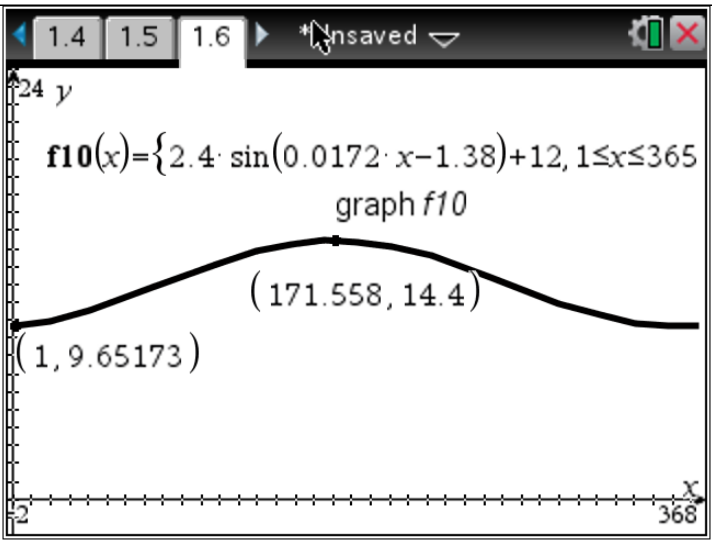
HL P2 2018

Section A

<table>
<tr><th>QN</th><th>Solution</th><th>Remarks</th></tr>
<tr><td>1a.</td><td>


[Sketch of graph is optional here.]
Max sunlight = 14.4h
Min sunlight = 9.65h
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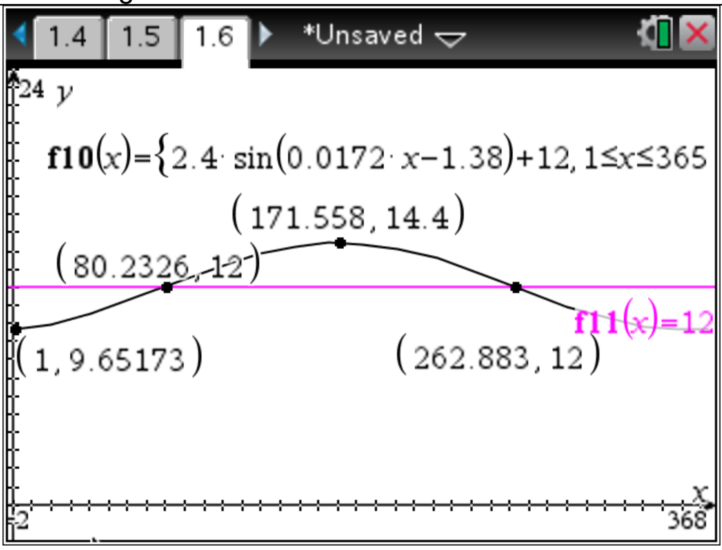
<tr><td>1b.</td><td>


Intersection points: x = 80.2, x=262.9

No. of days = $(80.23-1)+(365-262.9)$ days
= 181 days

OR No. of days = 364 – (263-80) = 181 days
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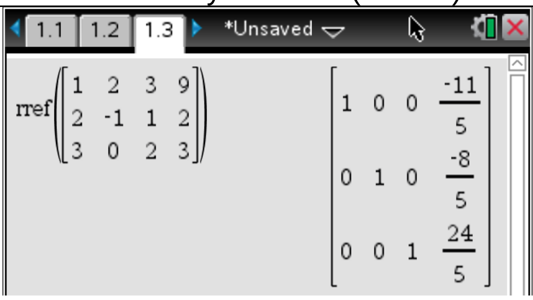
<tr><td>2a</td><td>


Point of intersection is $\left(-\frac{11}{5},-\frac{8}{5},\frac{24}{5}\right)$
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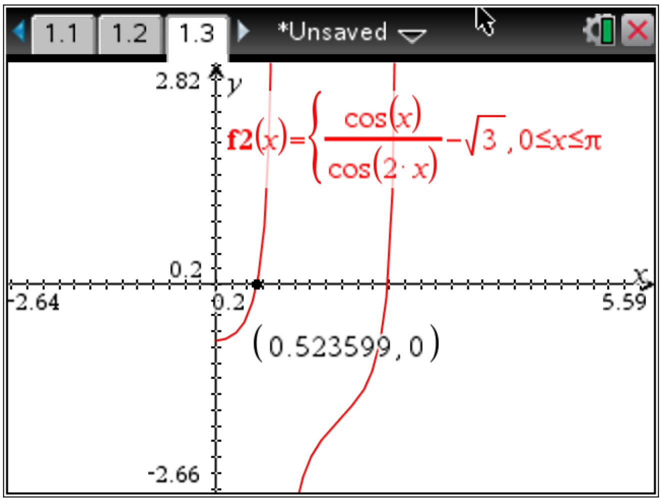
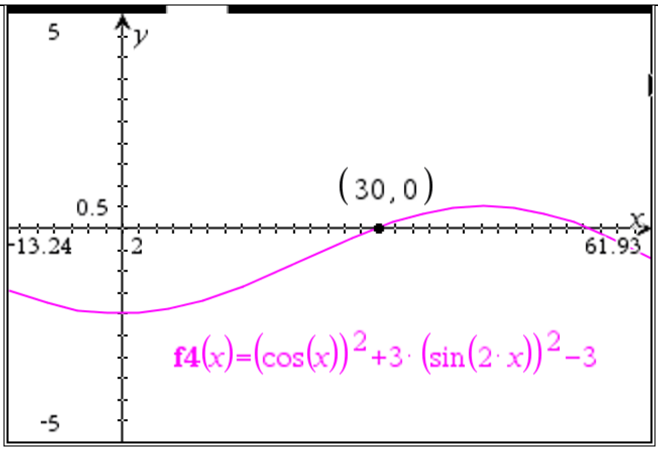
2b	$\text{rref}\left(\begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 3 \\ 3 & 0 & 3 & 3 \end{bmatrix}\right) \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ No solution. The planes are non-parallel and do not intersect.	
3a	$\text{cPolyRoots}\left(x^2+(7+i)\cdot x+24+7\cdot i, x\right)$ $\quad \{-4+3\cdot i, -3-4\cdot i\}$ $z=-4+3i, -3-4i$	Use GDC to solve.
3b	$z=-4+3i, -3-4i$ are represented by (-3, -4) and (-4, 3). $z_3 = iz_1 = i(-3-4i) = 4-3i$ $z_4 = -iz_2 = -i(-4+3i) = 3+4i$	Common mistake: $z_3 = z_1^*$
4a	$OA = \cos\alpha$ $BA = \sin\alpha$	
4b	From triangle COA $OA = OC\cos 2\alpha$ $OC\cos 2\alpha = \cos\alpha$ $OC = \dfrac{\cos\alpha}{\cos 2\alpha} = \sqrt{3}$ [Use GDC to solve.]  OR $CA = \sqrt{3}\sin 2\alpha$ By Pythagoras Theorem, $OC^2 = OA^2 + CA^2$ $\cos^2\alpha + 3\sin^2 2\alpha = 3$ [Use GDC to solve.]	

	 $\alpha = 0.523599 rad = 30.0^{\circ}$ Hour hand $2\alpha = 60.0^{\circ}$ Minute hand On the clock the time is 2:05.	
5a	By Sine rule, $\frac{\sin(180^{\circ} - A)}{8} = \frac{\sin 40^{\circ}}{6}$ $180^{\circ} - A = \arcsin\left(\frac{8}{6}\sin 40^{\circ}\right) = 58.98^{\circ} = 59.0^{\circ}$ $A = 180^{\circ} - 59.0^{\circ} = 121^{\circ}$	Common mistake: Not realising ACB is obtuse.
5b	$BCD = 59^{\circ}$ By Cosine rule, $6^2 = 6^2 + CD^2 - 2 \times 6 \times CD \times \cos 59.0^{\circ}$ $CD^2 - 12\cos 59^{\circ} \times CD = 0$ $CD(CD - 12\cos 59^{\circ}) = 0$ $CD = 12\cos 59^{\circ} = 6.18$	
6a	For $x\sqrt{9-x^2}$, $-3 \le x \le 3$ and for $2\arcsin\left(\frac{x}{3}\right)$, $-3 \le x \le 3$ Hence the domain is $-3 \le x \le 3$ Or	

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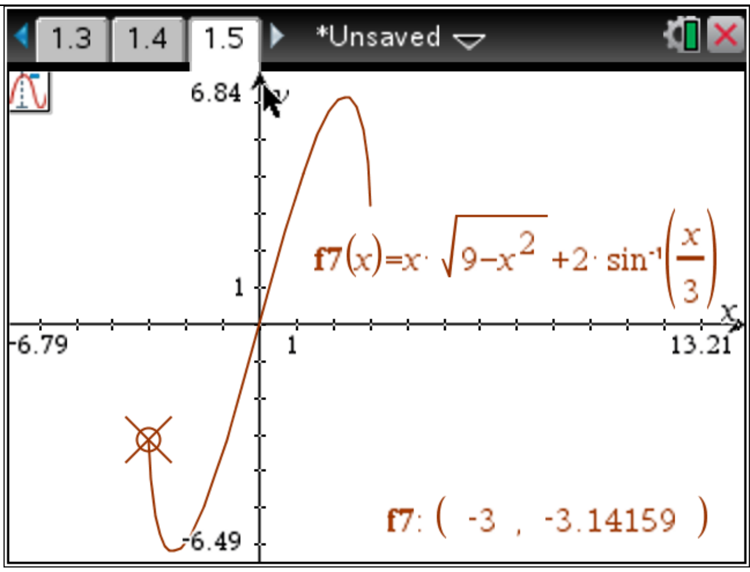

Hence the domain is – 3 ≤ x ≤ 3
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<td>6b</td>
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$$\frac{dy}{dx}=(9-x^2)^{\frac{1}{2}}-\frac{x^2}{(9-x^2)^{\frac{1}{2}}}+\frac{\frac{2}{3}}{\sqrt{1-\frac{x^2}{9}}}$$

[Chain Rule and formula to differentiate arcsinf(x)]

$$=(9-x^2)^{\frac{1}{2}}-\frac{x^2}{(9-x^2)^{\frac{1}{2}}}+\frac{2}{(9-x^2)^{\frac{1}{2}}}$$

$$=\frac{9-x^2-x^2+2}{(9-x^2)^{\frac{1}{2}}}$$

$$=\frac{11-2x^2}{\sqrt{9-x^2}}$$
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Recall:

$\frac{d}{dx}sin^{-1}f(x)=\frac{1}{\sqrt{1-f(x)^2}}\times f'(x)$
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<td>7a</td>
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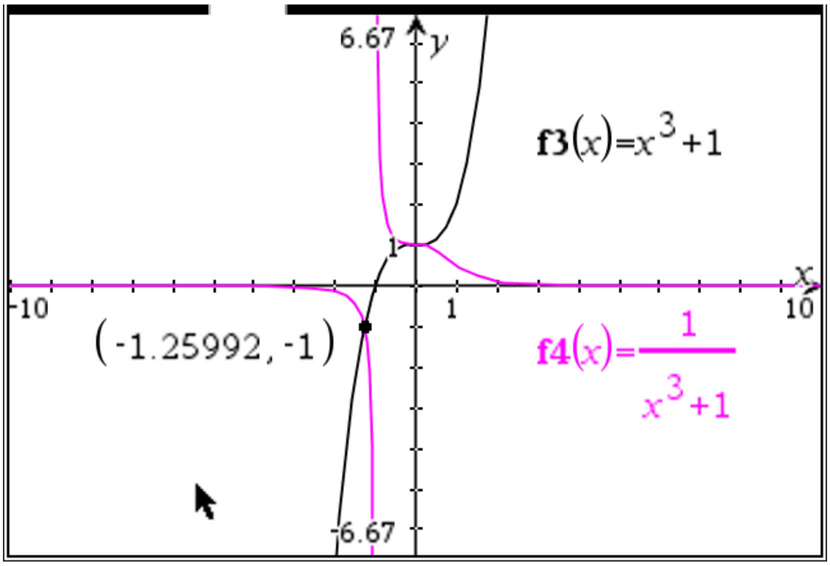
$$x^3+1=\frac{1}{x^3+1}$$



[Sketch of graph is optional here.]

$P(-1.26,-1)$ or $P(-\sqrt[3]{2},-1)$
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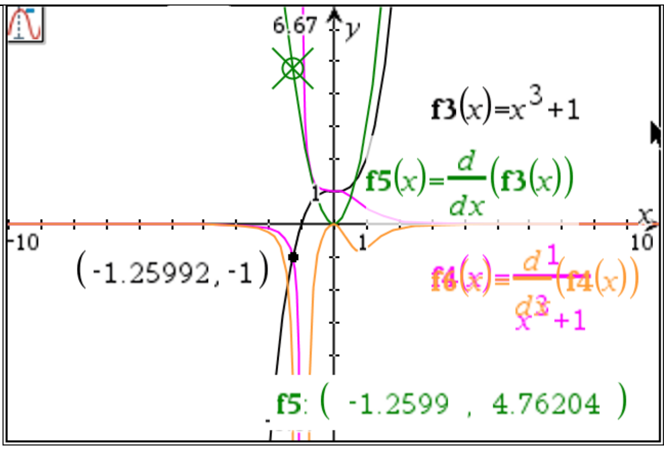
7b

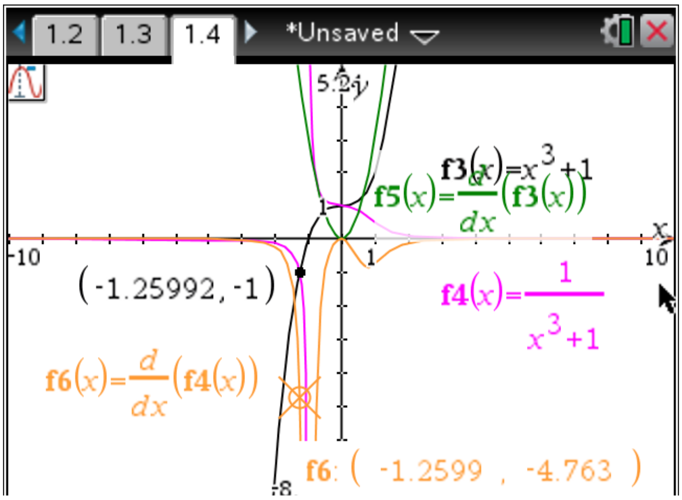


[Purpose of graph is to show how to find the value of the gradient at a point graphically. Sketch of graph is optional here.]

From GDC, $f'(-1.2599)=\dfrac{d}{dx}\left(x^3+1\right)=4.762$

Angle between tangent of f(x) to the horizontal,

$\alpha_1=\tan^{-1}\dfrac{y}{x}=\tan^{-1}f'(-1.2599)=\tan^{-1}4.762=78.14°$



$g'(-1.2599)=\dfrac{d}{dx}\left(\dfrac{1}{x^3+1}\right)=-4.763$

Angle between tangent of g(x) to the horizontal,

$\alpha_2=\tan^{-1}\dfrac{y}{x}=\tan^{-1}g'(-1.2599)=\tan^{-1}(-4.763)$

$=180°-78.14°=101.86°$

Angle between the two tangents =

$\alpha_2-\alpha_1=101.86°-78.14°=23.7°$

OR

From GDC, $f'(-1.2599)=\dfrac{d}{dx}\left(x^3+1\right)=4.762$ and

$g'(-1.2599)=\dfrac{d}{dx}\left(\dfrac{1}{x^3+1}\right)=-4.763$

Direction vector of the tangent of *f(x)* at P = $\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}=\begin{pmatrix}1\\ 4.762\end{pmatrix}$ and

Direction vector of the tangent of *g(x)* at P = $\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}=\begin{pmatrix}1\\ -4.762\end{pmatrix}$.

This is a difficult question and difficult to solve without the help of the GDC.

Recall:
Gradient = $\frac{\Delta y}{\Delta x}$

Tip:
Use GDC to solve for the gradient at a point
Ie.
$f'(-1.2599)=4.762$

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Angle between the two tangents $= \cos^{-1}\left(\frac{\binom{1}{4.762}\cdot\binom{1}{-4.762}}{\left|\binom{1}{4.762}\right|\left|\binom{1}{-4.762}\right|}\right) = 23.7°$

OR

Note: Let θ_1 be the angle between the tangent of f(x) and the vertical.

And let θ_2 be the angle between the tangent of g(x) and the vertical.

From GDC, $\frac{d}{dx}\left(x^3+1\right) = 4.762$ [A1]

$\frac{d}{dx}\left(\frac{1}{x^3+1}\right) = -4.763$ [A1]

Required angle =

$\arctan\theta_1 + \arctan\theta_2 = \arctan\frac{1}{\left|\frac{d}{dx}f(x)\right|} + \arctan\frac{1}{\left|\frac{d}{dx}g(x)\right|}$ [M1]

$= \arctan\frac{1}{4.762} + \arctan\frac{1}{|-4.763|}$ [M1]

$= 11.86° + 11.86° = 23.7°$ [A1]

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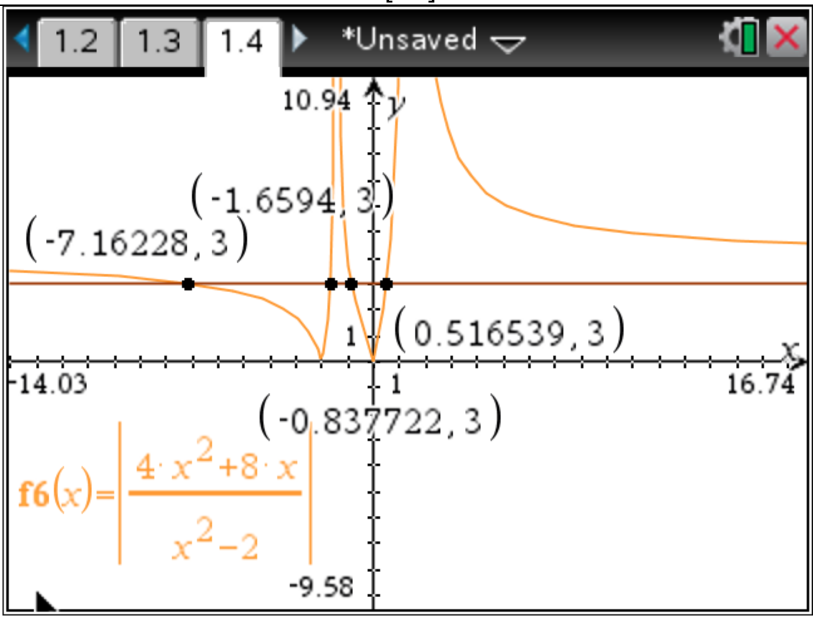

Asymptotes at:

$x^2-2=0 \Rightarrow x=\pm\sqrt{2} \Rightarrow x=\pm1.41$

[Since denominator = 0 at vertical asymptote]

$y=4$

[Since $\left|\frac{4x^2+8x}{x^2-2}\right| = \left|4+\frac{8x+8}{x^2-2}\right| \to 4$ as $x\to\infty$.]

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<td>A straight forward question that is done well.</td>
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From the graph,

Intersections at: x = -7.16, -1.66, -0.838, 0.517 (3 s.f)

$-15 < x < -7.16, \quad -1.66 < x < -0.838, \quad 0.517 < x < 15, \quad x \neq \pm1.41$

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<td>Most didn't remember to exclude the vertical asymptote.</td>
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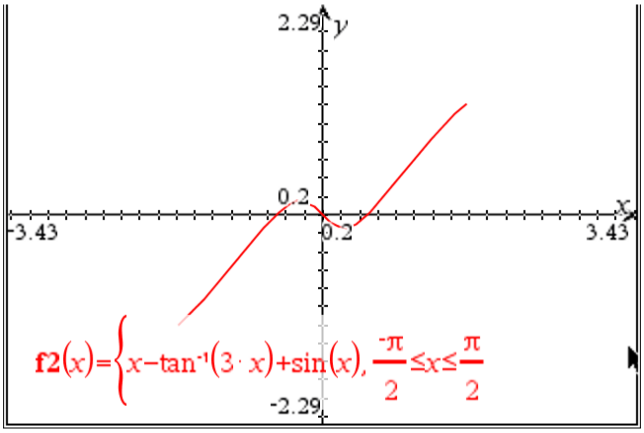
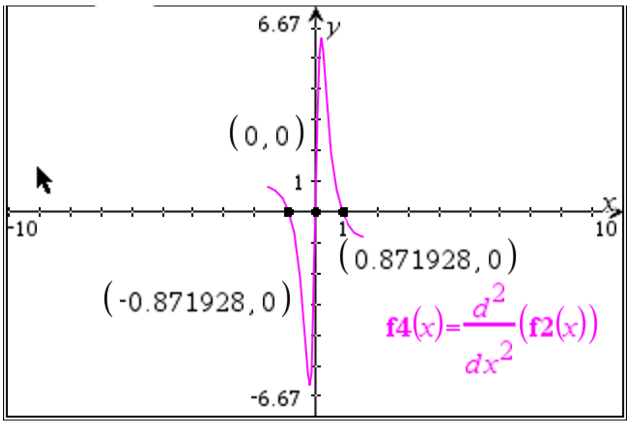
Section B

QN	Solution	Remarks
9a	$f(-x) = -x - \arctan(-3x) + \sin(-x)$ $= -x + \arctan 3x - \sin x$ $= -\left(x - \arctan 3x + \sin x\right)$ $= -f(x)$ OR  From graph, f(x) is symmetrical about the origin.	Note: A function that is not even does <u>not</u> imply that it must be odd. Recall: Odd function: f(x) = -f(-x) Even function: f(x) = f(-x)
9b	 The inflexion points are when f''(x) = 0. OR By differentiation, $f''(x) = -\sin x + \dfrac{54x}{(1+9x^2)^2}$ Equate $f''(x) = 0$, then $\sin x = \dfrac{54x}{(1+9x^2)^2}$.	Recall: Inflexion points for functions of degree ≤ 3 are at f''(x) = 0 OR at the turning points of f'(x). Tip: Use GDC to graph f''(x) by using the $\frac{d^2}{dx^2}(\ \ )$ template. Then find the zeros to solve f''(x) = 0.

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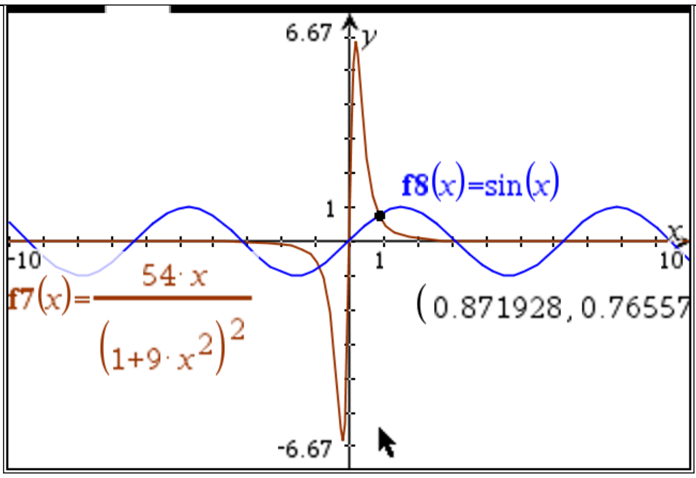

x-coordinates of inflexion points: x = -0.8719, 0, 0.8719

f2(-0.8719) -0.43181763
f2(0) 0
f2(0.8719) 0.43181763

Coordinates: (-0.872, -0.432), (0,0), (0.872, 0.432)
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<td>9c</td>
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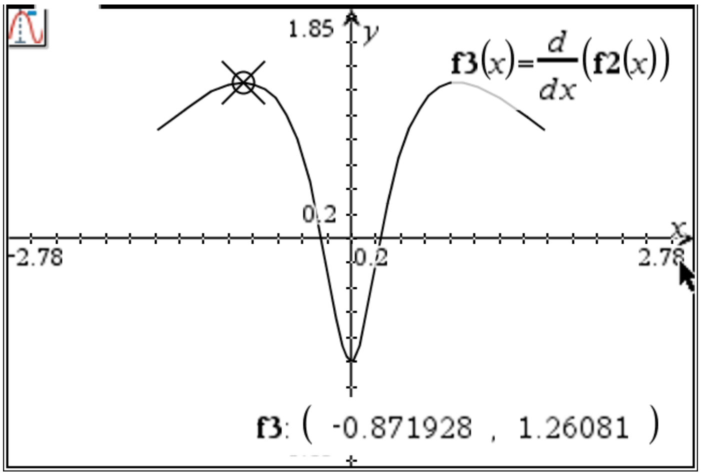

OR

$$f'(x) = 1 + \cos x - \frac{3}{(1+9x^2)^2}$$

Since $f'(-0.872) \neq 0$, $f'(0) \neq 0$, $f'(0.872) \neq 0$, they are all non-stationary points of inflexion.
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Recall:
Non-stationary point, x occurs when the gradient $f'(x) \neq 0$.

Hence non-stationary inflexion point, x occurs when $f'(x) \neq 0$ and $f''(x) = 0$.
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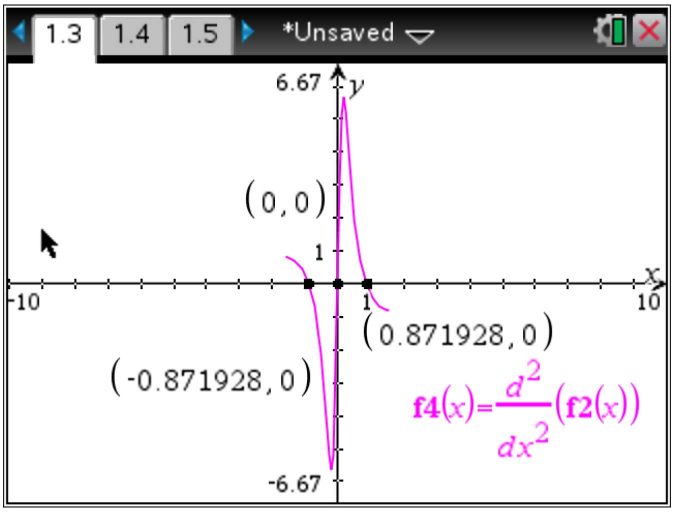
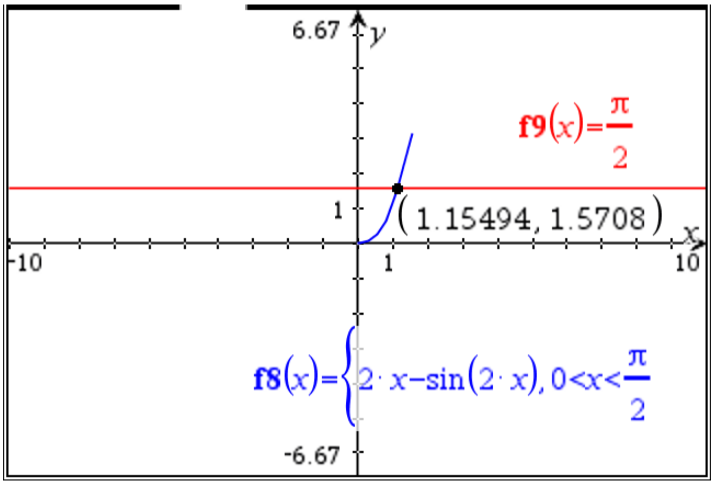
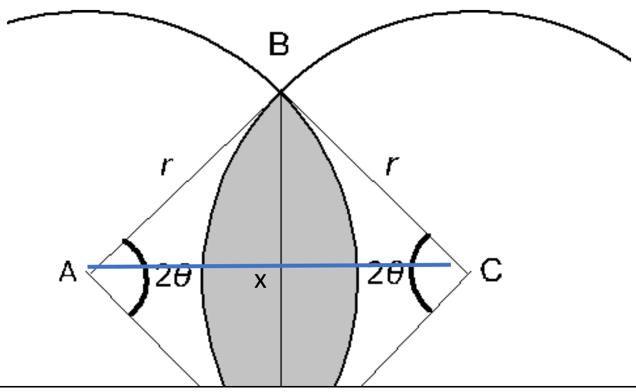
9d	 f(x) is concave up when f''(x) > 0. Hence, $-\frac{\pi}{2}<x<-0.872$ or $0<x<0.872$.	Recall: The inflexion points mark a change in curvature. f''(x) > 0, concave upwards f''(x) < 0, concave downwards
10a	$S=2\left[\frac{1}{2}r^2 2\theta-\frac{1}{2}r^2\sin 2\theta\right]$ $=2\theta-\sin 2\theta$	Question is well done.
10b	$S=(2\theta-\sin 2\theta)=\frac{1}{2}\pi r^2=\frac{\pi}{2}$ $2\theta-\sin 2\theta=\frac{\pi}{2}$  θ=1.15 rad	Use the GDC to solve.
10c		Note: Since both circles are equal and the diagram is symmetrical, the problem can be simplified by just considering one half of the figure. Ie. Consider triangle ABD expanding as X approaches A.

	Since AC decreases at a rate of 2 cm/min, $\frac{dAX}{dt}=-1$ From triangle ABX, $\cos\theta=\frac{adj}{hyp}=\frac{AX}{1}$ Differentiating with respect to time $\frac{d}{dt}\cos\theta=\frac{d}{dt}\frac{AX}{1}$ $-\sin\theta\frac{d\theta}{dt}=\frac{dAX}{dt}$ $\frac{d\theta}{dt}=\frac{-1}{\sin\theta}\times(-1)=\frac{1}{\sin\theta}$ OR $\cos\theta=\frac{adj}{hyp}=\frac{AX}{1} \quad \Rightarrow AX=\cos\theta$ $\frac{dAX}{d\theta}=-\sin\theta$ $\frac{d\theta}{dt}=\frac{\frac{dAX}{dt}}{\frac{dAX}{d\theta}}=\frac{-1}{-\sin\theta}=\frac{1}{\sin\theta}$	
10d	$S=(2\theta-\sin 2\theta)$ $\frac{dS}{d\theta}=(2-2\cos 2\theta)=2(1-\cos 2\theta)$ $\frac{dS}{dt}=\frac{dS}{d\theta}\times\frac{d\theta}{dt}=2(1-\cos 2\theta)\times\frac{1}{\sin\theta}$ $=2r(1-\cos 2\theta)\times\frac{1}{\sin\theta}$ $=2\frac{1-1+2\sin^2\theta}{\sin\theta}=4\sin\theta$	Done well. Recall: $\cos 2\alpha=1-2sin^2\alpha$
11a	At the x-z plane, y = 0 $\begin{pmatrix}1+2t\\-t\\-1+t\end{pmatrix}=\begin{pmatrix}x\\0\\z\end{pmatrix} \quad \Rightarrow t=0$ Coordinates of the point: $\begin{pmatrix}1\\0\\-1\end{pmatrix}$	Done well.
11b	$\begin{pmatrix}2\\-1\\1\end{pmatrix}\neq k\begin{pmatrix}1\\2\\-1\end{pmatrix}$ Hence the lines are not parallel.	Remember to check if the lines are parallel. Note: There is no need to find the point of intersection of the question doesn't specifically asks for it.

	$\begin{pmatrix}1+2t\\-t\\-1+t\end{pmatrix}=\begin{pmatrix}2+s\\-3+2s\\1-s\end{pmatrix}$ From equations 2 and 3, $-1=-2+s \quad \Rightarrow s=1$ $-t=-3+2(1) \quad \Rightarrow t=1$ Substituting into equation 1, LHS = $1+2(1)=3$ RHS = $2+1=3$ The lines are intersecting.	
11c	Let F be the foot of the perpendicular from P to L_1. $\overrightarrow{OF}=\begin{pmatrix}1+2t\\-t\\-1+t\end{pmatrix}$ $\overrightarrow{PF}=\begin{pmatrix}1+2t\\-t\\-1+t\end{pmatrix}-\begin{pmatrix}-17\\9\\9\end{pmatrix}=\begin{pmatrix}18+2t\\-9-t\\-10+t\end{pmatrix}$ $\begin{pmatrix}18+2t\\-9-t\\-10+t\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$ $36+4t+9+t-10+t=0$ $6t+35=0$ $t=\frac{-35}{6}$ $\overrightarrow{OF}=\begin{pmatrix}1+2\left(\frac{-35}{6}\right)\\-\left(\frac{-35}{6}\right)\\-1+\left(\frac{-35}{6}\right)\end{pmatrix}=\begin{pmatrix}\frac{-32}{3}\\\frac{35}{6}\\\frac{-41}{6}\end{pmatrix}$ Let *P'* be the image of *P* reflected. $\overrightarrow{OP'}=\overrightarrow{OP}+2\overrightarrow{PF}$ [By vector addition.] $=\begin{pmatrix}-17\\9\\9\end{pmatrix}+2\begin{pmatrix}18+2t\\-9-t\\-10+t\end{pmatrix}$ $=\begin{pmatrix}-17\\9\\9\end{pmatrix}+2\begin{pmatrix}18-\frac{35}{3}\\-9+\frac{35}{6}\\-10-\frac{35}{6}\end{pmatrix}=\begin{pmatrix}-\frac{13}{3}\\\frac{8}{3}\\-\frac{68}{3}\end{pmatrix}$	Some well confused with the method used for finding the foot of perpendicular from point to plane. Recall that both methods of solving are very different. Recall that the foot of the perpendicular from point to line, is the intersection point of the given line with a perpendicular line passing through the given point.

11d	$\underset{\sim}{n} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ -24 \end{pmatrix} = 4\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$ $\pi : \underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 12$	Most were able to prove this.
11e	$\theta = 90^\circ - \cos^{-1} \dfrac{\left\lvert \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \right\rvert}$ $= 90^\circ - \cos^{-1}\left(\dfrac{2}{\sqrt{6}\sqrt{49}} \right) = 6.70^\circ$	Common mistake: Taking the angle as lying between the normal of the plane and the direction vector of the line is incorrect. Ie $\theta = \cos^{-1} \dfrac{\left\lvert \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \right\rvert}$ (incorrect)
11f	$\mathrm{rref}\left(\begin{bmatrix} 3 & 2 & -6 & 12 \\ 1 & 2 & -1 & -3 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & \frac{-5}{2} & \frac{15}{2} \\ 0 & 1 & \frac{3}{4} & \frac{-21}{4} \end{bmatrix}$ From GDC $\begin{matrix} x - \frac{5}{2}z = \frac{15}{2} \\ y + \frac{3}{4}z = -\frac{21}{4} \end{matrix}$, $x = \frac{15}{2} + \frac{5}{2}t$ $y = -\frac{21}{4} - \frac{3}{4}t, \quad t \in \mathbb{R}$ $z = t$	Tip: The line of intersection are the solutions to the system of 2 linear equations. Use GDC to solve.