Additional Notes

Topic 14: Current of Electricity

Content

A Electric Current
B Potential Difference
C Resistance and Resistivity
D Power and Heating Effect
E Sources of Electromotive Force

Learning Outcomes

Candidates should be able to:

(a) show an understanding that electric current is the rate of flow of charge

(b) derive and use the equation $I = nAvq$ for a current-carrying conductor, where n is the number density of charge carriers and v is the drift velocity

(c) recall and solve problems using the equation $Q = It$

(d) recall and solve problems using the equation $V = \frac{W}{Q}$

(e) recall and solve problems using the equations $P = VI$, $P = I^2R$ and $P = \frac{V^2}{R}$

(f) define the resistance of a circuit component as the ratio of the potential difference across the component to the current passing through it, and solve problems using the equation $V = IR$

(g) sketch and explain the I-V characteristics of various electrical components such as an ohmic resistor, a semiconductor diode, a filament lamp and a negative temperature coefficient (NTC) thermistor

(h) sketch the resistance-temperature characteristic of an NTC thermistor

(i) recall and solve problems using the equation $R = \rho l / A$

(j) distinguish between electromotive force (e.m.f.) and potential difference (p.d.) using energy considerations

(k) show an understanding of the effects of the internal resistance of a source of e.m.f. on the terminal potential difference and output power

Additional Notes

A Electric Current

A.1 Electric Charge, *Q*

- The electric charge is a property of some elementary particles (such as electrons and protons) that gives rise to an interaction between them / causing them to attract and repel each other.

- The charge of an atom arises as a result of an excess or deficit of electrons with respect to protons in the atom. If the atom gains or loses an electron, it becomes charged and is called an "ion".

The data for the elementary charge, $e = 1.60\times10^{-19}$ C, is given in the list of data in exam.

- The smallest known electric charge is the elementary charge *e*.
  proton charge, $e = 1.60 \times 10^{-19}$ C
  electron charge, $-e = -1.60 \times 10^{-19}$ C

Note that *N* must be a positive integer.

- Subsequently, any charge *Q* can be quantized as a multiple of the elementary charge,

$$Q = Ne$$ where *N* = number of charge particles

In the definition of a *coulomb*, a unit, it is defined in terms of other units.
The definition of *electric charge*, a physical quantity, is in terms of other physical quantities.

- The SI unit of charge is coulomb (C).

One **coulomb** is that charge flowing per second past a point in a circuit in which the current is 1 ampere.

The **electric charge** ΔQ passing one point in a circuit in an interval of time Δt is given by $\Delta Q = I\,\Delta t$ for a constant electric current I in the circuit.

charge in coulombs = (current in amperes) × (time in seconds)

➔ $$\Delta Q = I\,\Delta t$$

Graphically, the charge, ΔQ, can be found from the area under I–t graph.

- If the current is not constant in a circuit then the charge which flows can be found by using the area beneath the current-time graph.

  In calculus terms this can be written as

➔ $$\Delta Q = \int I\,\mathrm{d}t$$

Example 1
Determine the number of electrons that will make up –1.0 C.

Example 2

The current through an electrical component is reduced uniformly from 100 mA to 20 mA over a period of 8.0 s. Determine the charge that flows through the electrical component during the 8.0 s.

A.2 Electric Current, I

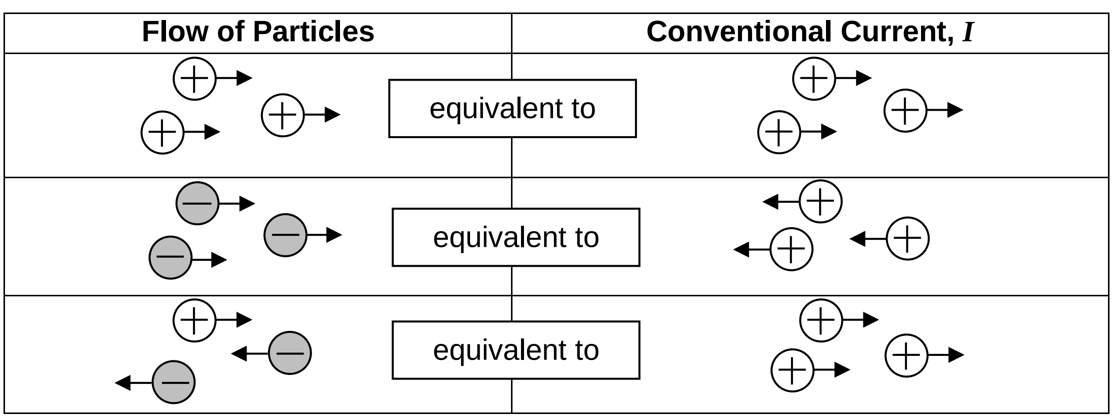
> Electric current (I) is the rate of flow of charge (which may be positively or negatively charged).

- The current I can be mathematically expressed as

$$I = \frac{dQ}{dt}$$

where dQ = amount of charge that passes through cross-section
dt = amount of time in which dQ passes through

- The S.I. unit for current is ampere (A), which is also a base unit.
- By convention, current I refers to the flow of positive charges.
- When negatively charged particles flow (i.e. in an electric circuit) in a given direction, it is equivalent to a conventional current I flowing in the opposite direction.

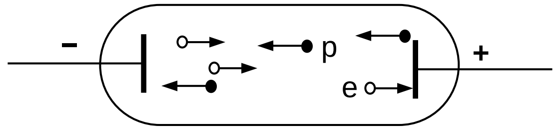


Additional Notes

memorise

For steady current, the amount of charge passing through a point in a given time is constant. Hence, the equation becomes $I = Q / t$.

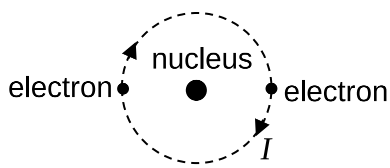
Example 3



The diagram above shows a hydrogen discharge tube.
A high potential difference is applied between the electrodes of the discharge tube so that the hydrogen gas is ionized.
Electrons then move towards the positive electrode and protons towards the negative electrode. In each second, 5.0×10^{18} electrons and 2.0×10^{18} protons pass a cross-section of the tube.
Calculate the current flowing in the discharge tube.

Example 4

The diagram shows a model of an atom in which two electrons move round a nucleus in a circular orbit. The electrons complete one full orbit in 1.0×10^{-15} s.



Calculate the current caused by the motion of the electrons in the orbit.

A.3 Microscopic Model of Current

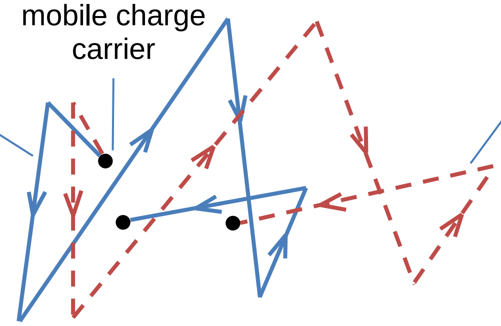
- In the absence of an external electric field, the mobile charge carriers (which can be either positively or negatively charged particles) in a conductor move randomly in all directions, somewhat like the molecules of a gas.

- Since the motion of the charge carriers is random, there is no net flow of charge in any direction and hence, no current.

- When a constant, steady electric field is applied across the conductor, the mobile charge carriers are subjected to a steady electric force. **IF** they are *moving in a vacuum*, this steady force will cause them to have a steady acceleration in the direction of the force.

- But a mobile charge carrier in a conductor undergoes frequent collisions with the massive atomic cores of the material. In each such collision, the carrier's direction of motion undergoes a random change.

- The net effect of the electric field is that there is a net motion or drift of the mobile charge carriers in the direction of the electric force, on top of the random motion.
  This motion is termed the drift velocity of the charge carriers. As a result, there is net current in the conductor. (Refer to animation in slideshow)



Additional Notes

An electric field is a region of space where charges experience an electric force, similar to how a mass experiences a gravitational force in a gravitational field. One method to obtain an electric field in a conductor is to connect its ends to plates of different potential as in Example 3.

In an electric field, the positive *mobile charge carriers* will have a drift velocity in the same direction as that of the field. For negative charge carriers, the direction of drift velocity is opposite to the direction of the electric field.

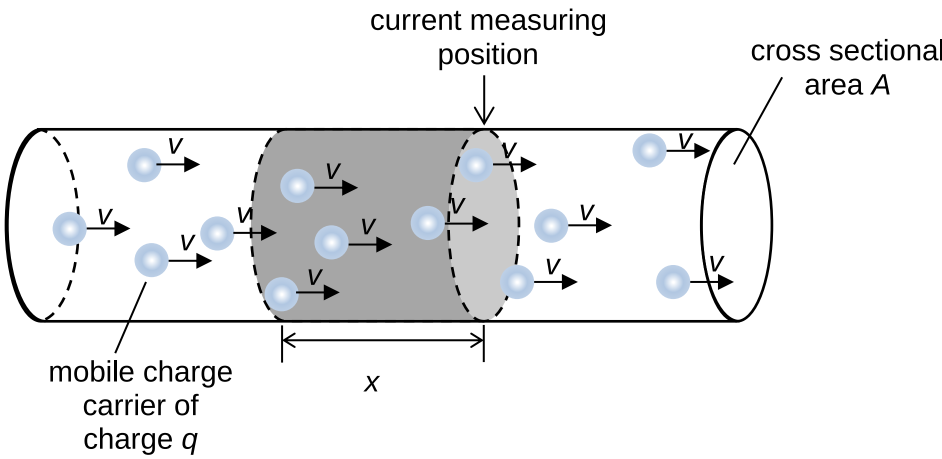
A.4 Derivation of $I = nAvq$ for a current-carrying conductor

- Consider a current-carrying conductor of uniform **cross-sectional area *A***, and having a **number density of charge carriers *n***.

- Each of the charge carrier has a **charge *q***.

- Due to the presence of an electric field, the charge carriers have a **drift velocity *v***.

Number density of charge carriers, *n*, is the number of charge carriers per unit volume of the conductor.



- In a length x of the conductor (shaded volume in diagram),

  number of charge carriers = nAx

  amount of charge that passes through current measuring position, $Q = nAxq$

  time for the charge to pass through current measuring position, $t = x / v$

- Hence, using $I = Q / t$,

  current through the conductor, $I = \dfrac{Q}{t} = \dfrac{nAxq}{x / v}$

$$\boxed{I = nAqv}$$

Example 5

A copper wire has 8.5×10^{28} free electrons per m^3. The wire has a current of 2.0 A and a cross-sectional area of 1.2 mm^2.

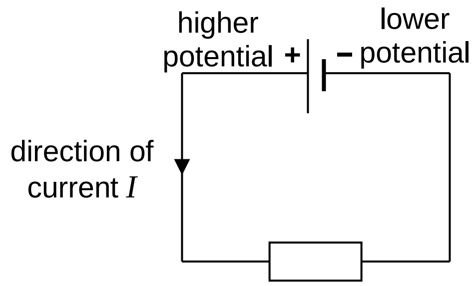
Calculate the average drift speed of the electrons.

Additional Notes

B Potential Difference

- To establish a current, an electrical potential difference must exist between two points in a conductor to set up an electric field.

- Electric current flows from a point of higher potential in an electric circuit to a point of lower potential.



Analogy:
Water flows when there is a difference in height; it flows from a higher level (*higher gravitational potential*) to a lower level (*lower gravitational potential*).

The potential difference (V) between two points in a circuit is the energy per unit charge converted from electrical energy to other forms of energy when charge passes from one point to the other.

memorise

- The potential difference V can be mathematically expressed as

$$V = \frac{W}{Q}$$

where W = electrical energy converted (J)

Q = charge flow (C)

The abbreviation “p.d.” is frequently used to denote “potential difference”.

- The S.I. unit for potential difference is volt (V).

One volt is the potential difference between two points in a circuit when the energy converted per unit charge is one joule per coulomb as charge passes from one point to the other.

Example 6

A battery is connected in series with a resistor R. The battery drives 200 C of charge round the complete circuit. During this process, 3000 J of energy is dissipated in the resistor R. Calculate the potential difference across the resistor R.

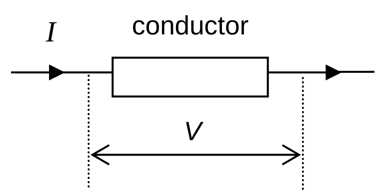
C Resistance and Resistivity

C.1 Resistance, *R*

- For a given potential difference applied to a circuit, the resistance in the circuit determines the amount of current that flows.

memorise

> The resistance (*R*) of a circuit component is the <u>ratio</u> of the potential difference across it <u>to</u> the current flowing through it.



- The resistance *R* of a component can be mathematically expressed as

$$R = \frac{V}{I}$$

where V is the potential difference across the conductor (V)

I is the current flowing through the conductor (A)

- The S.I. unit for resistance is ohm (Ω).

> One <u>ohm</u> is the resistance of a conductor through which the current is <u>one ampere</u> when a potential difference of <u>one volt</u> is applied across it.

C.2 Resistivity, *ρ*

The reciprocal of resistivity is conductivity. Good conductors have larger conductivity and lower resistivity.

- The resistivity of a material is the ability of the material to inhibit the flow of an electrical current. It is the relationship between the dimensions of a specimen of a material and its resistance that is constant at constant temperature.

- It enables comparisons to be made between the conducting ability of different materials. A good conductor has low resistivity and vice versa.

- The resistance, *R* of a given conductor <u>at a given temperature</u> depends on its
  (i) length (L),
  (ii) cross-sectional area (A), and
  (iii) the type of material i.e. its resistivity (ρ).

- The S.I. unit for resistivity is ohm metre (Ω m).

Additional Notes

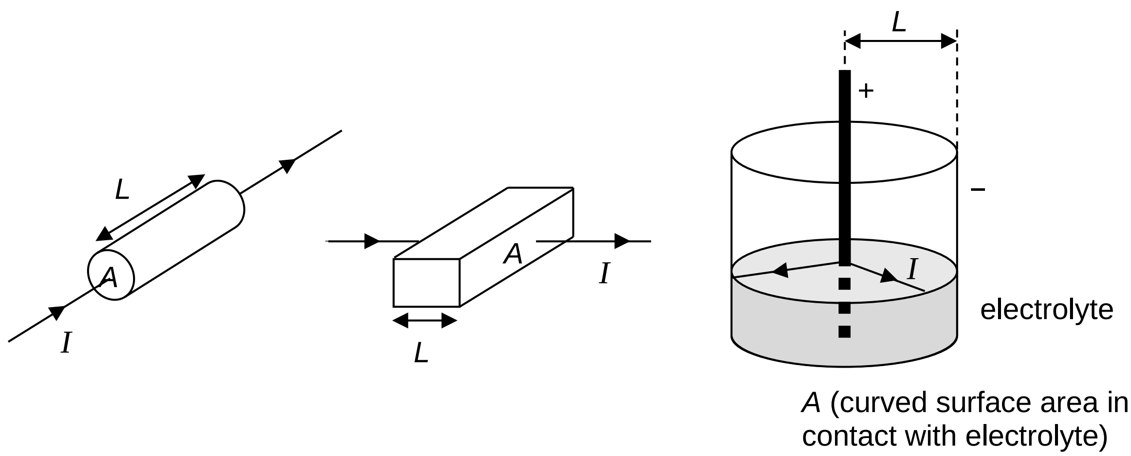
- For a piece of conductor as shown, the resistance R is given by

$$R = \frac{\rho L}{A}$$

where ρ = <u>resistivity</u> of material (Ω m)

L = length of material <u>parallel to the current</u> (m)

A = cross-sectional area of the material <u>perpendicular to the current</u> (m^2)



It is possible to state the resistance of a particular piece of copper wire (object), but not possible to state the resistance of copper (material). We can instead state the resistivity of copper (material).

- Below is a table of resistivity of some materials at 20 ºC.

Silver	1.47×10^{-8} Ω m
Copper	1.72×10^{-8} Ω m
Constantan	49×10^{-8} Ω m
Pure Germanium	0.60 Ω m
Pure Silicon	2300 Ω m
Glass	$10^{10} - 10^{14}$ Ω m
PET	10^{20} Ω m

- Resistivity is useful in comparing various materials on the basis of their ability to conduct electric currents. High resistivity ⇒ poor conductors

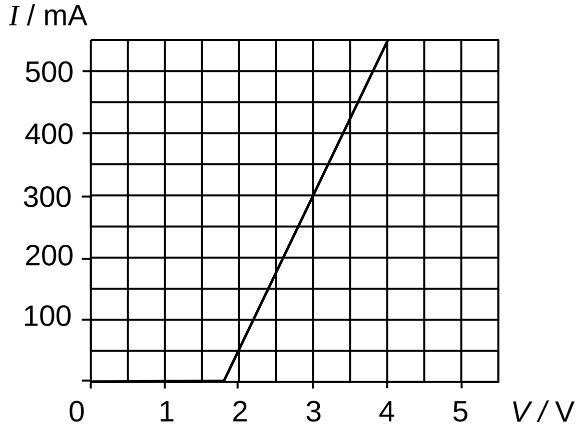
Additional Notes

Example 7

The current-voltage characteristic of a component is shown in the graph below.

(a) Calculate the resistance of the component at V = 3.0 V.

(b) Describe how the resistance of the component varies for V > 2.0 V.



Example 8

An electricity supply cable consists of a steel cylindrical core of diameter 7.98 mm. Calculate the resistance of 120 m of the cable. (Take ρ_{steel} = 9.0 × 10^{-8} Ω m)

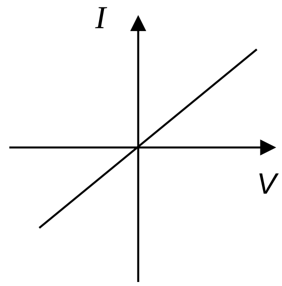
In calculations of R, you are expected to recall formulae of standard geometrical shapes.

Additional Notes

C.3 Ohm's Law

Ohm's Law states that the current flowing in a metallic conductor is proportional to the potential difference applied across it, provided that the physical conditions (e.g. temperature, pressure) are constant.

- This implies that at constant physical conditions (e.g. temperature),
  - resistance *R* is constant
  - *V* is proportional to *I* (straight line graph passing through the origin)



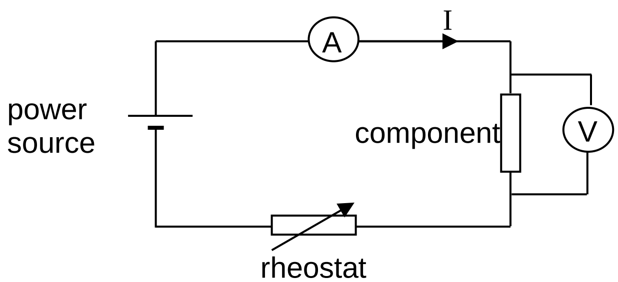
In Example 7, as the graph does not pass through the origin, the component does not obey Ohm's Law and is not ohmic. However, we can still use the relationship $V = IR$ to find the resistance of the component.

C.4 *I* –*V* Characteristics

- When a potential difference is applied across an electrical component, the potential difference causes an electric current in the component.

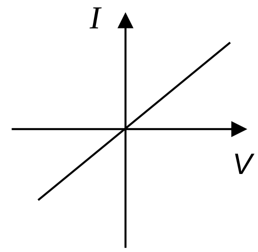
- To study the variation of electric current through the component with the potential difference applied across it, a current-potential difference (*I*-*V*) graph is typically plotted to determine its *I*-*V* characteristic. This is done with an electrical circuit setup as shown below.

The *I*−*V* graph is also called the *characteristics graph* of the component.



- Adjusting the resistance of the rheostat will vary the potential difference *V* across the component and hence the current *I* through the component.

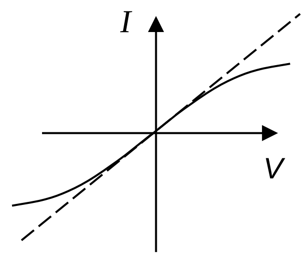
C.4.1 *I*−*V* Characteristic of an ohmic resistor



- The *I*−*V* graph is a straight line passing through the origin, showing that the current *I* increases proportionately with increase in potential difference *V*.
  The resistance *R* (i.e. ratio of *V*/*I*) remains constant.

- Pure metals are ohmic conductors, if the temperature remains constant.

C.4.2 *I*−*V* Characteristic of a filament lamp (e.g. tungsten)



- At low potential difference, the *I*−*V* characteristic is a straight line.
  The *I*−*V* curve concave downwards (becomes less steep) as potential difference *V* increases, showing that resistance *R* (i.e. the ratio *V*/*I*) increases.

From the definition of potential difference on pg. 14-7, an *increase* in potential difference will result in an *increase* in the energy per unit charge converted.

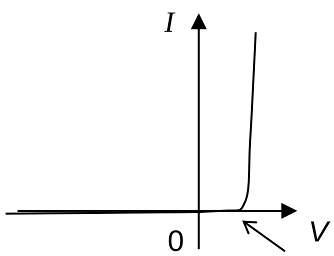
- As potential difference across the filament lamp increases, the current through the tungsten filament wire increases.
  The increase in current results in more collisions of the mobile electrons with the lattice ions (in the tungsten metal), hence transferring energy to them and increases the temperature of the filament lamp.

- When the temperature of the metal <u>increases</u>, two observations were made:
  1. the *number of free electrons per unit volume* <u>increases very slightly</u>
     - ➢ this results in <u>very slight decrease</u> in resistivity due to having *more* free electrons
  2. the *amplitude of vibration* of the lattice ions <u>increases</u>
     - ➢ this results in <u>increase</u> in resistivity due to *more frequent collisions* of the free electrons with the lattice ions

Additional Notes

- The increase in resistivity resulting from increased amplitude of vibration of lattice ions is more significant than the decrease in resistivity due to slight increase in free electrons. Hence, resistance of metal increases with potential difference.

C.4.3 I−V Characteristic of a semiconductor diode



- A diode conducts current in one direction only. This direction, called the forward direction, allows conduction much more easily than in the reverse direction.

- A typical diode requires a p.d. of around 0.3 − 0.7 V to conduct. Once the applied p.d. is above 0.3 − 0.7 V, the current rises very quickly with p.d., and the diode has very small resistance.

- It has a very low resistance in this forward direction, as seen by the near vertical line of the graph when the p.d. is positive (forward direction).

- In the reverse direction, the diode has a very high resistance. The current through the diode is almost zero, as seen by the near horizontal line of the graph when the p.d. is negative (reverse direction).

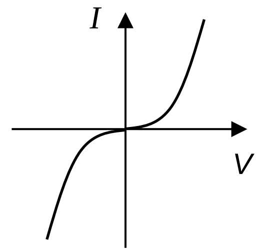
- A semiconductor has an electrical resistivity that is intermediate between a good conductor and an insulator. The mobile charge carriers in a semiconductor are the free electrons and holes. See *Appendix A* for details on the structure of a semiconductor.

The diode is said to be *forward bias* when connected such that current flows through. When it is connected such that no current flows, it is said to be *reverse bias*.

This one-way property allows the diode to be used as a rectifier for changing alternating currents to direct currents. More details can be found in the later topic of Alternating Current.

Additional Notes

C.4.4 I−*V* Characteristic of a negative temperature coefficient (NTC) thermistor



The response of a thermistor is rapid and most sensitive to changes in temperature in the region of 200 K to 600 K.

- The I-*V* curve concave upwards (becomes more steep) as potential difference *V* increases, showing that resistance *R* (i.e. the ratio *V*/I) decreases. The term NTC implies that the resistance decreases as temperature increases.

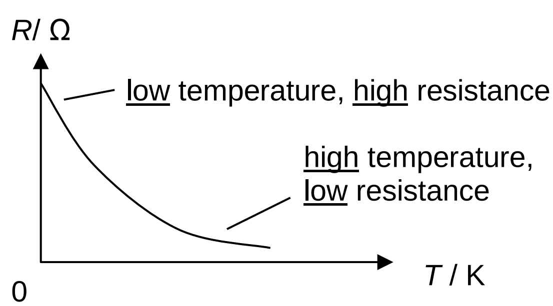
- As potential difference across the thermistor increases, the current through the thermistor increases.
  The increase in current results in more collisions of the conduction electrons with the atomic cores (in the semiconductor), hence transferring energy to them and increases the temperature of the thermistor.

A hole is a site within the atom where there is an absence of an electron. Refer to Appendix A for more details on the structure of a semiconductor.

- When temperature of the thermistor increases, two phenomena take place:
  1. the *number of mobile charge carriers* (holes and free electrons) *per unit volume* increases
     - ➢ this results in an decrease in resistivity due to having *more* charge carriers (holes and free electrons)
  2. the *amplitude of vibration* of the atomic cores increases
     - ➢ this results in increase in resistivity due to *more frequent collisions* of the mobile charge carriers with the atomic cores

- The decrease in resistivity due to more mobile charge carriers is more significant than the increase in resistivity due to the increased amplitude of vibration of atomic cores. Hence, resistance of thermistor decreases with potential difference.

- As the temperature *T* of the thermistor increases, its resistance *R* decreases.



Additional Notes

- The thermistor is commonly used at very low temperatures as the large variation in resistance at low temperatures enables more accurate measurements to be made. It can also be used to operate fire alarms and thermostats.

D Power and Heating Effect

- If the potential difference across a device is V and a charge Q passes through it, the electrical energy W converted by the device is

$$W = QV$$

This equation came from the definition of potential difference on pg. 14-7.

- Recall that when charge Q flows through steadily in time t, the current I through the device is

$$I = \frac{Q}{t}$$

This equation came from the electric current equation on pg. 14-3.

- Hence, the power P converted by the device is

$$P = \frac{W}{t} = \frac{QV}{t}$$

i.e. $$\boxed{P = IV}$$

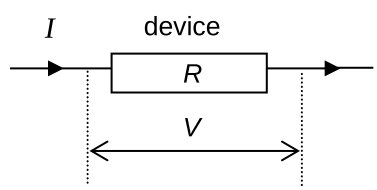
- The power of an electrical device is the rate at which it converts energy from one form to another.

- The S.I. unit for power is the watt (W).

- Substituting $V = IR$ into equation for power gives:

$$\boxed{P = I^2 R} \quad \text{or} \quad \boxed{P = \frac{V^2}{R}}$$

- **Any** of the 3 equations can be used to determine the power P dissipated by a device. The quantities used in the equation must be the potential difference V across the device, the current I through the device and the resistance R of the device.



All 3 formulae for calculation of power are useful and should be considered according to the quantities given in the questions.

- Across an electrical load (resistor or simple conductor), the energy is dissipated as heat due to collisions of electrons with lattice ions within it. The heating effect of a current is called joule heating.

Example 9

A current of 0.25 A flows in a conductor when a potential difference of 50 V is applied across it. Calculate

(a) the resistance of the conductor;

(b) the power dissipated in the conductor;

(c) the energy dissipated in 20 s; and

(d) amount of charge flow in 20 s.

The electrical rating here means that when the potential difference across the kettle is 240 V, the power dissipated at kettle is 2.0 kW. Hint: For an electrical appliance operating at a different condition (power or potential difference) from its rating, the **resistance** is assumed to be constant, not current or potential difference.

Example 10

An electrical kettle is rated at 2.0 kW when operating on a 240 V supply. Calculate the power that the kettle will draw if the supply is reduced to 220 V.

Additional Notes

E Sources of Electromotive Force

E.1 Electromotive Force, *E*

- In a battery or generator, energy is converted from other forms (e.g. chemical, mechanical, solar etc.) into electrical energy.

The electromotive force (e.m.f. or *E*) of a source is the energy converted per unit charge from other forms of energy into electrical energy when charge is moved round a complete circuit.

memorise

- It can be mathematically expressed as

$$E = \frac{W}{Q}$$

where W = electrical energy converted (J)

Q = charge (C)

Electromotive force is not a force!

- The S.I. unit for e.m.f. is volt (V).

- The e.m.f. of a source is said to be one volt when one joule per coulomb of other forms of energy (e.g. chemical, mechanical, solar, etc) is converted into electrical energy.

- Important note: The difference between e.m.f. and potential difference is that e.m.f. is the amount of energy converted from other forms to electrical per unit charge, whereas potential difference is the amount of energy converted from electrical to other forms per unit charge.

Difference lies in the direction of conversion between electrical and other forms of energy.

- Examples of e.m.f. sources:
  dry cell (chemical → electrical); solar cell (solar → electrical);
  dynamo (mechanical → electrical)

- Inside a battery, positive charges are moved from low potential (–ve terminal) to high potential (+ve terminal) and vice versa. This motion is against the electric field and therefore the battery does work on the charges. Energy conversion must take place within the battery and typically, chemical energy is transferred into electrical energy.

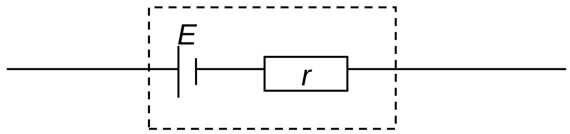
E.2 Internal resistance, *r*

- An ideal source of e.m.f. delivers all its electrical energy to the external circuit, i.e. no electrical energy is dissipated within the source itself.

- In reality, some of the electrical power delivered by a source is dissipated within the source itself. The reason is that the charges moving through the material of any real source experience resistance.

- The source is said to have *internal resistance*. All real sources of e.m.f. have internal resistance.

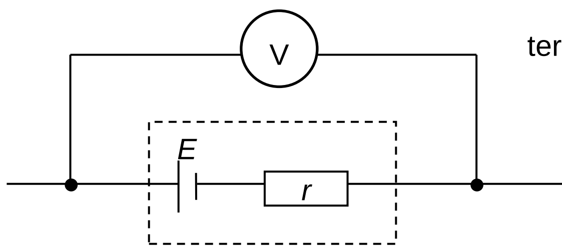
- Symbol of a cell with e.m.f. *E* and internal resistance *r*.



E.3 Terminal Potential Difference, V_T

- The terminal p.d. V_T of an e.m.f. source is the potential difference between the terminals of the e.m.f. source.

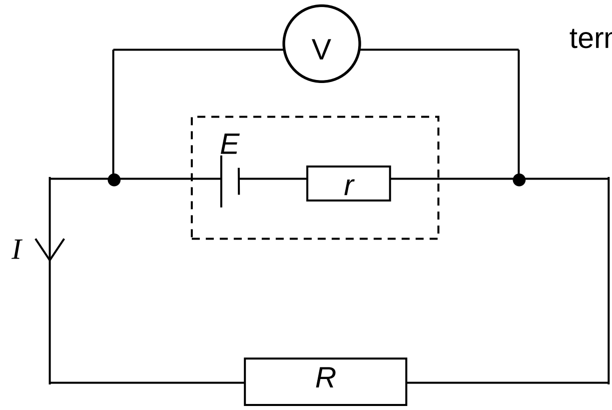
- In an open circuit where there is no current flowing through the e.m.f. source, the terminal p.d. has the same value as the e.m.f. of the source, even if the source has internal resistance.



terminal p.d., $V_T = E$

- When the e.m.f. source is connected to an external resistor *R* to form a closed circuit, a current *I* flows through the circuit.

- The terminal p.d. now has a value lower than the e.m.f. of the source as not all the energy supplied per unit charge by the cell (i.e. *E*) is changed in the external circuit to other forms of energy (e.g. heat). This inefficiency is due to the cell itself having some resistance.



terminal p.d., $V_T < E$

Additional Notes

The internal resistance can be treated like any other resistor connected in series with the source, when solving an electrical circuit mathematically. In reality, there is no additional resistor in the circuit.

- By Conservation of Energy,

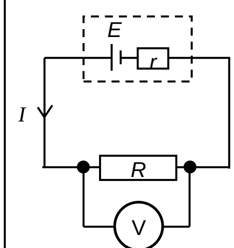
energy supplied per unit charge by e.m.f. source	=	energy converted per unit charge by total external resistance	+	energy wasted per unit charge due to internal resistance of e.m.f. source
e.m.f.	=	p.d. across *R*	+	potential drop due to *r*
E	=	IR	+	Ir

- Since terminal p.d. V_T = p.d. across *R*, using the above equation, we have

terminal p.d. $V_T = E - Ir$

- The above equation can be used to deduce that:
  - When there is <u>no current</u> passing through a real e.m.f. source, the terminal potential difference is equal to the e.m.f. <u>($V_T = E$)</u>.
  - When there is current through the source, the terminal potential difference is less than its e.m.f. <u>($V_T < E$)</u>. The "lost voltage" appears as the potential drop due to the internal resistance *r* of the e.m.f. source <u>($V_{loss} = Ir$)</u>.

By connecting the voltmeter across *R* in the circuit diagram, we can see that the p.d. across *R* is the same as the terminal p.d. of the source.



Example 11
A battery of e.m.f. 1.50 V has a terminal p.d. of 1.25 V when a resistor of 25 Ω is connected to it. Calculate
(a) the current through the resistor,
(b) the internal resistance *r* and
(c) the terminal p.d. when a resistor of 10 Ω replaces the 25 Ω resistor.

E.4 The Maximum Power Theorem *(see Appendix B)*

Proof of Maximum Power Theorem is not required. At maximum power, the efficiency is not maximum but only 50%. Also, note that efficiency can still be increase (or decrease) by varying the external resistance.

- The theorem states that:

> A source of e.m.f. delivers maximum power to the load when the load resistance R is equal to the internal resistance r of the source.

- This corresponds to an efficiency of 50%.
  i.e. when $R = r$, P_{out} is maximum and efficiency $\eta = 0.50$

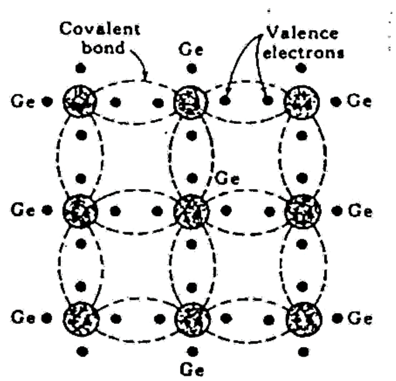
Appendix A

Structure of Semiconductor

There are different types of semiconductors: the intrinsic (pure) and the extrinsic (doped) semiconductors.

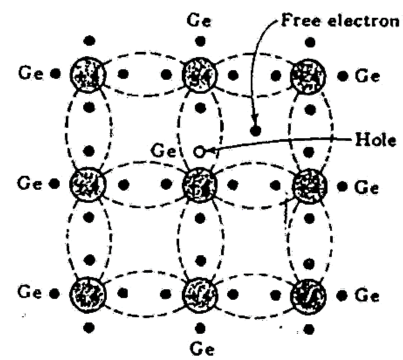
Intrinsic semiconductors are semiconductors made up purely of one type of element or compound such as Silicon (Si) and Germanium (Ge). Extrinsic semiconductors are intrinsic semiconductors doped with small amounts of impurities to increase their conductivity.

The figures below show the crystal structure of pure Ge at 0 K and room temperature. The 4 valence electrons of each atom form 4 covalent bonds with 4 surrounding atoms in tetrahedral shape.



Crystal structure of germanium, illustrated symbolically in two dimensions.

At temperature of 0 K



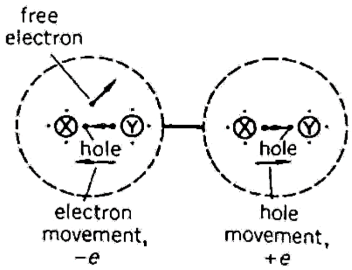
Germanium crystal with a broken covalent bond.

At room temperature

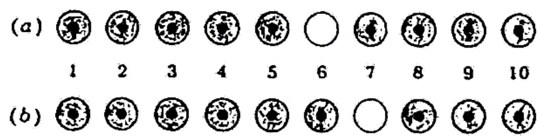
At temperature of **0 K**, all the valence electrons are firmly bound to the nucleus of their particular atoms. There is no free moving electron and it behaves as an insulator.

At **room temperature**, the thermal energy of a valence electron may become greater than the energy binding it to its nucleus. The covalent bond is then broken. An electron leaves the atom X, and becomes a **free electron**. This leaves atom X with a vacancy or **hole** (which is positively charged due to loss of electron). An electron in a neighbouring atom Y may then be attracted to fill up this positive hole. Thus the hole appears to move from atom X to atom Y.

Additional Notes



Semiconductor.
Electron (-) and Hole (+) movement



The mechanism by which a hole contributes to the conductivity.

If a battery is connected across the semiconductor, the holes drift in the direction of the field and the free electrons drift in the opposite direction. Thus the mobile charge carriers are free electrons and holes (in equal numbers for an intrinsic semiconductor).

Note:

The current is usually small due to limited mobile charge carriers, i.e. free electron-hole pairs. But as temperature increases, concentration of free electron-hole pairs increases. Thus conduction of current also increases.

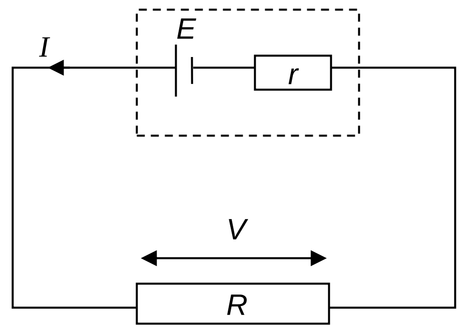
Additional Notes

Appendix B

Maximum Power Theorem – derivation and determining of efficiency at maximum power

At the load R, $P_{out} = IV = I^2R$

Since $I = \dfrac{E}{R+r}$, then $P_{out} = \left(\dfrac{E}{R+r}\right)^2 R$.



Maximum P_{out} occurs when $\dfrac{dP_{out}}{dR} = 0$.

Solving,

$$\left(\frac{E}{R+r}\right)^2 + 2R\left(\frac{E}{R+r}\right)\left[\frac{-E}{(R+r)^2}\right] = 0$$

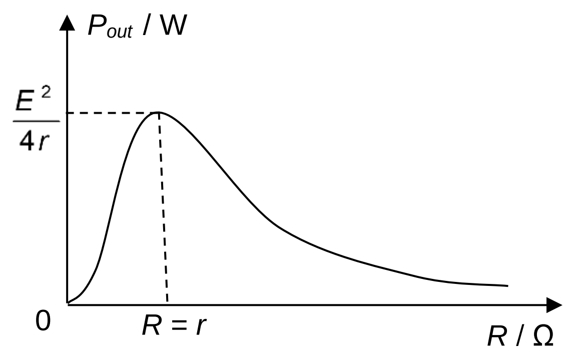
$$\left(\frac{E}{R+r}\right)^2 = \frac{2RE^2}{(R+r)^3} \rightarrow 2R = R + r$$

Hence, $R = r$.

Therefore, maximum power is delivered to the load when the load resistance R is equal to the source internal resistance r. This is the Maximum Power Theorem.

Substituting $R = r$ into $P_{out} = \left(\dfrac{E}{R+r}\right)^2 R$,

$$P_{out,max} = \left(\frac{E}{r+r}\right)^2 r = \frac{E^2}{4r}$$



At the source, $P_{supplied} = IE$

The ratio of the power output to the power supplied is the efficiency η of the circuit:

Additional Notes

$$\eta = \frac{P_{out}}{P_{supplied}} = \frac{IV}{IE} = \frac{V}{E} = \frac{IR}{I(R+r)} = \frac{R}{R+r}$$

i.e. when maximum power is delivered to the load (at $R = r$), the efficiency of the circuit is only 50%.