Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2019

YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS

HIGHER LEVEL

Paper 1 **SOLUTIONS**

SECTION A

Qn	Solution	
1.	Find the value of k if $$\sum_{r=0}^{\infty} \frac{k}{2^r} = 5.$$	
	$\sum_{r=0}^{\infty} \frac{k}{2^r} = k\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$	Expand to find the terms
	$= k\left(\frac{1}{1-\frac{1}{2}}\right) = 2k$	GP, Sum to infinity: a = k , r = ½
	$2k = 5$ => $k = 2.5$	
	Remarks: Well attempted	
2.	Consider the equation $(h-1)x^2 - xh + (h+1) = 0$, where h is a real constant. Find the set of values of *h* for which this equation has real roots.	

Qn	Solution
	Since the equation has real roots, $b^2-4ac\geq 0$ $h^2-4(h-1)(h+1)\geq 0$ $h^2-4h^2+4\geq 0$ $h\leq\frac{4}{3}$ $-\frac{2}{\sqrt{3}}\leq h\leq\frac{2}{\sqrt{3}}$
	Remarks: Well attempted. Common mistake: b^2-4ac > 0 applies only for real and equal (repeated) roots.
3. a	The sum of the first *n* terms of a sequence {u_n} is given by $S_n=n^2+r^2n-\frac{3}{4}n$, where $n\in Z^+$ and $r\in R$. Write down the value of u_1 in terms of *r*.
	$u_1=S_1=1^2+1^2n-\frac{3}{4}=\frac{1}{4}+r^2$ sum of one term = first term.
b	Prove that {u_n} is an arithmetic sequence, stating clearly its common difference.
	$u_n=S_n-S_{n-1}=n^2+r^2n-\frac{3}{4}n-(n-1)^2-r^2(n-1)+\frac{3}{4}(n-1)$ $=\left(n^2+r^2n-\frac{3}{4}n\right)-(n^2-2n+1)-r^2n+r^2+\frac{3}{4}n-\frac{3}{4}$ $=-\frac{7}{4}+2n+r^2$ $u_n-u_{n-1}=\left(-\frac{7}{4}+2n+r^2\right)-\left(-\frac{7}{4}+2(n-1)+r^2\right)$ $=2=$ common difference The common difference is a constant, hence {u_n} is an arithmetic sequence
	Alternative solution is to show that Sn satisfy the equation for sum of an AP where $S_n=\frac{n}{2}[u_1+u_n]$ Ie. $S_n=n^2+r^2n-\frac{3}{4}n=\frac{n}{2}\left(2n+2r^2-\frac{3}{2}\right)=\frac{n}{2}\left(\left(\frac{1}{4}+r^2\right)+\left(-\frac{7}{4}+2n+r^2\right)\right)$
	Remarks: Common mistake, students only show $U_3-U_2=U_2-U_1=2$ which shows that only U_1, U_2 and U_3 are in AP.

Qn	Solution
4.	Use Mathematical Induction to prove that $(1+x)^n \geq 1+nx$ for $\{n: n \in Z^+\}$ where $x > 1$.
	Let Pn be the proposition: $(1+x)^n \geq 1+nx$ for $\{n: n \in Z^+\}$ where $x > 1$. To prove P1 is true: LHS = $1+x$ RHS = $1+x$ Hence P1 is true. Assume Pn is true for some $n = k$, $(1+x)^k \geq 1+kx,\ k \in Z^+$ To prove Pk+1 is true: $(1+x)^{k+1} \geq 1+(k+1)x$ LHS $= (1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x)$ Sub. Pk+1 $= 1+x+kx+kx^2$ Expand RHS terms $\geq 1+x+kx$ (Since $x > 1$, $kx^2 \geq 0$) Justification for the inequality $= 1+(k+1)x$ = RHS Since P1 is true and Pk is true, implies Pk+1 is true, by Mathematical Induction $(1+x)^n \geq 1+nx$ for $\{n: n \in Z^+\}$ where $x > 1$ is true.
	Remarks: Most students understood the method of proof using MI but a number of students do not write the statements or the proof properly. A number of students did not prove that P_1 is true but proved that P_2 is true instead.
5. a	Express the binomial coefficient $\binom{n+1}{n-2}$ as a polynomial in *n*.
	$\binom{n+1}{n-2} = \frac{(n+1)!}{(n-2)!3!}$ Note: Given in the formula booklet, $\binom{n}{n} = \frac{n!}{(n-r)!r!}$ $= \frac{(n+1)(n)(n-1)}{6} = \frac{(n^2-1)n}{6}$ $= \frac{1}{6}(n^3-n)$
b	Hence find the coefficient of x^{5-n} in the expansion of $\left(x+\frac{1}{x}\right)^{n+1}$ in terms of n.

Qn	Solution
	$T_{r+1} = \binom{n+1}{r} x^{n+1-r} x^{-r} = \binom{n+1}{r} x^{n+1-2r}$ General term formula (not in formula booklet. Memorize this.) $x^{n+1-2r} = x^{5-n}$ $n + 1 - 2r = 5 - n$ Equating the index of x $2n - 4 = 2r$ $r = n - 2$ Sub. *r* into $\binom{n+1}{r}$ Coefficient = $\binom{n+1}{n-2} = \frac{1}{6}(n^3 - n)$ Use 5a. to solve this part.
	Remarks: Well attempted.
6. a	Show that $log_{r^2} x = \frac{1}{2}\log_r x$.
	$log_{r^2} x = \frac{log_r x}{log_r r^2}$ Change of base $= \frac{1}{2} log_r x$
b	Hence express $y = log_3 x + log_9 x$ in terms of $ln(x)$ and describe the transformation that will map $y = ln(x)$ into $y = log_3 x + log_9 x$.
	$y = log_3 x + log_9 x = log_3 x + \frac{1}{2} log_3 x$ Using result in (a) $= \frac{3}{2} log_3 x$ $= \frac{3}{2ln3} lnx$ Stretch parallel to y-axis, by factor $\frac{3}{2ln3}$.
	Remarks: well attempted.
7. a	Consider the equations of the lines: $l_1: \ \mathbf{r} = \begin{pmatrix} -11 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ and $l_2: \ \frac{x+2}{3} = y - 1 = 1 - z$. Determine whether or not, l_1 and l_2 intersect.

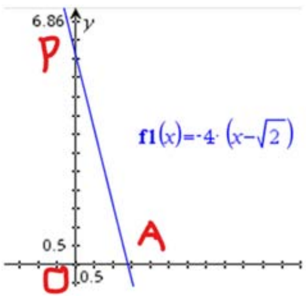
Qn	Solution
	Check if lines are parallel: $\begin{pmatrix}3\\-1\\1\end{pmatrix} \neq k\begin{pmatrix}3\\1\\-1\end{pmatrix}$ The direction vectors are not parallel, l_1 and l_2 are **not parallel**. Check if lines intersects by equating the two lines: $\begin{pmatrix}-11+3\lambda\\2-\lambda\\\lambda\end{pmatrix}=\begin{pmatrix}-2+3\mu\\1+\mu\\1-\mu\end{pmatrix}\Rightarrow\begin{pmatrix}-9+3\lambda=3\mu\\1-\lambda=\mu\\-1+\lambda=-\mu\end{pmatrix}$ Equating the first two equations and solving: $-9+3\lambda=3\mu$ and $1-\lambda=\mu$ $-6=6\mu\Rightarrow\mu=-1$ or $\lambda=2$ Check for consistency of the values: Sub. into third equation, LHS = $\lambda=2$ and RHS = $1-\mu=2$ Since values of λ and μ are consistent, sub either values into the line equation to find the point of intersection. Point of intersection (-5, 0, 2)
b	Find a vector that is perpendicular to the two lines.
	Since the two lines intersects, they lie on the same plane. A vector perpendicular to the two lines is the normal vector of that plane. Normal Vector = $\begin{pmatrix}3\\1\\-1\end{pmatrix}\times\begin{pmatrix}3\\-1\\1\end{pmatrix}$ Cross product of the dir. vectors $=\begin{pmatrix}1-1\\-(3+3)\\-3-3\end{pmatrix}=\begin{pmatrix}0\\-6\\-6\end{pmatrix}$ Also accept any other any vector parallel to $\begin{pmatrix}0\\1\\1\end{pmatrix}$.
	Remarks: well attempted.

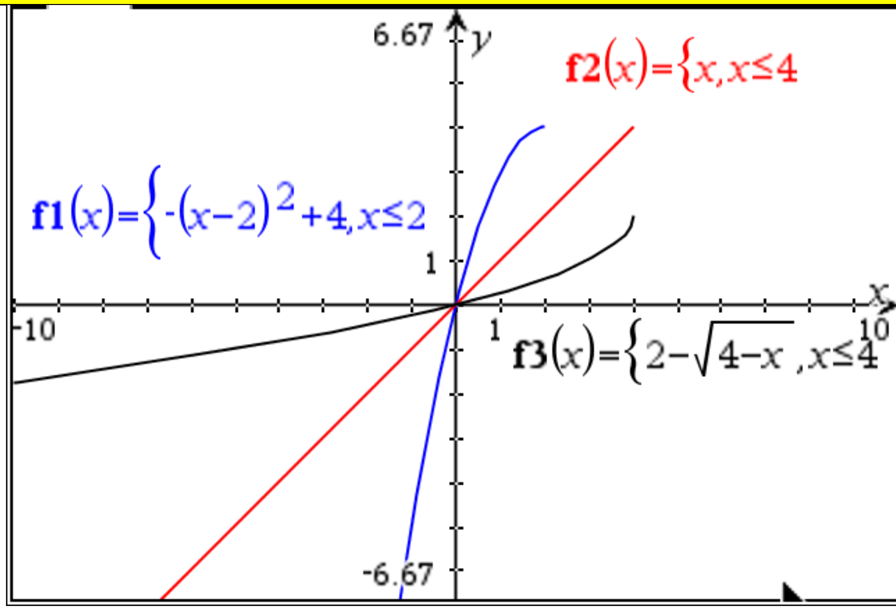
SECTION B

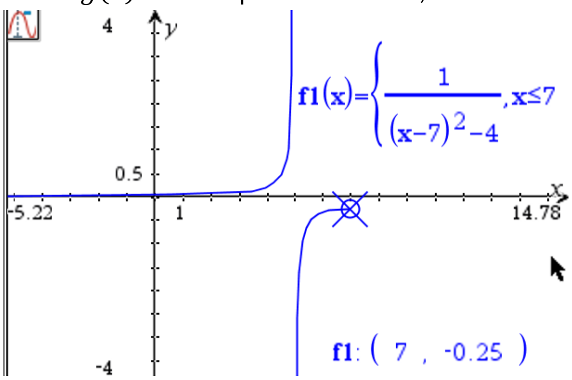
Qn	Solution
8. (i)	
a	Expand and simplify $(x-1)(x^5+x^4+x^3+x^2+x+1)$.
	$(x-1)(x^5+x^4+x^3+x^2+x+1)$ Expand the terms. $=x^6+x^5+x^4+x^3+x^2+x-x^5-x^4-x^3-x^2-x-1$ $=x^6-1$
b	Given that *w* is a root of the equation $z^6-1=0$ which does not lie on the real axis in the Argand diagram, show that $w^5+w^4+w^3+w^2+w+1=0$.
	$z^6-1=(z-1)(z^5+z^4+z^3+z^2+z+1)=0$ From 8ai, let *z* = *x*. $z-1=0 \quad \text{or} \quad z^5+z^4+z^3+z^2+z+1=0$ Since *w* is a root which does not lie on the real axis, $w\neq 1$. Hence, $w^5+w^4+w^3+w^2+w+1=0$.
	Remarks: Some students were not able to see the link from **a)** to **b)**.
c	Find the roots of the equation $z^6=1$, giving your answers exactly in the form $re^{i\theta}$, where $-\pi<\theta\leq\pi$.
	$z^6=1=e^{i(0+2k\pi)}$ Note that $r=\lvert 1\rvert=1$ and $\arg(1)=0$. $z=e^{i\frac{2k\pi}{6}}=e^{i\frac{k\pi}{3}},\ k=0,\pm1,\pm2,3$ $z=e^0,e^{i\frac{\pi}{3}},e^{-i\frac{\pi}{3}},e^{i\frac{2\pi}{3}},e^{-i\frac{2\pi}{3}},e^{i\pi}$ Alternatively, $z^6=1=\cos(0+2k\pi)+i\sin(0+2k\pi)$ $z=\cos\frac{2k\pi}{6}+i\sin\frac{2k\pi}{6}, \quad k=0,\pm1,\pm2,3$ $z=\cos 0+i\sin 0,\ \cos\frac{\pi}{3}+i\sin\frac{\pi}{3},\cos\frac{\pi}{3}-i\sin\frac{\pi}{3},\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3},\ \cos\pi+i\sin\pi$ Convert from Trigo to exponential form using $z=\cos\theta+i\sin\theta=e^{i\theta}$ $z=e^0,e^{i\frac{\pi}{3}},e^{-i\frac{\pi}{3}},e^{i\frac{2\pi}{3}},e^{-i\frac{2\pi}{3}},e^{i\pi}$
	Remarks: Some found the wrong arg(1) or *r*. Use **arg(z)** for principal argument (and not **A**rg(z)). Sketch an Argand diagram if necessary to find these values accurately. Students who solved using trigo form were les successful and showed lengthy workings.
(ii)	Consider the complex numbers $z_1=1-\sqrt{3}i$ and $z_2=2-2i$
a	Find the exact value of the argument of $\frac{z_1}{z_2}$.

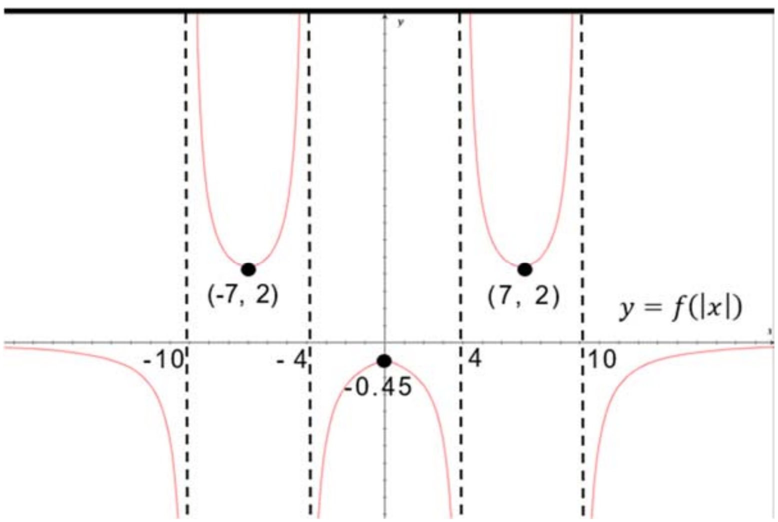
Qn	Solution
	$z_1 = 1-\sqrt{3}i$ is in the 4th <u>quadrant</u>, $$\arg(z_1) = -tan^{-1}\left\|\frac{\sqrt{3}}{1}\right\| = -\frac{\pi}{3}$$ $z_2 = 2-2i$ is in the 4th <u>quadrant</u>, $$\arg(z_2) = -tan^{-1}\left\|\frac{2}{2}\right\| = -\frac{\pi}{4}$$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = -\frac{\pi}{3} + \frac{\pi}{4} = -\frac{\pi}{12}$
	Remarks: Knowledge of the trigo ratios of the basic angles is required. Some students were unable to locate the quadrant that the complex number is in.
b	Express $\frac{z_1}{z_2}$ in the form $a+ib$, where $a, b \in R$.
	$\frac{z_1}{z_2} = \frac{1-\sqrt{3}i}{2-2i} \times \frac{2+2i}{2+2i}$ Multiply by the complex conjugate of the denominator. $= \frac{1}{8}\left((1-\sqrt{3}i)(2+2i)\right)$ $= \frac{1}{4}(1+\sqrt{3}) + \frac{1}{4}(1-\sqrt{3})i$
c	Using your results, find the exact value of $tan\frac{-\pi}{12}$, giving your answer in the form $\sqrt{a}-b$, $a, b \in Z^+$.
	From a) and b), $\frac{z_1}{z_2} = cos\left(-\frac{\pi}{12}\right) + isin\left(-\frac{\pi}{12}\right) = \frac{1}{4}(1+\sqrt{3}) + \frac{1}{4}(1-\sqrt{3})i$ Equating real part, $cos\left(-\frac{\pi}{12}\right) = \frac{1}{4}(1+\sqrt{3})$ and equating imaginary part, $sin\left(-\frac{\pi}{12}\right) = \frac{1}{4}(1-\sqrt{3})$. Hence, $tan\frac{-\pi}{12} = \frac{sin\frac{-\pi}{12}}{cos\frac{-\pi}{12}} = \frac{\frac{1}{4}(1-\sqrt{3})}{\frac{1}{4}(1+\sqrt{3})}$ $= \frac{1-\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$ $= \sqrt{3}-2$
	Remarks: Solving 8c is depended solving 8a and 8b correctly. This question requires the use of the previous results, hence solving for $tan\frac{-\pi}{12}$ will require establishing the link to 8a and 8b.
9ai	The curve *C* is defined by equation $2x^2 - y^2 += 4$. Find $\frac{dy}{dx}$ in terms of *x* and *y*.
	$2x^2 - y^2 + xy = 4$ Differentiating wrt x, (Implicit differentiation) $4x - 2y\frac{dy}{dx} + [x\frac{dy}{dx} + y] = 0$ Chain rule and product rule $\frac{dy}{dx} = \frac{4x+y}{2y-x}$

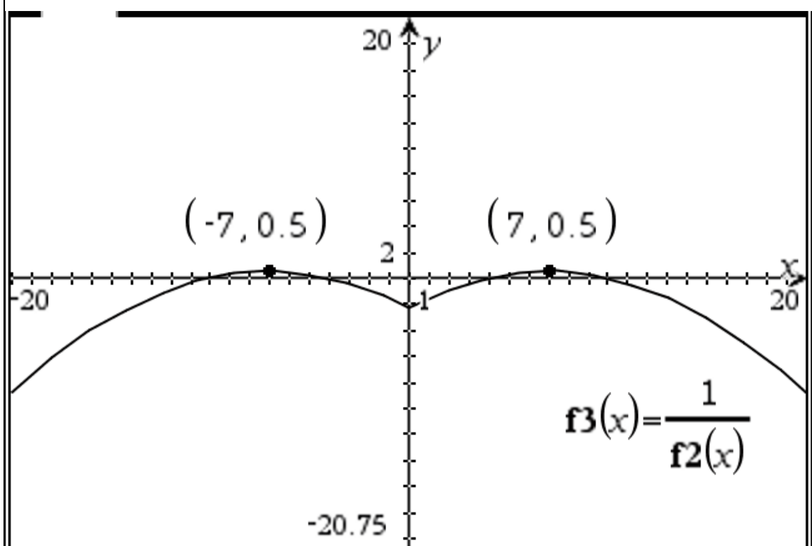
Qn	Solution
	Remarks: A number of students differentiated the RHS wrongly. Ie $\frac{d}{dx}4 = 4$. Some students did not apply product rule correctly for $\frac{d}{dx}xy$.
aii	Explain clearly whether *C* has any turning point/s.
	At turning point, $\frac{dy}{dx} = \frac{4x+y}{2y-x} = 0$ Hence $4x + y = 0$ Ie. Any turning point (x,y) must not only satisfy the equation of the function but also this equation. Sub. $y = -4x$ into the equation of the original function. $2x^2 - (-4x)^2 + x(-4x) = 4$ $2x^2 - 16x^2 - 4x^2 = 4$ $x^2 = -\frac{2}{9}$ Since $x^2 > 0$, there is no real solution, hence C has no turning point.
	Remarks: Most were unable to explain why there are no turning point clearly.
aiii	Find the equation of the tangent to the curve *C* at the point $P(\sqrt{2}, 0)$.
	At $P(\sqrt{2}, 0)$, Gradient of tangent = $\frac{4\sqrt{2}}{-\sqrt{2}} = -4$ Equation of tangent: $y - 0 = -4(x - \sqrt{2})$
	Remarks: For this question, taking the equation of the tangent as $y - y_1 = m(x - x_1)$ is a better method that finding the y-intercept for $y = mx + c$.
aiv	The tangent to the curve *C* at the point $P(\sqrt{2}, 0)$ meets the y-axis at the point *A*. Find the area of triangle *PAO*.

Qn	Solution
	The tangent intersects the y-axis at $A(0, 4\sqrt{2})$. [Diagram not required]  Area, POA, formed is a triangle = $\frac{1}{2}(4\sqrt{2}) \times \sqrt{2} = 4\ units^2$
bi	Consider the function f defined by $f(x) = e^x cosx,\ 0 \le x < \pi$. Find an expression for $f'(x)$.
	Differentiating $f(x) = e^x cosx$ $\frac{dy}{dx} = e^x(-sinx) + e^x(cosx)$ Product rule $= -e^x(sinx) + e^x(cosx)$
bii	Express $f''(x)$ in the form $Ae^x sinx$, where A is a constant.
	$\frac{d^2y}{dx^2} = e^x(-cosx) - e^x sinx + e^x(-sinx) + e^x(cosx)$ Product rule $= -2e^x sinx$
biii	Find the coordinates of the point of inflexion/s of the graph of f. (The proof of inflexion is not required.)
	At inflexion, $\frac{d^2y}{dx^2} = 0 = -2e^x sinx$ $2e^x \neq 0$ hence $sinx = 0$ $x = 0$ (since $0 \le x < \pi$) Only one value of x for that domain. Coordinates $(0, e^0(cos0)) = (0,1)$
	Remarks: Part b was well attempted.
10	The functions f and g are defined by $f : x \mapsto -(x-2)^2 + 4,\ x \in \mathbb{R},\ x \le 2,$ $g : x \mapsto \dfrac{1}{(x-7)^2 - 4},\quad x \in \mathbb{R},\ x \le k,\ x \neq 5.$ Sketch the graphs of f, f^{-1} and ff^{-1} on the same axes, showing the relationship between the three graphs and their domains.

Qn	Solution
ai	 **As seen from graph, domain of f is x≤2 and domain of f^{-1} and ff^{-1} are both x≤4.**
	Remarks: A number of students were not able to sketch the quadratic function $y = -(x-2)^2 + 4$. Most also did not take into account the domain of $x \leq 2$. It is clear from the question that the inverse function of f(x) exists hence f(x) is a one to one function. The inverse function of f(x) is found by reflecting f(x) about the line y = x. Students who realised this relation were able to sketch f^{-1}easily. There were a number of students who attempt to solve for the equation of f^{-1} into order to find ff^{-1}. This is not necessary as conceptually, the $ff^{-1}(x) = x$. This is as f^{-1} map x to y, which f will then map y back to x. The domain of ff^{-1} = domain of f^{-1}, as we apply the rule to find domain of the composite function.
aii	If $f(a) = f^{-1}(a)$, show that $a^2 - 3a = 0$.
	$f(a) = f^{-1}(a)$, hence $f(a) = a$. $-(a-2)^2 + 4 = a$ $-a^2 + 4a - 4 + 4 = a$ $a^2 - 3a = 0$ (Shown)
	Remarks: A number of students didn't know that $f(a) = f^{-1}(a) = a$ as a is the point of intersection of the graphs. Refer to the graph in 10ai) to see this relationship.
aiii	State the maximum value of *k* such that the inverse of g exists.
	The inverse of g exists if g can be restricted to a one-one function. Hence, k = 7, since the local maximum of g(x) is at x = 7.
aiv	Hence find the inverse of g, stating its domain.

Qn	Solution
	Let $y = g(x) = \frac{1}{(x-7)^2-4}$ $(x-7)^2 = 4 + \frac{1}{y}$ $x = 7 \pm \sqrt{4+\frac{1}{y}} = 7 - \sqrt{4+\frac{1}{y}}$ Since $x \leq 7$ (reject the -ve) $g^{-1}(x) = 7 - \sqrt{4+\frac{1}{x}}, \quad x \leq -\frac{1}{4}$ or $x > 0$
	Remarks: A number of students didn't realize that there are two possible set of values for x. Ie $x = 7 \pm \sqrt{4+\frac{1}{y}}$. Some also rejected the wrong function. The domain of the inverse function is found using the range of the original function. Since $g(x)$ is a reciprocal function,  range of g(x): $y \leq -\frac{1}{4}$ or $y > 0$, hence domain of $g^{-1}(x)$: $x \leq -\frac{1}{4}$ or $x > 0$.

Qn	Solution
bi	A rational function is defined by $f(x) = \frac{a}{(x-b)(x-c)}$ where the parameters $b, c > 0$ and a, b and $c \in Z$. The following diagram represents the graph of $y = f(\lvert x \rvert)$. $f(\lvert x \rvert)$ has a maximum point (0, -0.45) and minimum points (-7, 2) and (7, 2). The horizontal asymptote of $f(\lvert x \rvert)$ is $y = 0$ and the vertical asymptotes of $f(\lvert x \rvert)$ are $x = -10$, $x = -4$, $x = 4$ and $x = 10$.  Using the information on the graph, find the values of *a*, *b* and *c*.
	Read off asymptotes from the graph, $b = 4$ and $c = 10$ (or $b = 10$ and $c = 4$) Sub. (7, 2) to find $a = 2(7-4)(7-10) = -18$
	Remarks: Students who couldn't solve this question did not realize the significance of the denominator in finding the vertical asymptotes for sketching.
bii	Sketch the graph of $\frac{1}{f(\lvert x \rvert)}$.

Qn	Solution
	To find the graph of $\frac{1}{f(\lvert x \rvert)}$, we make use of the graph of $f(\lvert x \rvert)$ and the transformation $(x, y) \rightarrow \left(x, \frac{1}{y}\right)$. The vertical asymptotes $x = -10$, $x = -4$, $x = 4$ and $x = 10$ becomes the x-intercepts. y-int $\left(0, \frac{-20}{9}\right)$ Max points: $\left(7, \frac{1}{2}\right)$ and $\left(-7, \frac{1}{2}\right)$ 
	Remarks: Most were unable to get the correct shape of the function. Note that the y-intercept should be a sharp point as this is a $\lvert x \rvert$ function.

HL Math Paper2 Review, Group 5(II)

23 Oct, Wednesday 1340 – 1500, LT1

***Please bring your GDC, exam scripts and writing materials.**