RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 6

Tutorial S3: Normal Distribution

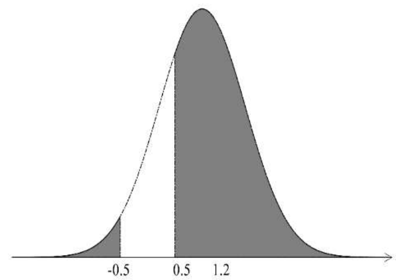
1 Given that $X \sim \mathrm{N}(1.5, 2.5)$, evaluate

(a) $\mathrm{P}(X < 0.7)$, **(b)** $\mathrm{P}(X \geq 3.7)$, **(c)** $\mathrm{P}(1.1 < X < 2.3)$,

(d) $\mathrm{P}(|X| \leq 0.5)$, **(e)** $\mathrm{P}(|X| > 0.5)$ **(f)** $\mathrm{P}(|X - 1.5| < \sqrt{2.5})$.

Solution

(a) to **(d)** can be evaluated from GC directly.

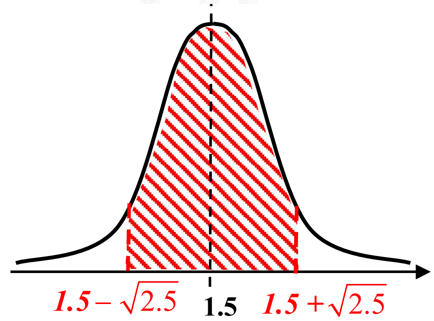


(e) $\mathrm{P}(|X| > 0.5)$

$= 1 - \mathrm{P}(-0.5 \leq X \leq 0.5)$

$= 0.839$ (3 s.f)

Or: $\mathrm{P}(|X| > 0.5) = \mathrm{P}(X < -0.5) + \mathrm{P}(X > 0.5)$



(f) $\mathrm{P}(|X - 1.5| < \sqrt{2.5})$

$= \mathrm{P}(-\sqrt{2.5} < X - 1.5 < \sqrt{2.5})$

$= \mathrm{P}(1.5 - \sqrt{2.5} < X < 1.5 + \sqrt{2.5})$

$= 0.683$ (3 s.f)

Or: $\mathrm{P}(|X - 1.5| < \sqrt{2.5}) = \mathrm{P}\left(\left|\dfrac{X - 1.5}{\sqrt{2.5}}\right| < 1\right) = \mathrm{P}(|Z| < 1) = 0.683$

2 Given that $Y \sim \mathrm{N}(22, 5)$ and $Z \sim \mathrm{N}(0, 1)$, find the value(s) of k such that

(a) $\mathrm{P}(Y < k) = 0.35$,

(b) $\mathrm{P}(Y > k) = 0.8$,

(c) $\mathrm{P}(k < Y < 25) = 0.46$,

(d) $\mathrm{P}(|Z| > k) = 0.27$,

(e) $\mathrm{P}(22 - k < Y < 22 + k) = 0.67$,

(f) $\mathrm{P}(Y < k) > 0.3$.

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<td>

Solution

$Y \sim \mathrm{N}(22,5)$ and $Z \sim \mathrm{N}(0,1)$

(a) $\mathrm{P}(Y < k) = 0.35$

From G.C, $k = 21.1$ (3 s.f)

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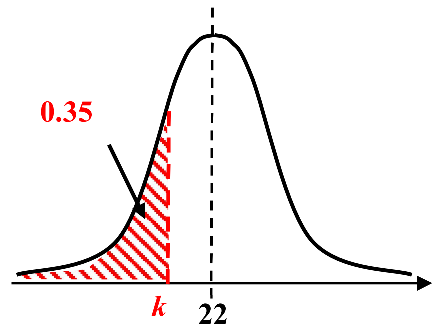

$k = \text{invNorm}(0.35, 22, \sqrt{5}, \text{LEFT}) = 21.1$

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(b) $\mathrm{P}(Y > k) = 0.8$

From G.C, $k = 20.1$ (3 s.f)

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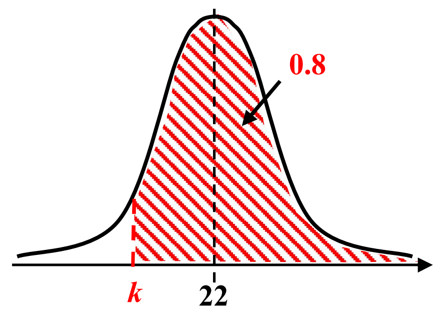

$k = \text{invNorm}(0.8, 22, \sqrt{5}, \text{RIGHT}) = 20.1$

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(c) $\mathrm{P}(k < Y < 25) = 0.46$

From the diagram,

$\mathrm{P}(Y < 25) - \mathrm{P}(Y < k) = 0.46$

$0.91014 - \mathrm{P}(Y < k) = 0.46$

$\mathrm{P}(Y < k) = 0.45014$

From G.C, $k = 21.7$ (3 s.f)

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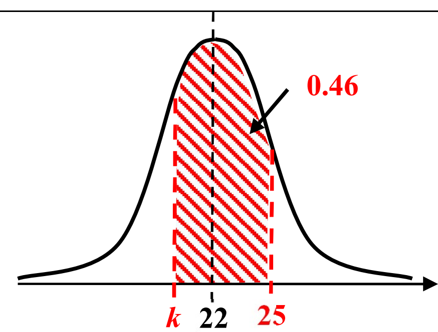

$k = \text{invNorm}(0.4501, 22, \sqrt{5}, \text{LEFT}) = 21.7$

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(d) $\mathrm{P}(|Z| > k) = 0.27$

From the diagram,

$\mathrm{P}(|Z| \le k) = 0.73$

From G.C, $k = 1.10$ (3 s.f)

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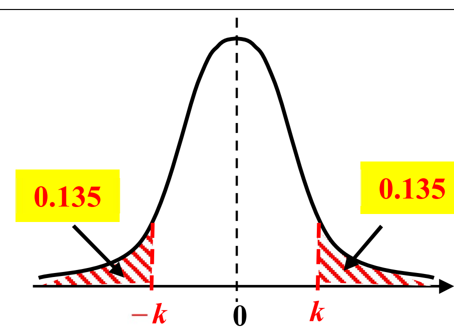

$k = \text{invNorm}(0.73, 0, 1, \text{CENTER}) = 1.10$

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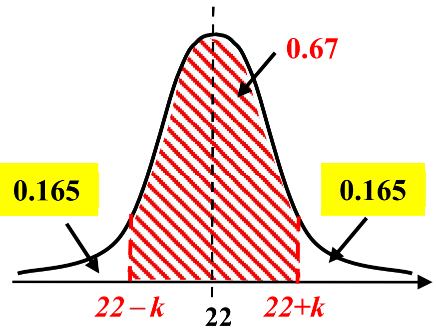
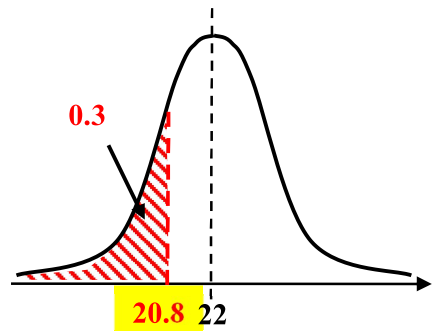
</table>

(e) $\mathrm{P}(22-k<Y<22+k)=0.67$ From G.C, $22-k=19.822$ and $22+k=24.178$ $\therefore k=2.18$ (3 s.f)	 using invNorm(0.67,22, $\sqrt{5}$,CENTER), $22-k=19.822$ and $22+k=24.178$
(f) $\mathrm{P}(Y<k)>0.3$ From G.C, $\mathrm{P}(Y<20.8)=0.3$ From the diagram, $k>20.8$ (3 s.f)	 invNorm(0.3,22, $\sqrt{5}$,LEFT) = 20.8

3 The random variable X follows a normal distribution with mean μ and standard deviation σ. Given that $\mathrm{P}(X<20)=0.14$ and that $\mathrm{P}(X<50)=0.65$, calculate the values of μ and σ.

Solution

$X \sim \mathrm{N}(\mu,\sigma^2)$.

$\mathrm{P}(X<20)=0.14$

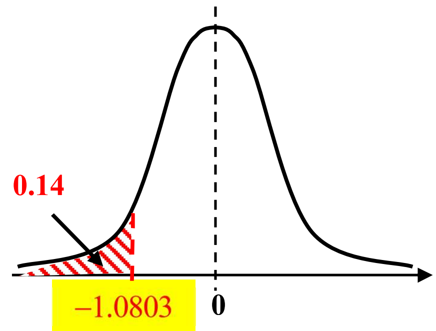
$\Rightarrow \quad \mathrm{P}\left(Z<\dfrac{20-\mu}{\sigma}\right)=0.14$

From GC,

$\mathrm{P}(Z<-1.0803)=0.14$

$\Rightarrow \quad \dfrac{20-\mu}{\sigma}=-1.0803 \;\; (5 \text{ s.f.})$

$\Rightarrow \quad 1.0803\sigma-\mu=-20$ ----- (1)



invNorm(0.14,0,1,LEFT) = – 1.0803

$\mathrm{P}(X<50)=0.65$

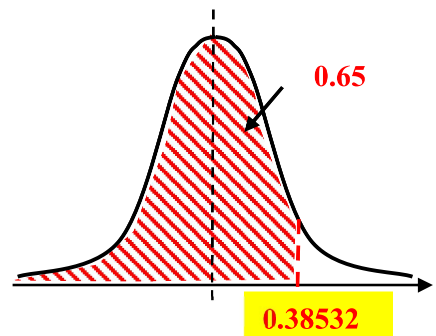
$\Rightarrow \quad \mathrm{P}\left(Z<\dfrac{50-\mu}{\sigma}\right)=0.65$

From GC,

$\mathrm{P}(Z<0.38532)=0.65$

$\Rightarrow \quad \dfrac{50-\mu}{\sigma}=0.38532 \;\; (5\text{s.f.})$

$\Rightarrow \quad 0.38532\sigma+\mu=50$ ----- (2)



invNorm(0.65,0,1,LEFT) =0.38532

Solving (1) and (2), $\mu=42.1$ (3 s.f.) and $\sigma=20.5$ (3 s.f.).

4 The independent random variables X and Y are each normally distributed with means 6 and 8 respectively, and variances 9 and 16 respectively. Find

(a) **(i)** $\mathrm{P}(X < Y)$ **(ii)** $\mathrm{P}(|X-Y| < 1.5)$.

(b) X_1, X_2 and X_3 are three independent observations of X and Y_1 and Y_2 are two independent observations of Y. Find

(i) $\mathrm{P}(X_1 + X_2 + X_3 < 15)$,

(ii) $\mathrm{P}(X_1 + X_2 > Y_1 + Y_2)$,

(iii) $\mathrm{P}(2X > Y_1 + Y_2)$.

Solution

Given $X \sim \mathrm{N}(6, 3^2)$ and $Y \sim \mathrm{N}(8, 4^2)$

(a)(i) $\mathrm{E}(X-Y) = \mathrm{E}(X) - \mathrm{E}(Y) = 6 - 8 = -2$

$\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 9 + 16 = 25$

$\therefore \; X - Y \sim \mathrm{N}(-2, 25)$

$\mathrm{P}(X < Y) = \mathrm{P}(X - Y < 0) = 0.655$ (3 s.f)

(ii) $\mathrm{P}(|X-Y| < 1.5)$

$= \mathrm{P}(-1.5 < X - Y < 1.5)$

$= 0.218$ (3 s.f)

(b)(i) $\mathrm{E}(X_1 + X_2 + X_3) = 3\mathrm{E}(X) = 18$

$\mathrm{Var}(X_1 + X_2 + X_3) = 3\mathrm{Var}(X) = 27$

$\therefore \; X_1 + X_2 + X_3 \sim \mathrm{N}(18, 27)$

$\mathrm{P}(X_1 + X_2 + X_3 < 15) = 0.282$ (3 s.f)

(ii) $\mathrm{E}\left[X_1 + X_2 - (Y_1 + Y_2)\right] = 2\mathrm{E}(X) - 2\mathrm{E}(Y) = 12 - 16 = -4$

$\mathrm{Var}\left[X_1 + X_2 - (Y_1 + Y_2)\right] = 2\mathrm{Var}(X) + 2\mathrm{Var}(Y) = 2(9) + 2(16) = 50$

$\therefore \; X_1 + X_2 - (Y_1 + Y_2) \sim \mathrm{N}(-4, 50)$

$\mathrm{P}(X_1 + X_2 > Y_1 + Y_2)$

$= \mathrm{P}(X_1 + X_2 - (Y_1 + Y_2) > 0) = 0.286$ (3 s.f)

(iii) $\mathrm{E}\left[2X - (Y_1 + Y_2)\right] = 2\mathrm{E}(X) - 2\mathrm{E}(Y) = 12 - 16 = -4$

$\mathrm{Var}\left[2X - (Y_1 + Y_2)\right] = 2^2\,\mathrm{Var}(X) + 2\mathrm{Var}(Y) = 4(9) + 2(16) = 68$

$\therefore \; 2X - (Y_1 + Y_2) \sim \mathrm{N}(-4, 68)$

$\mathrm{P}(2X > Y_1 + Y_2)$

$= \mathrm{P}(2X - (Y_1 + Y_2) > 0) = 0.314$ (3 s.f)

5 HCI Prelim 9740/2007/02/Q7 (ii) (modified)

The random variables X and Y are normally distributed with known means and variances.

A student was asked to find the probability that sum of 3 independent observations of X differ from 3 times an observation of Y by at least 1. State an assumption that the student has to make to solve the question.

Part of the student's solution to this question is given as follows:

$$3X-3Y \sim \mathrm{N}\left(3\mathrm{E}(X)-3\mathrm{E}(Y), 9\mathrm{Var}(X)-9\mathrm{Var}(Y)\right)$$

$$\mathrm{P}(3X-3Y \geq 1)=....$$

Point out 3 mistakes that the student has made. **[3]**

Solution

The student has to assume that X and Y are independent.

(1) It should be $X_1+X_2+X_3$ instead of $3X$

(2) It should be $3\mathrm{Var}(X) + 9\ \mathrm{Var}(Y)$ instead of $9\mathrm{Var}(X)-9\mathrm{Var}(Y)$

(3) It should be $\mathrm{P}\left(|X_1+X_2+X_3-3Y| \geq 1\right)$ instead of $\mathrm{P}(3X-3Y \geq 1)$

6 9233/2005/02/Q23

The pineapples in a large warehouse have masses with a normal distribution. The mean of this distribution is 1.52 kg and the standard deviation is 0.070 kg. Six randomly chosen pineapples are packed in a box of mass 2.15 kg. Find the probability that the total mass of the filled box is less than 11.2 kg.

Find the probability that, out of 3 randomly chosen filled boxes, at least 1 has total mass less than 11.2 kg. **[4]**

[0.342 , 0.715]

Solution

Let X be the mass of a pineapple in kg. Then $X \sim \mathrm{N}(1.52, 0.07^2)$.

Let Y be the mass of a filled box of pineapples in kg.

Then $Y = X_1+X_2+\cdots+X_6+2.15$

$\mathrm{E}(Y)=6\mathrm{E}(X)+2.15=6(1.52)+2.15=11.27$

$\mathrm{Var}(Y)=6\mathrm{Var}(X)=6(0.07^2)=0.0294$

$Y \sim \mathrm{N}(11.27, 0.0294)$

$\mathrm{P}(Y<11.2)=0.34155\ (5\text{s.f.})=0.342\ (3\text{s.f.})$.

OR:

Let $T = X_1+X_2+\cdots+X_6$

Then $T \sim \mathrm{N}(9.12, 0.0294)$

$\mathrm{P}(T<11.2-2.15)$

$=\mathrm{P}(T<9.05)$

$=0.342\ (3\text{s.f.})$

Let W be the number of boxes, out of 3, with total mass less than 11.2kg.

$W \sim \mathrm{B}(3, 0.34155)$

$\mathrm{P}(W \geq 1)$

$=1-\mathrm{P}(W=0)$

$=0.715\ (3\text{ s.f.})$.

7 VJC Prelim 9758/2020/02/Q8

A company sells bags of cement in two sizes. The mass of a large bag of cement can be modelled by a normal distribution with mean 50 kg and standard deviation 2 kg. The mass of a small bag of cement can be modelled by a normal distribution with mean 30 kg and standard deviation 1.5 kg.

(i) Find the mass that is exceeded by 99% of the large bags. **[2]**

(ii) Find the probability that total mass of three small bags is less than twice the mass of one large bag. **[3]**

(iii) State an assumption needed for your calculation in part **(ii)**. **[1]**

Three large bags are selected at random.

(iv) Find the probability that two weigh more than 53 kg and one weighs less than 53kg. **[3]**

[(i) 45.3 **(ii)** 0.982 **(iv)** 0.0125**]**

Solution	
(i)	Let X be the mass of a large bag, in kg. Then $X \sim \mathrm{N}\left(50, 2^2\right)$ $\mathrm{P}(X > a) = 0.99$ From GC, $\mathrm{P}(X > 45.347) = 0.99$ $\therefore a = 45.3$ (3 s.f.)
(ii)	Let Y be the mass of a small bag, in kg. Then, $Y \sim \mathrm{N}\left(30, 1.5^2\right)$ Let $T = Y_1 + Y_2 + Y_3 - 2X$ $\mathrm{E}(T) = 3\mathrm{E}(Y) - 2\mathrm{E}(X) = -10$ $\mathrm{Var}(T) = 3\mathrm{Var}(Y) + 2^2\,\mathrm{Var}(X) = 22.75$ $T \sim \mathrm{N}(-10, 22.75)$ $\mathrm{P}(T < 0) = 0.982$ (3 s.f.)
(iii)	We assume that the mass of a bag of cement is independent of the mass of another bag of cement. OR: We assume that the masses of the bags of cement are independent of each other. Note: It is **insufficient** to say that “mass of a small bag of cement is independent of the mass of a large bag of cement”. This is because for the 3 small bags of cement, we also need the condition that the mass of a small bag of cement is independent of the mass of another small bag of cement.
(iv)	Required probability $= \left[\mathrm{P}(X > 53)\right]^2 \mathrm{P}(X < 53) \times {}^3\mathrm{C}_2 = 0.0125$ (3 s.f.)

8 **9758/2020/02/Q6**

In this question you should assume that T, W and D follow independent normal distributions.

James leaves home to go to work at T minutes past 8 am each day, where T follows the distribution $\mathrm{N}\left(5,\ 1.2^2\right)$.

(i) Sketch this distribution for the period from 8 am to 8.10 am. **[2]**

(ii) Find the probability that, on a randomly chosen day, James leaves for work later than 8.06 am. **[1]**

When the weather is fine, James walks to work. The time, W minutes, he takes to walk to work follows the distribution $\mathrm{N}\left(21,\ 3^2\right)$. James is supposed to start work at 8.30 am.

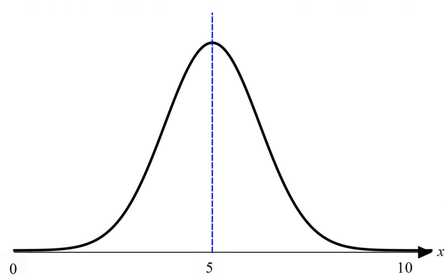
(iii) Find the probability that, on a randomly chosen day when James walks, he is late for work. **[2]**

When the weather is not fine, James drives to work. He still leaves at T minutes past 8 am each day; the time, D minutes, he takes to drive to work follows the distribution $\mathrm{N}\left(19,\ 6^2\right)$.

On average, the weather is fine on 70% of mornings.

(iv) One day, James is late for work. Find the probability that the weather is fine that day. **[5]**

[(ii) 0.202 **(iii)** 0.108 **(iv)** 0.606**]**

<table>
<tr><th colspan="2">Solution</th></tr>
<tr><td>(i)</td><td>

Note:
1) Distribution is symmetrical about 5
2) $\mathrm{P}(5-3(1.2) < X < 5+3(1.2)) = 0.997$
$\mathrm{P}(X < 1.4) = \mathrm{P}(X > 8.6) = 0.0015$
$\Rightarrow \mathrm{P}(X<0)\approx 0$ and $\mathrm{P}(X>10)\approx 0$
3) Curve should be very close to x axis at x=0 and x=10</td></tr>
<tr><td>(ii)</td><td>$\mathrm{P}\left(T > 6\right) = 0.202$ (3 s.f.)</td></tr>
<tr><td>(iii)</td><td>$T + W \sim \mathrm{N}\left(5+21,\ 1.2^2+3^2\right)$
i.e. $T+W \sim \mathrm{N}\left(26,\ 10.44\right)$
$\mathrm{P}\left(T+W > 30\right) = 0.108$ (3 s.f.)</td></tr>
<tr><td>(iv)</td><td>$T + D \sim \mathrm{N}\left(24,\ 37.44\right)$
$\mathrm{P}\left(\text{Day is fine} \mid \text{James is late}\right)$
$= \dfrac{\mathrm{P}\left(\text{Day is fine and James is late}\right)}{\mathrm{P}\left(\text{James is late}\right)}$
$= \dfrac{0.7 \times \mathrm{P}\left(T+W>30\right)}{0.7\times \mathrm{P}\left(T+W>30\right) + 0.3 \times \mathrm{P}\left(T+D>30\right)}$
$= 0.606$ (3 s.f.)</td></tr>
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9 9758/2018/02/Q10 (modified)

In this question you should state the parameters of any distributions that you use.

A manufacturer produces specialist light bulbs. The masses in grams of one type of light bulb have the normal distribution $\mathrm{N}\left(50, 1.5^2\right)$.

(i) Sketch the distribution for masses between 40 grams and 60 grams. **[2]**

(ii) Two light bulbs are randomly chosen. Find the probability that one of the light bulbs has mass less than 50.4 g and the other has mass more than 50.4 g. **[3]**

Each light bulb is packed into a randomly chosen box. The masses of the empty boxes have the distribution $\mathrm{N}\left(75, 2^2\right)$.

(iii) Find the probability that the total mass of 4 randomly chosen empty boxes is more than 297 grams. **[2]**

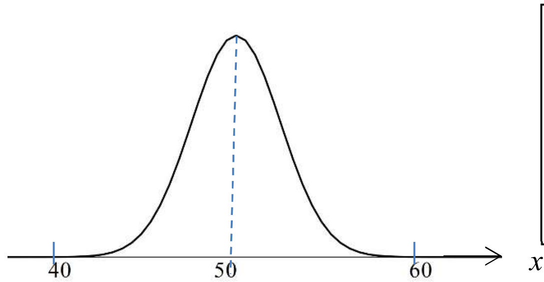
(iv) Find the probability that the total mass of a randomly chosen light bulb and a randomly chosen box is between 124.9 and 125.7 grams. **[3]**

In order to protect the bulbs in transit each bulb is surrounded by padding before being packed in a box. The mass of the padding is modelled as 30% of the mass of the bulb.

(v) The probability that the total mass of a box containing a bulb and padding is more than k grams is 0.9. Find k. **[4]**

(vi) Find the probability that the total mass of 4 randomly chosen boxes, each containing a bulb and padding, is more than 565 grams. **[3]**

[(ii) 0.478 **(iii)** 0.773 **(iv)** 0.126 **(v)** 137 **(vi)** 0.163**]**

Solution	
(i)	Let X be the mass of a light bulb in grams. Then $X \sim \mathrm{N}\left(50, 1.5^2\right)$  Note: 1) Distribution is symmetrical about 50 2) $\mathrm{P}(50-3(1.5) < X < 50+3(1.5))$ = Ie $\mathrm{P}(45.5 < X < 54.5)) = 0.997$
(ii)	$P\left(X < 50.4\right) = 0.60514$ $= 0.605$ (3 s.f.) Required probability $= \mathrm{P}\left(X < 50.4\right)\mathrm{P}\left(X > 50.4\right) \times 2$ $= \left(0.60514\right)\left(1-0.60514\right) \times 2$ $= 0.478$ (3s.f.)

(iii)	Let Y be the mass of a randomly chosen empty box, in g. Then $Y \sim \mathrm{N}\left(75, 2^2\right)$. Let $T = Y_1 + Y_2 + Y_3 + Y_4$ $\mathrm{E}(T) = 4(75) = 300$ $\mathrm{Var}(T) = 4\left(2^2\right) = 16$ $T \sim \mathrm{N}(300, 16)$ $P(T > 297) = 0.773$
(iv)	$X + Y \sim N(125, 6.25)$ $\mathrm{P}(124.9 < X + Y < 125.7) = 0.126$
(v)	Let $S = X_1 + Y + 0.3X_2$ (Note that $S \neq 1.3X + Y,$ since mass of padding is independent of mass of bulb) $\mathrm{E}(S) = 125 + 0.3(50) = 140$ $\mathrm{Var}(S) = 6.25 + 0.3^2(4) = 6.4525$ $S \sim \mathrm{N}(140, 6.4525)$ $\mathrm{P}(S > k) = 0.9$ From G.C, $\mathrm{P}(S > 136.74) = 0.9$ $\therefore k = 137$
(vi)	Let $W = S_1 + S_2 + S_3 + S_4$ $\mathrm{E}(W) = 4(140) = 560$ $\mathrm{Var}(W) = 4(6.4525) = 25.81$ $W \sim \mathrm{N}(560, 25.81)$ $P(W > 565) = 0.163$

10 RI Prelim 9740/2010/02/Q9 (modified)

Every weekday, the last train from Bedok Station leaves the station at 11.29 pm. Its travel time X (in minutes) from Bedok Station to Redbridge Station may be taken to follow a normal distribution with mean 30 and variance 9.

(i) Find the travel time exceeded by 90% of the trips made by the last train. **[2]**

At 11.40 pm every weekday, Abel walks from his workplace to Redbridge Station to catch the last train. The time Y (in minutes) he takes follows a normal distribution with mean 16 and variance 4.

(ii) Find the probability that the train arrives at Redbridge Station after 12 am and Abel arrives before 12 am. **[2]**

(iii) Find the probability that Abel will not miss the train. **[3]**

(iv) Explain why the answer to part **(iii)** is greater than the answer to part **(ii)**. **[1]**

The time W (in minutes) taken by Dina to walk from her workplace to Redbridge Station has a mean of 4 and a variance of 6.
Explain why W is unlikely to be normally distributed. **[1]**

[(i) 26.2 **(ii)** 0.361 **(iii)** 0.797**]**

Solution

Given $X \sim \mathrm{N}(30,9)$ where X is the travel time in mins from Bedok Station to Redbridge Station.

(i) $\mathrm{P}\left(X > t\right) = 0.9$

$t = 26.2$ (in minutes) (3 s.f.)

Given $Y \sim \mathrm{N}(16,4)$ where Y is the time taken for Abel to walk from his workplace to Redbridge Station

(ii) $\mathrm{P}\left(X > 31 \text{ and } Y < 20\right) = \mathrm{P}\left(X > 31\right) \times \mathrm{P}\left(Y < 20\right)$

$= 0.361$ (3 s.f.)

(iii) $\mathrm{P}\left(X > Y + 11\right) = \mathrm{P}\left(X - Y > 11\right)$ where $X - Y \sim \mathrm{N}\left(14, 13\right)$

$= 0.797$ (3 s.f.)

(iv) Part (iii) includes the case in (ii), as well as other cases in which Abel reaches the station before the train, e.g. train arrives before 12am but Abel arrives before the train and so does not miss the train.

If $W \sim \mathrm{N}\left(4, 6\right)$, then about 99.7% of the values would lie within $4 \pm 3\sqrt{6}$, which contains a significant range of negative values.
In fact, $\mathrm{P}(W < 0) = 0.0512$ (3 sf). Hence W is unlikely to be normally distributed.