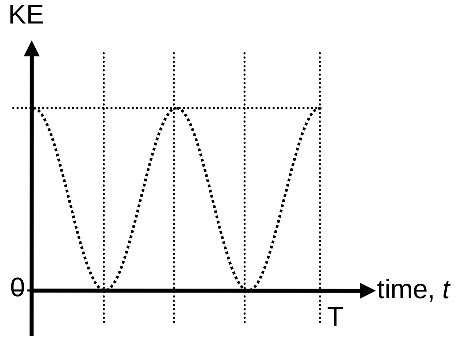
Take note of the shapes and understand how the shapes are obtained from information given in the question (especially information regarding the initial conditions). **Applicable for horizontal springs only**.

For vertical spring, need to include its GPE.



<u>Potential Energy</u>

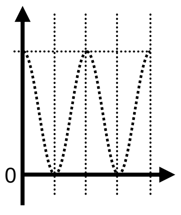
$$E_P = \frac{1}{2}m\omega^2 x^2$$
$$= \frac{1}{2}m\omega^2 {x_0}^2 \sin^2 \omega t$$



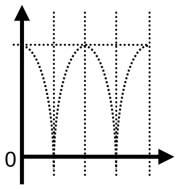
<u>Kinetic Energy</u>

$$E_K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m\omega^2 {x_0}^2 \cos^2 \omega t$$

Note that graph (i) and graph (ii) are different.

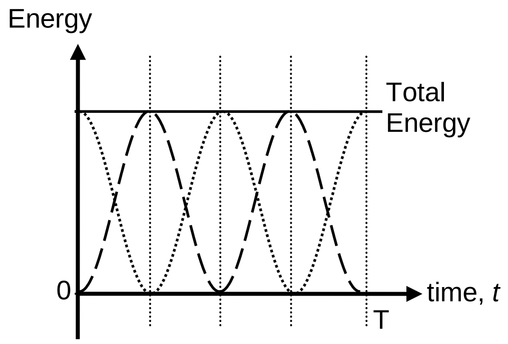


(i)



(ii)

The correct energy-time graph is graph (i).

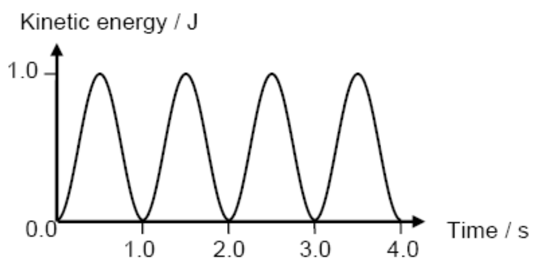


<u>Total Energy</u>

$$E_T = E_P + E_K = E_{P\max} = E_{K\max} = \frac{1}{2}m\omega^2 {x_0}^2$$

<u>Check Your Understanding 4</u>

The time variation of the kinetic energy of a particle undergoing simple harmonic motion with amplitude 3.0 cm is shown in the figure below.



What is the maximum velocity of the particle?

A	0.59 m s^{-1}	**B**	0.30 m s^{-1}
C	0.094 m s^{-1}	**D**	0.19 m s^{-1}

Additional Notes

Example 4

A mass m of 2.00 kg, connected to a light spring with spring constant $k = 2\pi^2$ N m^{-1}, oscillates on a frictionless horizontal surface. Its angular frequency ω is $\sqrt{\frac{k}{m}}$. The amplitude of the motion is 0.12 m.

(a) By considering the maximum speed of the mass, calculate the total energy of the system

(b) Calculate the kinetic energy of the system when the displacement is 0.080 m.

(c) Calculate the potential energy of the system when the displacement is 0.080 m.

[0.142 J, 0.0790 J, 0.0631 J]

D Damped Oscillations

D.1 Damped Oscillations

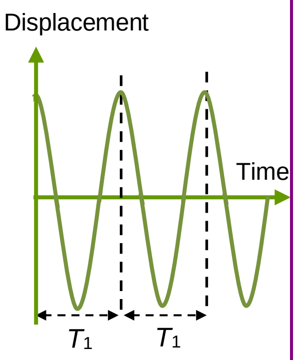
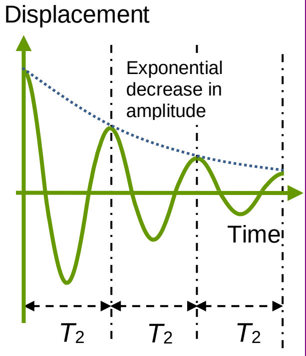
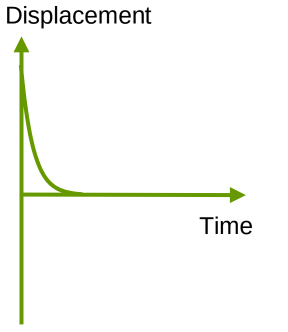
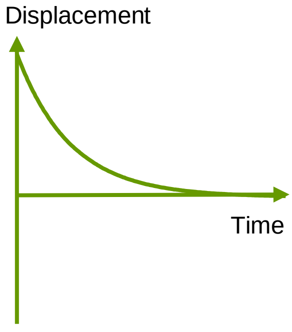
- In an ideal oscillating system, there are no dissipative forces acting on it; the total mechanical energy is constant. The system, once set into motion, continues oscillating forever, with no decrease in amplitude.

- In real life, the amplitude of oscillations (e.g. a swinging pendulum) gradually decreases to zero due to resistive / dissipative forces (e.g. air resistance, viscous forces) that reduce the total energy of the oscillations. The motion is not a simple harmonic motion and is said to be damped.

- **Damped oscillations** are oscillations in which the amplitude diminishes with time as a result of resistive forces that reduce the total energy of the oscillations.

D.2 Degrees of damping

- **Damping** is the dissipation of total energy of an oscillating system with time due to resistive forces. The behaviour of a mechanical system depends on the extent of damping.

Undamped (free) oscillations	**Light Damping**	**Critical Damping**	**Heavy Damping**
Without any dissipative force acting on the oscillating system, the system continues to oscillate with constant amplitude.	• definite oscillations but the **amplitude** of oscillation (as shown by dotted line) decreases with time **exponentially**. • frequency of vibration is usually slightly affected • e.g. simple pendulum in air	• As degree of damping increases from light to heavy damping, critical damping is the degree of damping when there is no real oscillation • the time taken for the displacement to become effectively zero is a minimum • e.g. pointer in ammeter & car suspension system	• no real oscillation • the system returns very slowly to the equilibrium position • e.g. simple pendulum in a very viscous liquid
			
Note: T_1 is constant, and T_2 is constant . However, $T_1 \leq T_2$.			

D.3 The Importance of Critical Damping:

- **Critical damping** occurs when the displacement of the body is reduced to zero in the minimum time possible without any oscillations occurring.

- The motion of many devices is critically damped on purpose. Too little damping results in a large number of oscillations while too much damping leads to there being too long a time when the system cannot respond to further disturbances.

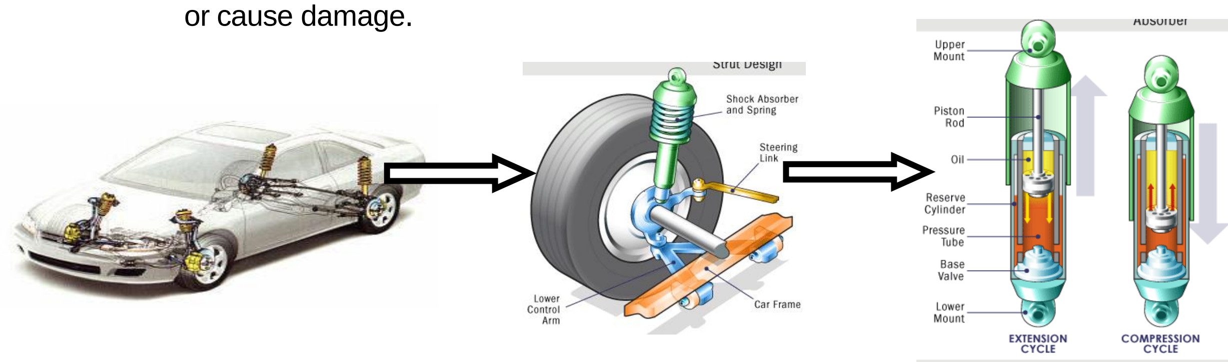
- **<u>Examples:</u>**

(a) <u>Damping in instruments</u>

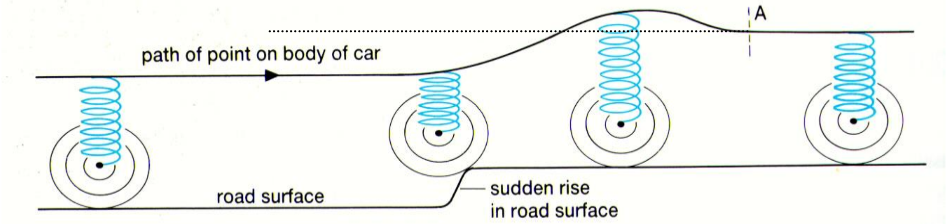
Instruments such as balances and electrical meters are critically damped (i.e. dead-beat) so that the pointer moves quickly to the correct position without oscillating.

(b) <u>Damping in the suspension system of a car</u>

- The suspension system of a car should ensure a comfortable ride for passengers when the car moves on a bumpy road.
- One method to damp oscillations in the springs of a suspension system of a car is to use oil dampers as shock absorbers, as shown in the figure below.
- The shock absorbers on a car critically damp the suspension of the vehicle. The motion of the springs are slowed down by the flow of oil through valves in piston dampers and so resist the setting up of vibrations which could make control difficult or cause damage.



- If the system is over-damped, it would be too slow in returning to equilibrium before the next bump. It is therefore not able to cushion the next shock effectively.
- A good suspension system is one that is slightly under critical damping so that the car will respond quickly to consecutive bumps.

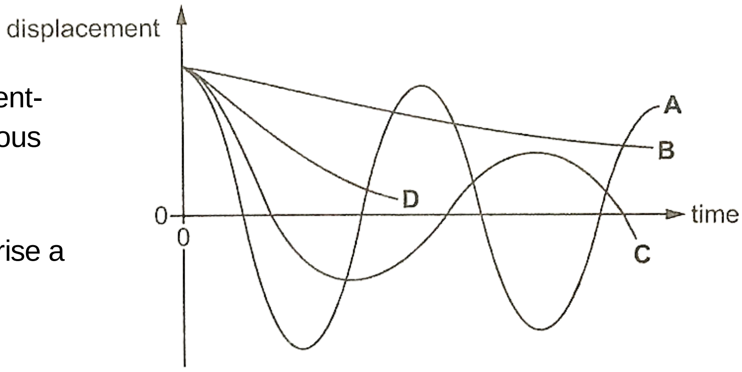


- When the dampers wear out, the system will become more under-damped. After going over a bump, the car would continue to oscillate for some time.

<u>Check Your Understanding 5</u>

The diagram shows four displacement-time curves for oscillations with various degrees of damping.

Which curve would ideally characterise a good car suspension system?



E Forced Oscillations and Resonance

E.1 Forced Oscillations, Natural Frequency, Resonance

- Since every oscillating system is damped to a certain extent, if *no external energy* is supplied, the system eventually comes to rest.
- For the oscillating system to maintain constant amplitude, it is necessary to apply an external oscillating force. The external applied force is the *driving force*, the oscillating system is the *driven oscillator,* and the resulting oscillations are *forced oscillations*.
- Imagine pushing a child on a swing. The pushes are an example of a **periodic force** – a force applied at regular intervals.
- **Forced oscillation** is produced when a body is acted upon by an external **periodic** driving force, causing the body to oscillate at the driving frequency, rather than at the **natural frequency** of the body (i.e. frequency of oscillation for a body that oscillates freely without any driving force)
- When a periodic force is applied to an oscillating system the response depends on the driving frequency.
- If the frequencies are not equal, the periodic force will build the amplitude up for a while, slow down the motion for a while (decreasing the amplitude), then build up the amplitude again, etc.
- If the driving frequency is equal to the natural frequency of the system, then each push can build up the amplitude further.
- The periodic force thus provides a means of supplying energy to the system, but energy can be taken in over many cycles only if the frequency of the periodic force matches (is equal to) the natural frequency. When not matched, energy is fed in for a few cycles, and then taken out again for a few cycles, etc.
- When the driving frequency equals the natural frequency, energy is fed in continuously and the amplitude builds up more and more. The system is said to be in **resonance**.
- **Resonance is the phenomenon that occurs when the driving frequency of a body is equal to its natural frequency, giving a maximum amplitude of oscillation.**

At resonance, it is incorrect to state that displacement is maximum. Rather, it is the amplitude that is maximum.

- At **resonance**, there is **maximum transfer of energy** from the driving system into the oscillating system.
- Thereafter, as the driving frequency is increased further, the amplitude of oscillation of the driven system decreases.

Additional Notes

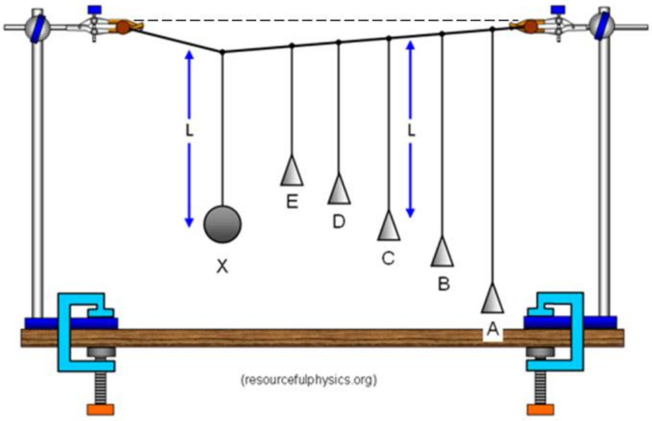
Check Your Understanding 6

Which of these pendulums A,B,C,D,E will have the largest amplitude of oscillation as X oscillates?

Hint: period of pendulum is given by:

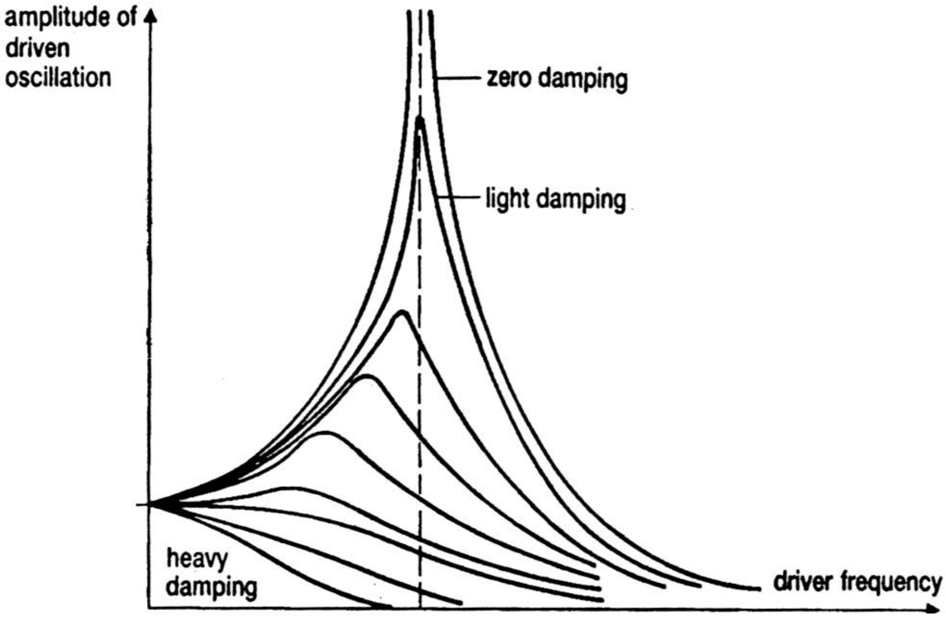
$$T = 2\pi\sqrt{\frac{l}{g}}$$

(More information on the Barton's pendulum can be found on pg 31)



E.2 Effect of Damping on Forced Oscillations

- The response of a damped system to a periodic driving force **depends on the degree of damping**.
- As the degree of damping increases,
  - the amplitude of vibration at all frequencies decreases.
  - the resonant peak becomes broader.
    The lighter the damping, the sharper the response (i.e. taller and narrower peaks).
  - the resonant peak occurs at lower frequencies (i.e. resonant frequency shifts slightly to the left.)

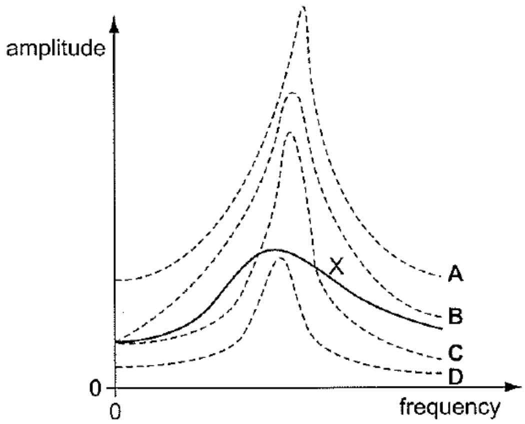


f_0 (natural frequency)

Check Your Understanding 7

The solid line X on the graph shows how the amplitude of an oscillating system varies with the frequency at which it is driven. The driving oscillator has a constant amplitude at all frequencies.

Which dashed line shows the response of the system under conditions where there is less damping?



E.3 Practical Examples of Resonance (for general knowledge)

Resonance can be useful or harmful (destructive).

Examples of Useful Resonance

1) Radio Receiver

Resonance not only occurs in mechanical oscillations but also in electric circuits. A radio receiver works on the principle of resonance. Our air is filled with radio waves of many different frequencies which the aerial (antenna) picks up. The tuner can be adjusted so that the natural frequency of the electrical oscillations in the circuits is the same as that of the radio wave transmitted from a particular station (the desired station). The radio waves of that particular frequency cause much larger oscillations (due to resonance) compared to the radio waves of other frequencies.

2) Musical Instruments

Brass instruments consist of a mouthpiece attached to a long tube filled with air. The vibrating air column inside the tube can be adjusted. The vibrations of the lips against the mouthpiece produce a range of frequencies. When one of the frequencies in the range of frequencies matches one of the natural frequencies of the air column inside of the brass instrument, this forces the air inside of the column into resonance vibrations.

3) Magnetic resonance imaging

Magnetic resonance imaging (MRI) is increasingly used in medical diagnosis to produce images similar to those produced by X-rays. Strong, electromagnetic fields of varying radio frequencies are used to cause oscillations in atomic nuclei. When resonance occurs, energy is absorbed by the molecules. By analysing the pattern of energy absorption, a computer-generated image can be produced. The advantage of MRI scanner is that no ionising radiation (as in the process of producing X-ray images) is involved.

Additional Notes

4) Quartz oscillator

Quartz is a specific form of a compound called silicon dioxide. It is also a piezoelectric material: that is, when a quartz crystal is subjected to mechanical stress (e.g. bending), it accumulates electrical charge across some planes. In a reverse effect, if charges are placed across the crystal plane, quartz crystals will bend. Quartz, directly driven by an electric signal, is used as a resonator. The frequency at which a quartz crystal oscillates depends on its shape, size, and the crystal plane on which the quartz is cut. If the crystal is accurately shaped and positioned, it will oscillate at a desired frequency. In nearly all quartz watches, the frequency is 32768 Hz, and the crystal is cut in a small tuning fork shape on a particular crystal plane. This frequency is a power of two ($32768 = 2^{15}$), just high enough so most people cannot hear it, yet low enough to permit inexpensive counters to derive a 1-second pulse.

Examples of Destructive Resonance

1) Earthquakes and tidal waves

An earthquake consists of many low-frequency vibrations range from 1 to 10 Hz. During an earthquake, when the frequencies of the vibration match with the natural frequencies of buildings, resonance may occur and result in serious damages. In regions of the world where earthquakes happen regularly, buildings may be built on foundations that absorb the energy of the shock waves. In this way, the vibrations are damped and the amplitude of the oscillations cannot reach dangerous levels.

An example is the collapse of building in Mexico City in1985 due to seismic waves from the earthquake. The angular frequency of the waves at 3 rad s^{-1} matches the natural frequency of the intermediate height buildings which collapsed.

2) Shattering of glass

When an opera singer projects a high-pitched note whose frequency matches the natural frequency of a wine glass, the resonance may cause the glass to vibrate at so large an amplitude that it breaks.

3) Vibrations in machines

If a loose part in a car rattles when the car is travelling at a certain speed, it is likely that a resonant vibration is occurring. A washing machine with an unbalanced load which has natural frequency matching the spinning frequency will get violent vibrations as resonance occurs.

4) Mechanical resonance in structures

The London Millennium Footbridge, which opened in June 2000, is a pedestrian-only steel suspension bridge crossing the River Thames in London, England, linking Bankside with the City. Londoners nicknamed the bridge the Wobbly Bridge after crowds of pedestrians felt an unexpected swaying motion (resonance where the forcing frequency of the pedestrians' movement equals to the natural frequency of the bridge) on the first two days after the bridge opened. The bridge was closed and modified, and further modifications eliminated the wobble entirely.

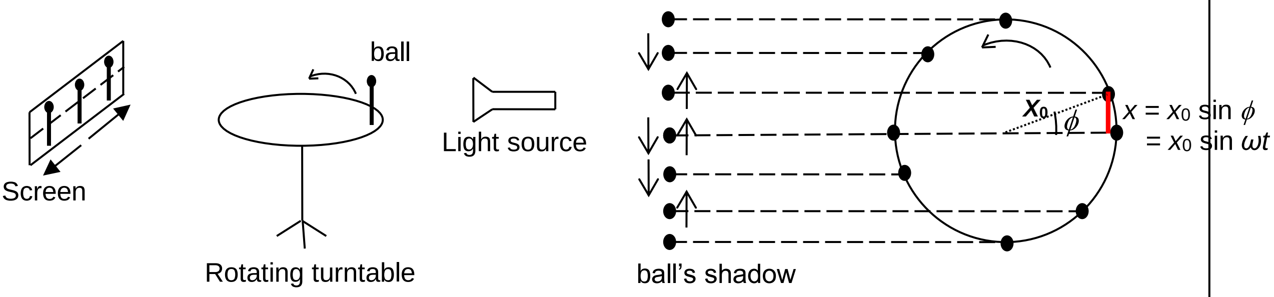
It was not an entirely unknown phenomenon: there was a sign dating back to 1873 on London's Albert Bridge warning military troops to break their usual lock-step motion when crossing. The culprit was not Millennium Bridge's design. Rather, it was due to a weird synchronicity between the bridge's lateral (sideways) sway and pedestrians' gaits.

It turns out that people walking on a bridge that starts to shift will instinctively adjust their stride to match the bridge's swaying motion as it lurches sideways. This will be familiar to anyone who has tried to walk on a fast-moving train and needed to find steady footing as the train wobbled from side to side. But on a bridge, this exacerbates the problem, giving rise to additional small sideways oscillations that amplify the swaying.

Additional Notes

Annex A: Relationship Between Uniform Circular Motion and SHM

Experiment to illustrate Connection between Uniform Circular Motion and Simple Harmonic Motion



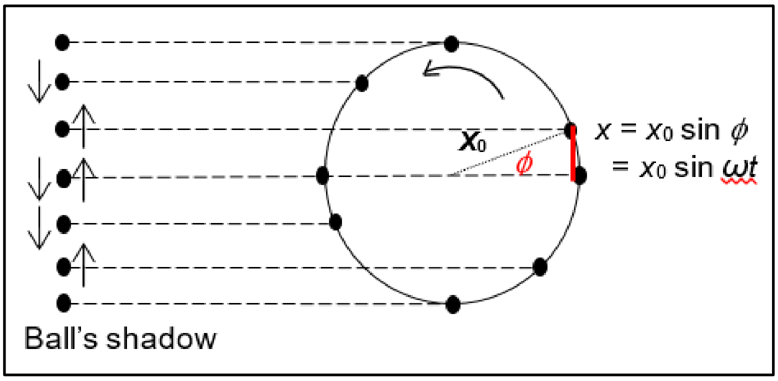
As the ball undergoes circular motion on the rotating turntable, the projected ball's shadow undergoes simple harmonic motion. The radius of the circular path of ball x_o is the amplitude of SHM of ball's shadow.

The quantity ω (angular frequency) is closely related to the angular velocity ω for circular motion but they are different.

SHM: ω is a constant of the motion which gives a measure of how rapidly oscillations are occurring (i.e. the more oscillations per unit time, the higher the value of ω).
ω is known as the **angular frequency** of the oscillation where ω = <u>$2\pi f$</u>

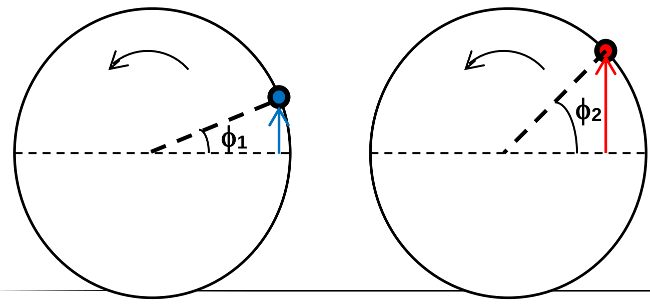
Circular motion: ω is the **angular velocity**, which is the rate of change of angular displacement.

- **Phase** ϕ is an angle either in degrees (°) or radians (rad) which gives a measure of the <u>fraction of a cycle</u> that has been completed by an oscillating particle or a wave.



- **Phase difference** $\Delta\phi$ is a measure of how much one oscillation (or wave) is out of step with another. A phase difference of **one cycle** corresponds to **360° or 2π rad**.

We can compute the phase difference between two oscillations **only if** they have the same ***same frequency/ period.***



$$\boxed{\Delta\phi = \phi_2 - \phi_1}$$

- If two oscillations are ***in step*** with one another
  $\Rightarrow$ <u>phase difference is zero (i.e. $\Delta\phi = 0$)</u>
  → they are said to be ***in phase*** with one another

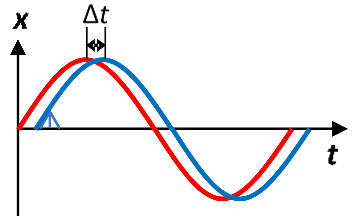
- If two oscillations are ***out of step*** with one another
  $\Rightarrow$ <u>phase difference is non-zero(i.e. $0 < \Delta\phi < 2\pi$)</u>
  → they are said to be ***out of phase*** with one another

- If two oscillations are ***always moving in opposite directions***, for instance if one moves up when the other moves down
  $\Rightarrow$ <u>phase difference $\Delta\phi = \pi$ rad</u>
  → they are said to be *in* ***antiphase***

- The phase difference $\Delta\phi$ is given by

  $$\Delta\phi = \frac{\Delta t}{T} \times 2\pi \qquad \text{or} \qquad \Delta\phi = \frac{\Delta position}{\lambda} \times 2\pi$$ (to be covered in Waves)

  where $0 \leq \Delta\phi < 2\pi$



This concept will be used in the next topic (Waves) where you will be asked to compare the oscillations of two particles.

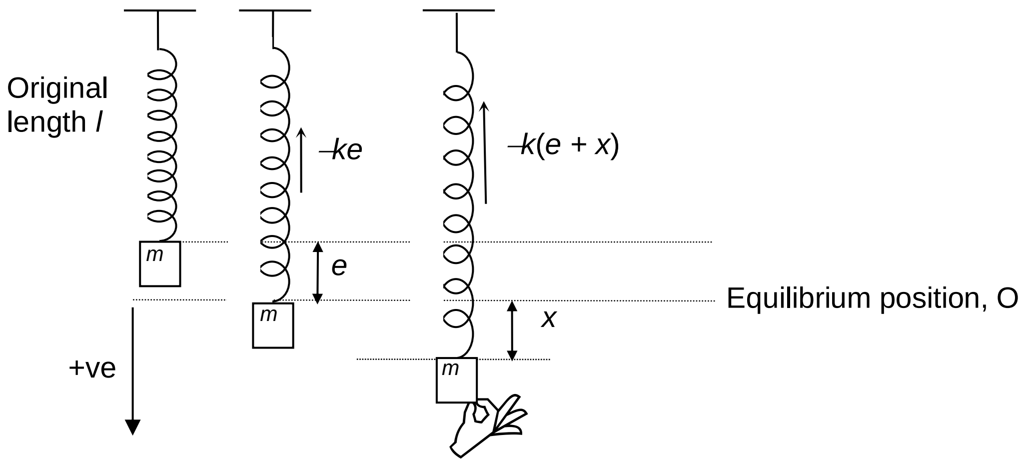
Additional Notes

Annex B: Applications of Simple Harmonic Motion

1 Oscillating Mass on Vertical Spring

- A mass m attached to the end of a spring experiences a constant downward force due to its weight, mg.

- At equilibrium, there must be an upward force provided by the spring, which has an extension e as shown in the diagram.



- Since the mass is in equilibrium, net force is zero.

$$mg + (-ke) = 0 \qquad \Rightarrow \qquad mg = ke$$

- Suppose the mass is now pulled a further distance x below its equilibrium position. The tension in the spring increases to $-k(e + x)$. The resultant force on the mass is now

$$\begin{aligned} F &= mg - k(e + x) \\ &= ke - k(e + x) \qquad \text{(since } mg = ke\text{)} \\ &= -kx \end{aligned}$$

- The resultant force is acting upwards, with a direction opposite to that of the displacement.

By Newton's 2nd law, $\quad F = ma = -kx \quad \Rightarrow \quad a = -\dfrac{k}{m}x$

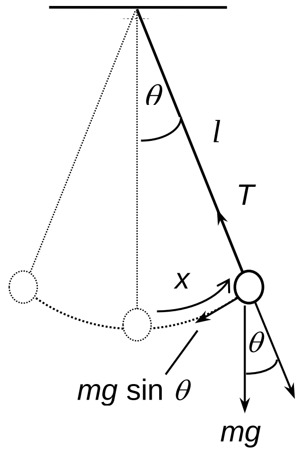
- Since k and m are fixed, the oscillation of a mass on spring is simple harmonic with angular frequency ω is given by $\omega = \sqrt{\dfrac{k}{m}}$

Note:
The amplitude of oscillation must be small because Hooke's law is not valid when the extension exceeds the <u>limit of proportionality</u> of the spring.

2 Simple Pendulum

- Motion of a small pendulum bob (radius of bob small compared to length of string) attached to a light inextensible string oscillating with a *small* angular displacement.

- In a simple pendulum, the restoring force is



$$F = -mg\sin\theta$$

- The negative sign indicates that it is in the opposite direction to the displacement.

Arc length $\qquad x = l\theta$

$$\Rightarrow \qquad \theta = \frac{x}{l}$$

- When the angular displacement θ is small, $\sin\theta \approx \theta$

$$\Rightarrow \qquad F = -mg\sin\theta \approx -mg\frac{x}{l}$$

- From Newton's 2nd law: $ma \approx -mg\dfrac{x}{l}$

$$\Rightarrow \qquad a \approx -\left(\frac{g}{l}\right)x$$

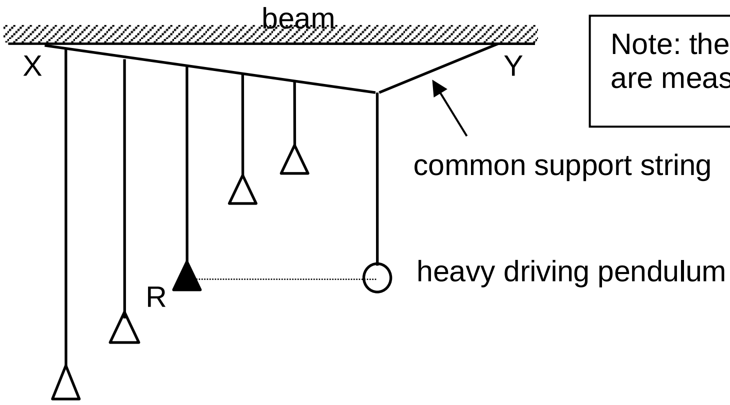
Hence, $\qquad a = -\omega^2 x \qquad$ for small x, $\qquad$ where $\omega^2 = \dfrac{g}{l}$

- The above shows that the movement of a pendulum is to a good approximation, a simple harmonic motion with period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{g}}$
- The motion is only simple harmonic for small amplitudes of oscillation.

Additional Notes

3 Forced Oscillation and Resonance - Barton's Pendulums

- Barton's pendulums (diagram shown below) consist of several pendulums of different lengths hung from a horizontal string. Each has its own natural frequency of oscillation.



Note: the lengths of the pendulums are measured from the beam

- The 'driver' pendulum at the end has a large mass, and its length is equal to that of pendulum R. When the driver is set swinging, its vibrations are transferred through the common support string to the other pendulums and cause the pendulums to oscillate.

- Observations show that:

  (i) All the pendulums vibrate with the same frequency as the driver.
  (ii) Pendulum R whose length matches that of the driver pendulum (therefore has the same natural frequency as the driver) has the largest amplitude. It is said to be resonating with the driver.

- Explanation:

  (i) All the pendulums are coupled together by the suspension. As the driver swings, it moves the suspension, which in turn moves the other pendulums.
  (ii) The matching pendulum is being pushed slightly once each oscillation, and the amplitude gradually builds up to maximum. The other pendulums are being pushed at a frequency that does not match their natural frequency, and they therefore oscillate with smaller amplitudes.

SUMMARY

Term	**Definition**
Amplitude	the maximum displacement of an oscillating body from its equilibrium position.
Angular frequency	a constant of a given oscillator and is related to its natural frequency f by $\omega = 2\pi f$.
Simple harmonic motion	For a body that undergoes simple harmonic motion (s.h.m.), its acceleration is proportional to its displacement from the equilibrium position AND is always directed towards that point (or in opposite direction to its displacement).
Damped oscillations	Oscillations in which the amplitude diminishes with time as a result of resistive forces that reduce the total energy of the oscillations.
Forced oscillations	Occurs when a body is acted upon by an external periodic driving force, causing the body to oscillate at the driving frequency, rather than the natural frequency of the body.
Resonance	A phenomenon that occurs when the driving frequency of a body is equal to its natural frequency, giving a maximum amplitude of oscillation.