Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2020

YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS

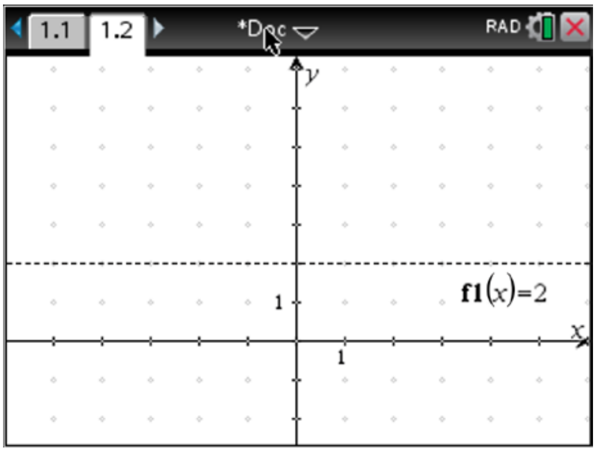
HIGHER LEVEL

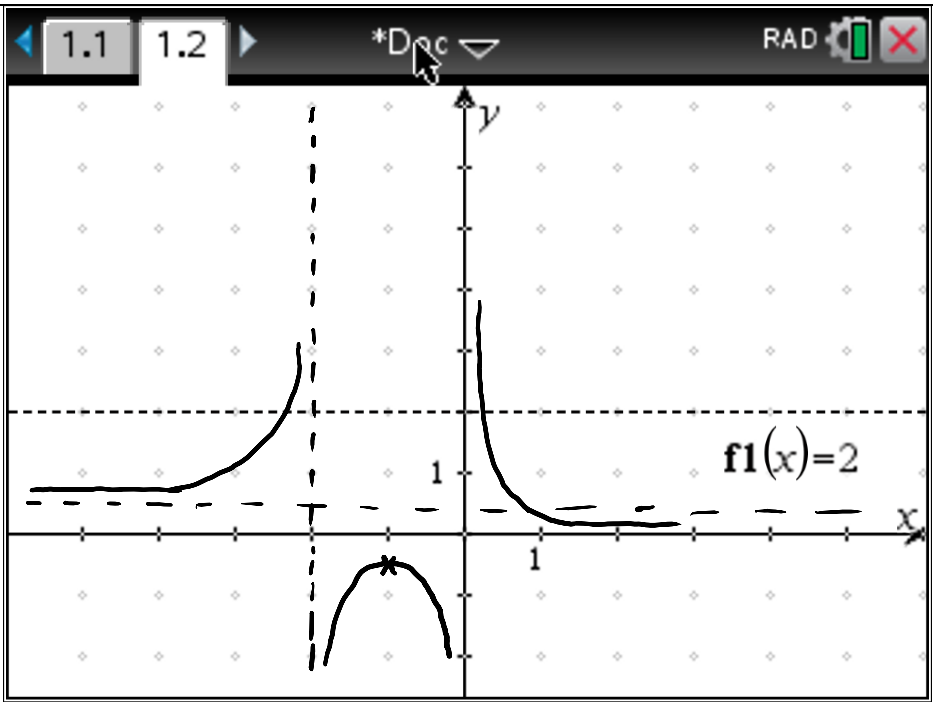
Paper 1 **SOLUTIONS**

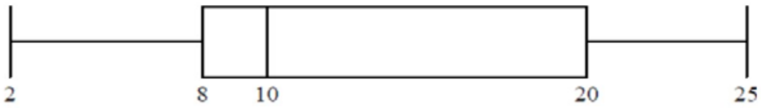
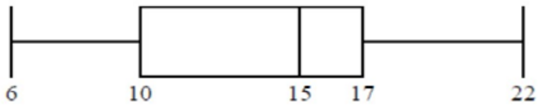
SECTION A

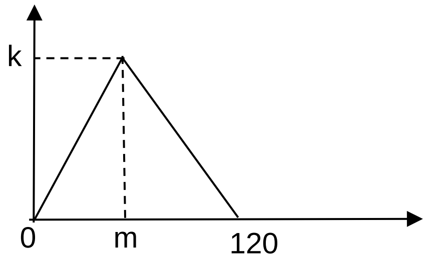
	Solution	
1	The lengths of two sides of a triangle are 4 cm and 5 cm. Let θ be the angle between the two given sides. The area of the triangle is $\frac{5\sqrt{15}}{2}$ cm^2. (a) Show that $\sin\theta = \frac{\sqrt{15}}{4}$ (b) Find the two possible values of the length of the third side.	Maximum [7 marks]
	$\frac{1}{2}\times 4\times 5\times \sin\theta = \frac{5\sqrt{15}}{2}$ $\sin\theta = \frac{\sqrt{15}}{4}$	Well done
	Let x be the length of the third side. $x^2 = 4^2 + 5^2 - 2(4)(5)\cos\theta$	Most did not realise 2 values for cosine

	$\cos\theta = \dfrac{\pm\sqrt{4^2 - \left(\sqrt{15}\right)^2}}{4} = \pm\dfrac{1}{4}$ $x^2 = 4^2 + 5^2 - 2(4)(5)\left(\pm\dfrac{1}{4}\right)$ $x^2 = 31 \text{ or } 51$ $x = \sqrt{31} \text{ or } \sqrt{51}$	
2	9, m, n are consecutive terms of a geometric progression and 9, 2m and 3n are consecutive terms of an arithmetic progression. Given that $\mathrm{m} \neq \mathrm{n}$, find the value of m and of n.	Maximum [6 marks]
	GP: 9, m, n $\Rightarrow m = 3\sqrt{n}$ ------------ (1) AP: 9, 2m, 3n $\Rightarrow 2m = \dfrac{9+3n}{2}$ --------(2) Subst (1) into (2) gives $12\sqrt{n} = 9 + 3n$ $144n = 81 + 54n + 9n^2$ $n^2 - 10n + 9 = 0$ $(n-9)(n-1) = 0$ $n = 9,\ 1$ $m = 3\sqrt{9} = 9 \text{ (NA) }, 3$ Hence $m = 3,\ n = 1$	Well done but some introduce 'r' and 'd', getting lost in the manipulation
3	Given that $\sum_{k=3}^{22}[k + log_2(x)] = 170, x$, find the value of x .	Maximum [6 marks]
	$(3+\log_2 x)+(4+\log_2 x)+(5+\log_2 x)+\ldots+(22+\log_2 x)=170$ $(3+4+5+\ldots+22)+20\log_2 x = 170$	Some got number of terms wrong

	$\frac{20}{2}(3+22)+20\log_2 x=170$ $20\log_2 x=-80$ $\log_2 x=-4$ $x=2^{-4}=\frac{1}{16}$	while others did not realise $log_2 x$ is repeated 20 times.
4	$\alpha,\ \beta$ and γ are roots of the equation $z^3+7z^2+20z+50=0$. Without solving for $\alpha,\ \beta$ and γ find the polynomial of degree 3 whose roots are $\alpha-1,\ \beta-1$ and $\gamma-1$.	Maximum [6marks]
	$z^3+7z^2+20z+50=0$ $(y-\alpha+1)(y-\beta+1)(y-\gamma+1)=0$ $(y+1-\alpha)(y+1-\beta)(y+1-\gamma)=0$ Substitute $z=y+1$ into $z^3+7z^2+20z+50$ gives $(y+1)^3+7(y+1)^2+20(y+1)+50$ $=y^3+3y^2+3y+1+7y^2+14y+7+20y+20+50$ $=y^3+10y^2+37y+78$	Mostly well done except some used $z=y-1$. Many calculated the coeff one by one.
5	The following diagram shows the graph of $y=f(x)$ with x-intercepts at -2 and 0, minimum point (-1, -2) and asymptote $y=2$. 	Maximum [7 marks]

	(a) Sketch the graph of $\frac{1}{f(x)}$ in the same diagram above indicating the x-intercepts, turning point and asymptote(s) clearly. (b) If the domain of another function $g(x)$ is $[0,8]$ and its range is $[4,11]$, write down the domain and range of $\frac{1}{g(x)}$ whenever possible.	
	 2 vertical asymptotes x = -2, x = 0[1 mark] 1 horizontal asymptote y = 0.5 [1 mark] Local max point (-1, -0.5) and shape [2 mark] Shape of 2 branches [1 mark]	Fairly well done except for some missing out on the horizontal asymptote and the positions of the branches were inaccurate
	Domain [0,8] Range $\left[\frac{1}{11},\frac{1}{4}\right]$	Many gave range as $\left[\frac{1}{4},\frac{1}{11}\right]$
6	The box-and-whisker diagram below shows the amount of salt used in different types of snacks in 2 confectionery shops. Shop 1 has 240 snacks and Shop 2 has 320 snacks.	Maximum [5 marks]

	SHOP 1  Amount of salt per snack in grams SHOP 2  Amount of salt per snack in grams (i) State what the number 10 represents in Shop 1. (ii) Write down the approximate number of snacks whose amount of salt ranges from 2 to 20 grams in Shop 1. (iii) A health inspector visited the two shops to check the snacks in the shop. Snacks sold in the shop are required to have less than 10 grams of salt. State, giving a reason, which shop has more snacks that **do not** meet the requirement.	
	Median	Mode & mean were wrong answers
	$\frac{75}{100}\times 240$ $=180$	Well done
	Shop 2 has more snacks that do not meet the requirement because Shop 1 has 50% of 240 = 120 snacks while Shop 2 has 75% of 320 = 240 snacks what do not meet the requirement.	Fairly well done except some could not give concrete reasons
7	A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = \frac{1}{4}$ and $u_{n+1} = u_n - \frac{6}{(3n+5)(3n+8)}$, for $n \in Z^+$. Use the principle of mathematical induction to prove that for all $n \in Z^+$, $u_n = \frac{2}{3n+5}$.	Maximum [7 marks]
	Let Pn be the proposition: $u_n = \frac{2}{3n+5}, n \in Z^+.$ To prove P1 is true: LHS = $u_1 = \frac{1}{4}$	Well done although some started with the proposition

	RHS = $\frac{2}{3n+5} = \frac{1}{4}$ Hence P1 is true. Assume Pn is true for n = k, $$u_k = \frac{2}{3k+5}, k \in Z^+$$ To prove Pk+1 is true: $$u_{k+1} = \frac{2}{3(k+1)+5}$$ LHS = $u_{k+1} = u_k - \frac{6}{(3k+5)(3k+8)}$ $$= \frac{2}{3k+5} - \frac{6}{(3k+5)(3k+8)} = \frac{2}{3k+5}\left(1 - \frac{3}{3k+8}\right)$$ $$= \frac{2}{3k+5}\left(\frac{3k+5}{3k+8}\right) = \frac{2}{3(k+1)+5}$$ Since P1 is true and Pk is true, implies Pk+1 is true, thus for all $n \in Z^+, u_n = \frac{2}{3n+5}$	$u_{k+1} = u_k - \frac{6}{(3k+5)(3k+8)}$
8	The diagram shows the probability density function of a continuous random variable, X whose median is m.  (a) Find the value of k and of m. (b) Hence state the value of its mode.	Maximum [5 marks]

	Area of triangle = Total probability = $\frac{1}{2}\times 120\times k=1$ $k=\frac{1}{60}$	Well done.
	Since m is the median, $\frac{1}{2}\times m\times\frac{1}{60}=\frac{1}{2}$ $m=60$	
	Mode = 60	
9	Consider the following system of equations $2x+2y+6z=0$ $4x+5y+14z=4$ $2x-2y+(\alpha-2)z=\beta-12$ (a) Find the values of α and β for which the system has an infinite number of solution. (b) In the case where the number of solutions is infinite, find the general solution of the system of equations in parametric form.	Maximum [6 marks]
	$\begin{pmatrix}2 & 2 & 6 & 0\\ 4 & 5 & 14 & 4\\ 2 & -2 & \alpha-2 & \beta-12\end{pmatrix}\rightarrow\begin{pmatrix}2 & 2 & 6 & 0\\ 0 & 1 & 2 & 4\\ 0 & 4 & 8-\alpha & 12-\beta\end{pmatrix}$ $\rightarrow\begin{pmatrix}2 & 2 & 6 & 0\\ 0 & 1 & 2 & 4\\ 0 & 0 & \alpha & \beta+4\end{pmatrix}$ For infinite solutions, $\alpha=0$ and $\beta=-4$	Some equated the rows of zeros to the matrix that was not in REF. This qn should be attempted using row operations on the matrix.
	From $2x+2y+6z=0$ and $y+2z=4$ Let $z=t,\text{where } t\in\mathbb{R}$ Make y and x the subject respectively, $y=4-2t$ and $x=-4-t$	

SECTION B

10. [Maximum mark: 21]*

a) A function f is defined by $f(x)=\dfrac{2x-1}{3}$, $a \le x \le \dfrac{5}{4}$.

(i) Given that the range of f is $-1 \le f(x) \le \dfrac{1}{2}$, find the value of a.

(ii) Find $f^{-1}(x)$ and state its range.

Another function g is defined by $g(x)=\dfrac{x}{x-5}$, $x \in \mathbb{R}$, $x \neq 5$.

(iii) Find $fg(x)$, given that it exists.

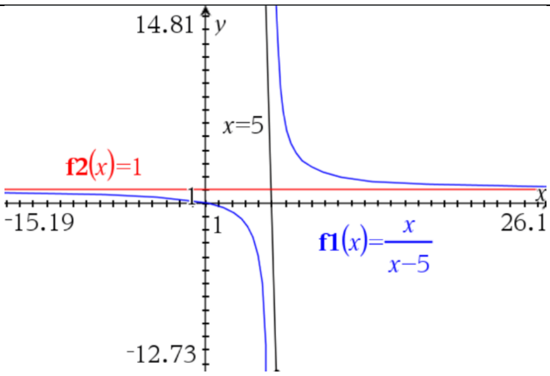
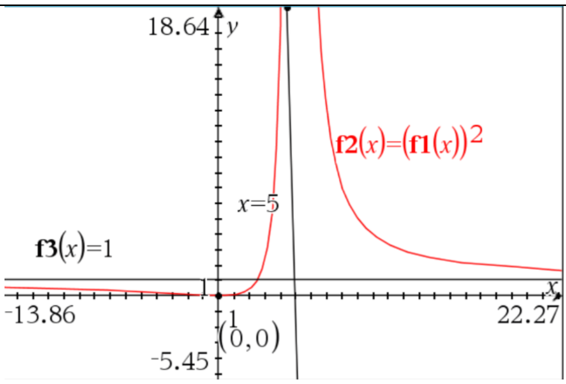
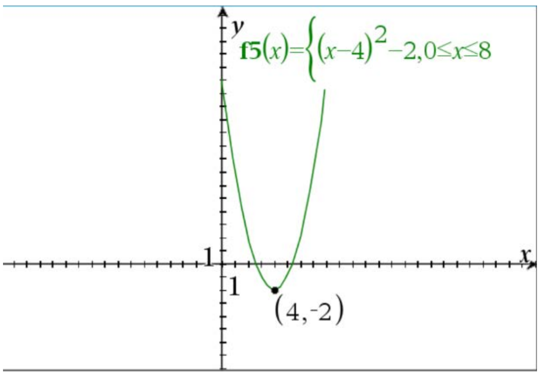
(iv) Sketch the graph of $g(x)$ clearly showing the asymptotes and any intercepts.

(v) Sketch on a separate clearly labeled diagram the graph of $y=\left(g(x)\right)^2$, showing any asymptotes, turning points and intercepts.

b) (i) Sketch the graph of $h(x)=x^2-8x+14,\ 0 \le x \le 8$, clearly showing the turning point of $h(x)$.

(ii) Describe, in order, three transformations that transforms the graph of $h(x)$ to the graph of $j(x)=-(x-2)^2+1, -2 \le x \le 6$..

ai	$f(x)=\frac{2x-1}{3}, a \le x \le \frac{5}{4}$ Solving $-1 \le f(x)$, $-1 \le \dfrac{2x-1}{3}$	Well done.
	$x \ge -1$ $a=-1$	
ii	Let $y=\frac{2x-1}{3}$ $3y=2x-1$ $3y+1=2x$ $x=\dfrac{3y+1}{2}$	
	$f^{-1}(x)=\dfrac{3x+1}{2}$	
	Range of $f^{-1}(x)$ = Domain of f(x) Range of $f^{-1}(x)$ = $-1 \le y \le \frac{5}{4}$	Common error: Writing range using x.
iii	$fg(x)=f\left(\dfrac{x}{x-5}\right)$	

	$$fg(x)=\frac{2\left(\frac{x}{x-5}\right)-1}{3}$$	
	$$fg(x)=\frac{\frac{2x-x+5}{x-5}}{3}$$	
	$$fg(x)=\frac{x+5}{3x-15}$$	
iv	 Horizontal asymptote at y =1 and Vertical asymptote at x = 5 Intercept (0, 0)	
v	 Shape Horizontal asymptote at y =1 Vertical asymptote at x = 5 Minimum point (0,0)	Poorly attempted.
bi	$y=x^2-8x+14=(x-4)^2-2$ (Complete the square)  Turning point (4, -2) Shape and domain	Most made the mistake of not restricting the function to the require domain. Read the question carefully, some gave much more information than necessary. Ie. Intercepts.
bii	$y=x^2-8x+14=(x-4)^2-2$	Poorly attempted. Students could

		not order the transformations correctly.
	A translation of 2 units in the negative x -direction $y = (x - 4 + 2)^2 - 2 = (x - 2)^2 - 2$	$f(x + 2)$
	A reflection about the x-axis $y = -[(x - 2)^2 - 2] = -(x - 2)^2 + 2$	$-f(x)$
	A translation of 1 unit in the negative y -direction $y = -(x - 2)^2 + 2 - 1 = -(x - 2)^2 + 1$	$f(x) - 1$

11. [Maximum mark: 14]

Given the function $f(x) = \frac{2}{x^2+6x+8}$,

(i) express $f(x)$ in partial fractions.

(ii) Hence show that the first three terms in the expansion of $f(x)$ in descending powers of x can be expressed by $\frac{2}{x^2} - \frac{12}{x^3} + \frac{56}{x^4}$.

[6]

(iii) State the range of values of x for which the expansion is valid.

(iv) Using the expansion of $f(x)$, evaluate $\frac{2}{168}$ corrected to 4 decimal places.

i)	$f(x) = \dfrac{2}{x^2 + 6x + 8} = \dfrac{2}{(x + 2)(x + 4)} = \dfrac{A}{x + 2} + \dfrac{B}{x + 4}$	Well done.
	Solving: $A = \frac{2}{(-2+4)} = 1$, $B = \frac{2}{(-4+2)} = -1$	
	$f(x) = \dfrac{1}{x + 2} - \dfrac{1}{x + 4}$	
aii	$f(x) = \dfrac{1}{x + 2} - \dfrac{1}{x + 4} = (x + 2)^{-1} - (x + 4)^{-1}$	Students did not link $\frac{1}{x+2}$ to $(x + 2)^{-1}$ to for the binomial expansion.

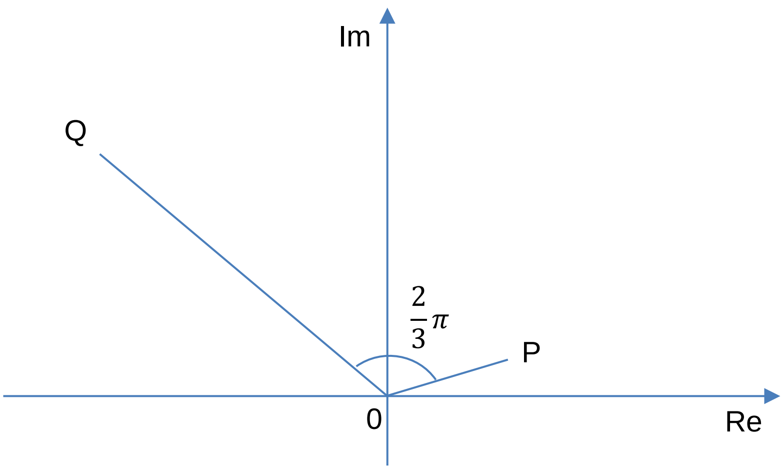
	$=\frac{1}{x}\left(1+\frac{2}{x}\right)^{-1}-\frac{1}{x}\left(1+\frac{4}{x}\right)^{-1}$	
	$=\frac{1}{x}\left(1-\frac{2}{x}+\frac{4}{x^2}-\frac{8}{x^3}\dots\right)-\frac{1}{x}\left(1-\frac{4}{x}+\frac{16}{x^2}-\frac{64}{x^3}\dots\right)$	
	$=\frac{1}{x}\left(\frac{2}{x}-\frac{12}{x^2}+\frac{56}{x^3}\dots\right)=\frac{2}{x^2}-\frac{12}{x^3}+\frac{56}{x^4}\dots$	
aiii	Valid for $\left\lvert\frac{2}{x}\right\rvert<1$ and $\left\lvert\frac{4}{x}\right\rvert<1$ => $\lvert x\rvert>2$ and $\lvert x\rvert>4$	Recall: For $(1+x)^n$, validity is $\lvert x\rvert<1$.
	From finding the common intersection of both validities $\therefore \lvert x\rvert>4$	
aiv	$\frac{2}{168}=\frac{2}{10^2+6(10)+8}$	
	$=\frac{2}{10^2}-\frac{12}{10^3}+\frac{56}{10^4}$	
	$=0.02$ −0.012 +0.0056= 0.0136 (to 4 decimal places)	

12. [Maximum mark: 20]

(a) $P(z)$ is a polynomial of degree four with real coefficients. Given that $2e^{i\frac{\pi}{3}}$ and $2ie^{i\frac{\pi}{3}}$ are roots of the equation. Express $P(z)$ as a product of 2 quadratic factors, $(z^2+az+b)(z^2+cz+d)$ such that a, b, c, d are real numbers.

(b) The following diagram shows the complex plane with origin O. The points P and Q represents the complex numbers z_1 and z_1z_2 respectively.

OP is a third of the length of OQ and the angle POQ is $\frac{2}{3}\pi$.



[Diagram not drawn to scale]

Given that $z_1 = \sqrt{3} + i$, find z_2, giving your answer in the form $a + ib$.

ci) Show that for complex numbers, z and w, $zw^* + z^*w = 2Re(zw^*)$.

ii) Prove by contradiction the triangle inequality for complex numbers, that is, $|z + w| < |z|+|w|$.

a	$2e^{i\frac{\pi}{3}} = 2cis\frac{\pi}{3} = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $= 1 + i\sqrt{3}$	Some were unaware that a polynomial with real coefficient has roots in conjugate pairs.
	The conjugate $1 - i\sqrt{3}$ is also a root.	Most do not know the trigo ratios for the basic angles.
	$2ie^{i\frac{\pi}{3}} = 2icis\frac{\pi}{3} = 2i\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = 2i\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $= i - \sqrt{3}$	
	The conjugate $-i - \sqrt{3}$ is also a root.	
	The quadratic factors are	
	$\left(z - (1 + i\sqrt{3})\right)\left(z - (1 - i\sqrt{3})\right) = z^2 - 2z(1) + \left(1^2 + \sqrt{3}^2\right)$	
	$= z^2 - 2z + 4$	
	$\left(z - (-\sqrt{3} + i)\right)\left(z - (-\sqrt{3} - i)\right)$ $= z^2 - 2z\left(-\sqrt{3}\right) + \left(\sqrt{3}^2 + 1^2\right)$	
	$= z^2 + \left(2\sqrt{3}\right)z + 4$	

	$P(z)=(z^2-2z+4)\left(z^2+\left(2\sqrt{3}\right)z+4\right)$	
OR	Since $P(z)$ is a polynomial with real coefficients, $2e^{-i\frac{\pi}{3}}$ is also a root.	
	$\left(z-2e^{i\frac{\pi}{3}}\right)\left(z-2e^{-i\frac{\pi}{3}}\right)=z^2-2z\left(e^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}\right)+2e^{i\frac{\pi}{3}}\times 2e^{-i\frac{\pi}{3}}$	
	$=z^2-2z\left(2\cos\frac{\pi}{3}\right)+4e^{i\left(\frac{\pi}{3}-\frac{\pi}{3}\right)}=z^2-2z\left(2\cos\frac{\pi}{3}\right)+4$	
	$=z^2-2z+4$	
	$2ie^{i\frac{\pi}{3}}=2i\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-\sqrt{3}+i$	
	Hence $-\sqrt{3}-i$ is also a root.	
	$\left(z-(-\sqrt{3}+i)\right)\left(z-(-\sqrt{3}-i)\right)$ $=z^2-2z\left(-\sqrt{3}\right)+\left(\sqrt{3}^2+1^2\right)$	
	$=z^2+\left(2\sqrt{3}\right)z+4$	
	$P(z)=(z^2-2z+4)\left(z^2+\left(2\sqrt{3}\right)z+4\right)$	
b	Since OP is a third of the length of OQ, $\lvert z_2\rvert=3$ because $z_1=r_1 cis\theta$ then $z_1z_2=r_1r_2cis\left(\theta+\frac{2}{3}\pi\right)$ makes $\frac{r_1r_2}{r_1}=3$ meaning $r_2=3$	
	$\arg(z_2)=\frac{2}{3}\pi$	
	$z_2=3\left(cos\frac{2}{3}\pi+isin\frac{2}{3}\pi\right)$	Many did not know that $cos\frac{2}{3}\pi=-\frac{1}{2}$.
	$=3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-\frac{3}{2}+i\frac{3\sqrt{3}}{2}$	
ci	$zw^*+z^*w=zw^*+(zw^*)^*$ $=2Re(zw^*)$ Note: z^*w is the conjugate of zw^*	Recall: $z+z^*=2Re(z)$ Note: $z^*=\frac{1}{z}$ only if $\lvert z\rvert=1$

OR	Let z= $x+iy$ and $w=a+ib$, $zw^*+z^*w=(x+iy)(a-ib)+(x-iy)(a+ib)$	
	$=(ax-iay+ibx+by)+(ax+iay-ibx+by)$	
	$=2(ax+by)=2Re(zw^*)$	
cii	Assume $\|z+w\|\geq\|z\|+\|w\|$, then $\|z+w\|^2\geq(\|z\|+\|w\|)^2$.	Most were unable to prove this.
	$(z+w)(z^*+w^*)\geq\|z\|^2+\|w\|^2+2\|z\|\|w\|$	
	$zz^*+zw^*+z^*w+ww^*\geq\|z\|^2+\|w\|^2+2\|z\|\|w\|$	
	$\|z\|^2+\|w\|^2+2Re(zw^*)\geq\|z\|^2+\|w\|^2+2\|z\|\|w\|$ $Re(zw^*)\geq\|z\|\|w\|$	
	Since $Re(zw^*)\leq\|zw^*\|\leq\|z\|\|w\|$	
	Contradiction, hence $\|z+w\|<\|z\|+\|w\|$.	
OR	Let z= $x+iy$ and $w=a+ib$,	
	$\|z+w\|=\|(x+iy)+(a+ib)\|=\|(x+a)+i(y+b)\|=$ $\sqrt{(x+a)^2+(y+b)^2}$	
	$\|z\|+\|w\|=\sqrt{x^2+y^2}+\sqrt{a^2+b^2}$	
	Assume $\|z+w\|\geq\|z\|+\|w\|$ $\sqrt{(x+a)^2+(y+b)^2}\geq\sqrt{x^2+y^2}+\sqrt{a^2+b^2}$	
	$(x+a)^2+(y+b)^2\geq x^2+y^2+2\sqrt{(x^2+y^2)}\sqrt{(a^2+b^2)}+a^2+b^2$	
	$xa+yb\geq\sqrt{x^2+y^2}\times\sqrt{a^2+b^2}$	
	$Re(zw^*)\geq\|z\|\|w\|$	
	Since $Re(zw^*)\leq\|zw^*\|\leq\|z\|\|w\|$	
	Contradiction, hence $\|z+w\|<\|z\|+\|w\|$.	