Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2022

YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS : ANALYSIS AND APPROACHES

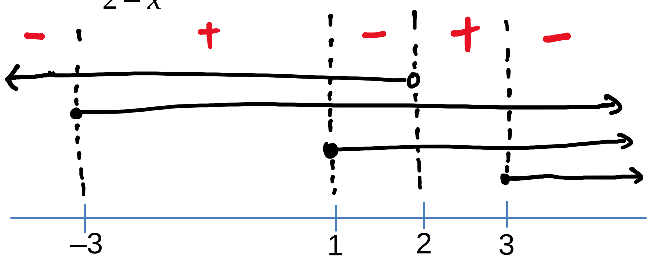
HIGHER LEVEL

Paper 1

Qn	Solution	Marker's Comments
1.		
	$9^x - 12\cdot 3^x + 27 = 0$ $3^{2x} - 12\cdot 3^x + 27 = 0$ Let $y = 3^x$ $y^2 - 12y + 27 = 0$ $(y-9)(y-3) = 0$ $y = 9 \quad or \quad 3$ $x = 2 \quad or \quad 1$	General well done with some careless mistakes.

2.		
	$\log_3 x = 1 + \log_3 y = \lg\frac{2}{9} - \lg\frac{9}{125} + 2\lg\frac{9}{5}$ $\lg\frac{2}{9} - \lg\frac{9}{125} + 2\lg\frac{9}{5} = \lg\frac{2\times125\times81}{9\times9\times25}$ $= \lg 10 = 1$ Therefore $\log_3 x = 1 \quad \Rightarrow x = 3$ $1 + \log_3 y = 1 \quad \Rightarrow \log_3 y = 0$ Hence $y = 1$	Many students made mistake when trying to simplify the log expression on the RHS.
3.		
(a)	$\ln x^3 - \ln x^2 = 3\ln x - 2\ln x$ $= \ln x$ $\ln x^2 - \ln x = 2\ln x - \ln x$ $= \ln x$ Since the common difference is the same, they form 3 consecutive terms in AP	Students are required to show. But many simply find the difference and did not explain why are they finding the difference. A handful find the general terms when the question clearly state that there are only three terms.

(b)	$\sum_{k=1}^{10} \ln x^k = \frac{10}{2}(\ln x + \ln x^{10})$ $= 5(11 \ln x)$ $55 \ln x = 55$ $\ln x = 1$ $x = e$	Some students used GP formula instead of AP. Did not realise that (a) and (b) are related. Some treat x as the dummy variable.

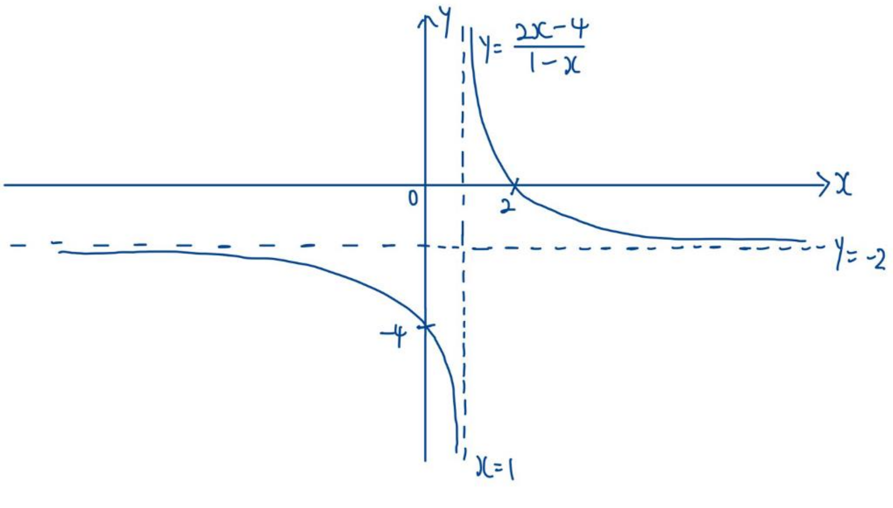
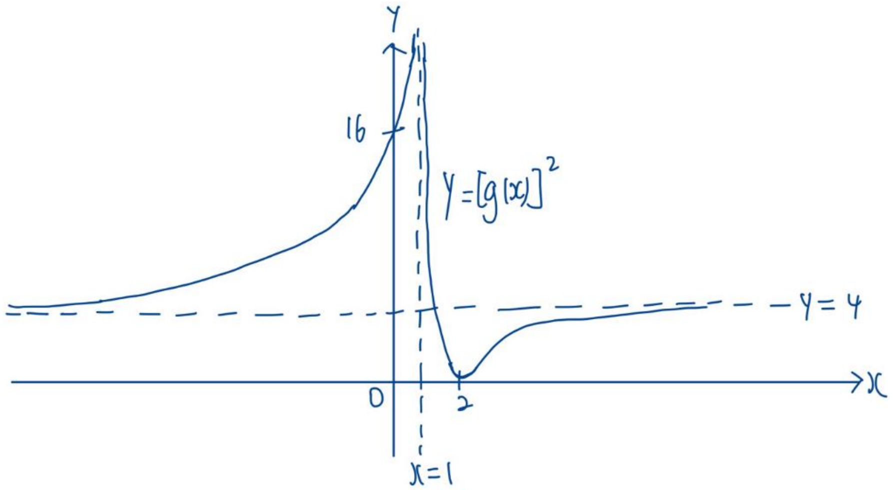
4.		
(a)	General term $=\binom{6}{r}(px^2)^{6-r}(x)^{-r}$ $=\binom{6}{r}(p^{6-r}x^{12-3r})$ Constant term: Power of x is 0 $\therefore 12-3r=0$ $r=4$ Term = $\binom{6}{4}p^2=15p^2$	Generally well done. Most errors are due to wrong computation of 6C4 or 6C2
(b)	$15p^2=60$ $p=-2\ or\ 2$	Generally well done. Most errors are due to carelessness.
5.		
(a)	Let $f(x)=x^3-x^2-9x+9$ $f(1)=0$ Therefore, by factor theorem, $x-1$ is a factor of $f(x)$. Let $x^3-x^2-9x+9=(x-1)(ax^2+bx+c)$ By inspection, $a=1$ and $c=-9$ Comparing coefficient of x^2: $-1=b-a$ $b=-1+a=-1+1=0$ $x^3-x^2-9x+9=(x-1)(x^2-9)=(x-1)(x-3)(x+3)$	Generally well done.
(b)	$\frac{x^3-x^2-9x+9}{2-x}\geq 0$ $\frac{(x-1)(x-3)(x+3)}{2-x}\geq 0$ using answer from (a)  $-3\leq x\leq 1$ and $2<x\leq 3$	Many students could not work with equations with inequalities. This is an area for further work.

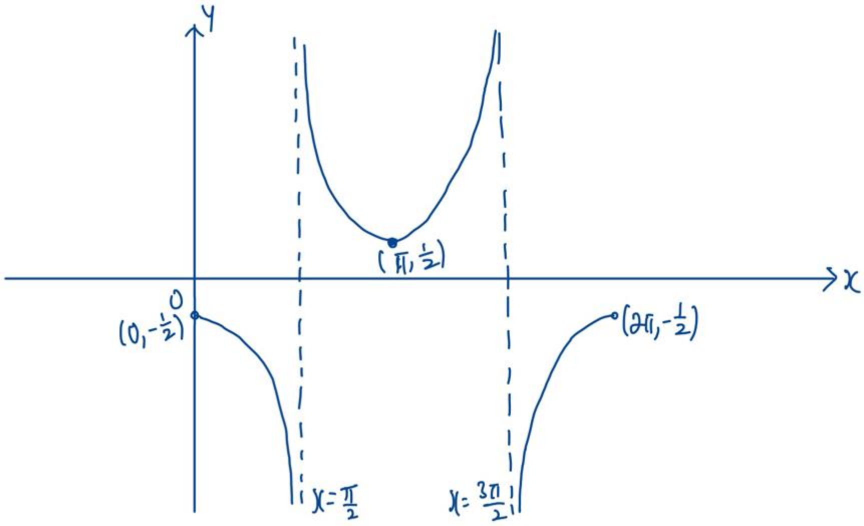
6.		
	$\dfrac{x+1}{2x^2-3x-2}=\dfrac{x-1}{(2x+1)(x-2)}$ Let $\dfrac{x+1}{2x^2-3x-2}=\dfrac{A}{2x+1}+\dfrac{B}{x-2}$ $x+1=A(x-2)+B(2x+1)$ $x=2:\ 3=5B\Rightarrow B=\dfrac{3}{5}$ $x=-\dfrac{1}{2}:\ \dfrac{1}{2}=-\dfrac{5}{2}A\Rightarrow A=-\dfrac{1}{5}$ $\dfrac{x+1}{2x^2-3x-2}=\dfrac{3}{5(x-2)}-\dfrac{1}{5(2x+1)}$	Generally well done
7.		
(a)	$\begin{pmatrix}1&1&-1&3\\2&4&2&3\\1&1&k^2-10&k\end{pmatrix}$ $\xrightarrow[R3-R1]{R2-2R1}\begin{pmatrix}1&1&-1&3\\0&2&4&-3\\0&0&k^2-9&k-3\end{pmatrix}$ In order to have no solution, $k^2-9=0$ and $k-3\neq 0$ Therefore, $k=-3$.	Mostly well done and students can do manual ref. Some still need to wok on identifying no solution and infinite solutions from looking at the ref final matrix.
(b)	For infinite number of solutions, $k=3$. Then we get $\begin{pmatrix}1&1&-1&3\\0&2&4&-3\\0&0&0&0\end{pmatrix}$. Let $z=\lambda,\lambda\in\mathbb{R}$ Then $2y+4z=-3\Rightarrow y=-\dfrac{3}{2}-2\lambda$ and $x+y-z=3\Rightarrow x=3-\left(-\dfrac{3}{2}-2\lambda\right)+\lambda=\dfrac{9}{2}+3\lambda$ Therefore, the general solution is $x=\dfrac{9}{2}+3\lambda$, $y=-\dfrac{3}{2}-2\lambda$ and $z=\lambda$ Accept: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4.5\\-1.5\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\1\end{pmatrix}$	Common mistakes due to conceptual error identified in 7a. Many wasted time doing ref again.

8.		
(a)	$\lvert \mathbf{a}+2\mathbf{b}\rvert = \lvert \mathbf{b}-3\mathbf{a}\rvert$ $\lvert \mathbf{a}+2\mathbf{b}\rvert^2 = \lvert \mathbf{b}-3\mathbf{a}\rvert^2$ $(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}+2\mathbf{b}) = (\mathbf{b}-3\mathbf{a})\cdot(\mathbf{b}-3\mathbf{a})$ $\mathbf{a}\cdot\mathbf{a}+4\mathbf{a}\cdot\mathbf{b}+4\mathbf{b}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{b}-6\mathbf{a}\cdot\mathbf{b}+9\mathbf{a}\cdot\mathbf{a}$ $\lvert\mathbf{a}\rvert^2+4\mathbf{a}\cdot\mathbf{b}+4\lvert\mathbf{b}\rvert^2 = \lvert\mathbf{b}\rvert^2-6\mathbf{a}\cdot\mathbf{b}+9\lvert\mathbf{a}\rvert^2$ Since $\mathbf{a}$ and $\mathbf{b}$ are unit vector, then $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = 1$. $10\mathbf{a}\cdot\mathbf{b}+5=10$ $\mathbf{a}\cdot\mathbf{b}=\frac{1}{2}$	Failure to use shorter method by using the concept u.u =|u|^2. There are attempts to prove using 2 dimensional vectors which is not general. For IB students, if this were the solution path to take, the least they can do is work on 3 dimensional vectors. But this tedious method is not recommended.
(b)	$\cos\theta = \frac{\frac{1}{2}}{1\times1} = \frac{1}{2}$ $\theta = \frac{\pi}{3}$ (accept $60°$)	Generally well done
9.		
(a)	$\tan(x+y) = \frac{2a+b}{2-ab}$ $\frac{\tan x+\tan y}{1-\tan x\tan y} = \frac{2a+b}{2-ab}$ $\frac{a+\tan y}{1-a\tan y} = \frac{2a+b}{2-ab}$ because $x=\arctan(a) \Rightarrow \tan x = a$ $2a+2\tan y-a^2b-ab\tan y = 2a+b-2a^2\tan y-ab\tan y$ $2\tan y(1+a^2) = b(1+a^2)$ $\tan y = \frac{b}{2}$	Most have a good start but did not finish solving to the end.

(b)	$\tan x = \dfrac{2-b}{2+b} = a$ Consider $\tan\left[\arctan\left(\dfrac{1}{a}\right)+\arctan\left(\dfrac{2}{b}\right)\right]$ $= \dfrac{\dfrac{1}{a}+\dfrac{2}{b}}{1-\dfrac{2}{ab}}$ $= \dfrac{b+2a}{ab-2}$ $= \dfrac{b+\dfrac{4-2b}{2+b}}{\dfrac{2b-b^2}{2+b}-2}$ $= \dfrac{2b+b^2+4-2b}{2b-b^2-4-2b}$ $= -\dfrac{b^2+4}{b^2+4} = -1$ Therefore $\tan\left[\arctan\left(\dfrac{1}{a}\right)+\arctan\left(\dfrac{2}{b}\right)\right] = -1$ Then $\arctan\left(\dfrac{1}{a}\right)+\arctan\left(\dfrac{2}{b}\right) = \arctan(-1)$ Since $\arctan\left(\dfrac{1}{a}\right)$ and $\arctan\left(\dfrac{2}{b}\right)$ are between 0 and π, $0 < \arctan\left(\dfrac{1}{a}\right)+\arctan\left(\dfrac{2}{b}\right) < \pi$. Therefore $\arctan\left(\dfrac{1}{a}\right)+\arctan\left(\dfrac{2}{b}\right) = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$	Poorly done. Students should spend more time being familiar with inverse trigonometric functions.
10.		
(a)(i)	Let $f(x) = a(x+2)(x-4)$. Substitute $(0,16)$: $16 = a(2)(-4)$ $a = -2$ $f(x) = -2(x+2)(x-4) = -2x^2+4x+16$	Surprisingly, many students were unable to find the correct expression for f.

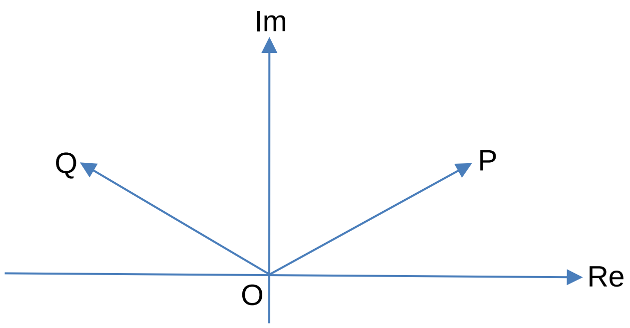
(ii)	$f(x) = -2x^2 + 4x + 16$ $= -2(x^2 - 2x) + 16$ $= -2\left[(x-1)^2 - 1\right] + 16$ $= -2(x-1)^2 + 18$	Although the diagram is not drawn to scale, many students assume turning is (1,18) which happens to be correct this time round. They should not assume that the diagram is drawn to scale. All turning will usually be indicated in the question itself.
(iii)	Sequence of transformation: 1. Vertical stretch of scale factor $\frac{3}{2}$. 2. Reflection in *x*-axis. 3. Translation of $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Note: The order is not important for this question.	Poorly attempted. For stretching, the factor must be positive (IB context). If it is negative, they should indicate by stating that there is a reflection. Many wrong words are used to describe the transformations. Words like shift and move are used instead of translation. Words like compression and enlargement are used instead of stretch. Should indicate vertical stretch or stretch parallel (the word "along" is not accepted) to the y-axis.
(iv)	$\left(a+3, -\frac{3b}{2}\right)$	Poor done due to the mistakes in (iii).

(b)(i)		A handful of students did not state the equation of the asymptotes. The curve must shows that it is asymptotic. Many students drawn the curve away from the asymptote.
(ii)		Poorly attempted. Note that the curve at the x-intercept should not be pointed. It should be a turning point at the x-intercept.

(c)	Note: reciprocal of $2\sin\left(x-\frac{\pi}{2}\right)$ is $\frac{1}{2}\text{cosec}\left(x-\frac{\pi}{2}\right)$ Sketching the reciprocal of $y=2\sin\left(x-\frac{\pi}{2}\right)$, we have  	Poorly attempted. Many students did not realise that cosec is the reciprocal sine function.
11.		
(a)	$l_{AD}: \mathbf{r}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}3\\1\\-1\end{pmatrix}, \mu\in\mathbb{R}$	Many students did not know how to write down a proper vector equation.
(b)	From the vector equation the line joining C and D, $\begin{pmatrix}-2\\1\\0\end{pmatrix}$ is parallel to l_{AB}. $l_{AB}: \mathbf{r}=\begin{pmatrix}-5\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\0\end{pmatrix}, \lambda\in\mathbb{R}$	Many students did not know how to write down a proper vector equation.

(c)	l_{AB} and l_{AD} intersect at the point A. Therefore, $-5-2\lambda=2+3\mu \Rightarrow 2\lambda+3\mu=-7$ $\lambda=-1+\mu \Rightarrow \lambda-\mu=-1$ $2=1-\mu \Rightarrow \mu=-1$. [Note: Can use first two equations to solve simultaneously for $\lambda=-2$ and $\mu=-1$ or simply use the last equation to get $\mu=-1$] Substitute $\mu=-1$ into l_{AD}: $\overrightarrow{OA}=\begin{pmatrix}2-3\\-1-1\\1+1\end{pmatrix}=\begin{pmatrix}-1\\-2\\2\end{pmatrix}$ $A(-1,-2,2)$	Many presented their answer as positive vector of A. It should be in coordinates form.
(d)	$\overrightarrow{AB}=\begin{pmatrix}-5\\0\\2\end{pmatrix}-\begin{pmatrix}-1\\-2\\2\end{pmatrix}=\begin{pmatrix}-4\\2\\0\end{pmatrix}$ $\overrightarrow{AD}=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}-1\\-2\\2\end{pmatrix}=\begin{pmatrix}3\\1\\-1\end{pmatrix}$ $\overrightarrow{AB}\times\overrightarrow{AD}=\begin{pmatrix}-4\\2\\0\end{pmatrix}\times\begin{pmatrix}3\\1\\-1\end{pmatrix}=\begin{pmatrix}-2\\-4\\-10\end{pmatrix}=-2\begin{pmatrix}1\\2\\5\end{pmatrix}$ $\therefore k=-2$	A handful of students thought that $\overrightarrow{AB}\times\overrightarrow{AD}$ is the same as $\overrightarrow{AD}\times\overrightarrow{AB}$. Many students also assume $\overrightarrow{AB}$ is equal to the direction vector of l_{AB}. Poor concept of vector of lines.
(e)	$d=1(-5)+2(0)+5(2)=5$ Cartesian equation of the plane: $x+2y+5z=5$	Generally well done but a handful did not know what is cartesian form of a plane. They presented their answers in vector equation form.
(f)	Since $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$, then C must also lies on the plane as A, B and D. Substitute $C(a,a,1)$ into the equation of the plane: $a+2a+5=5$ $a=0$	A simple question but some students made certain wrong assumption.

g)	Quadrilateral is formed by triangle ABD and triangle CBD. Area of triangle ABD $=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AD}\right\|$ $=\frac{1}{2}\left\|-2\begin{pmatrix}1\\2\\5\end{pmatrix}\right\|=\sqrt{1+4+25}=\sqrt{30}$ $\overrightarrow{CB}=\begin{pmatrix}-5\\0\\2\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-5\\0\\1\end{pmatrix}$ $\overrightarrow{CD}=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ $\overrightarrow{CB}\times\overrightarrow{CD}=\begin{pmatrix}-5\\0\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\5\end{pmatrix}$ Area of triangle CBD $=\frac{1}{2}\left\|\overrightarrow{CB}\times\overrightarrow{CD}\right\|=\frac{1}{2}\left\|\begin{pmatrix}1\\2\\5\end{pmatrix}\right\|=\frac{1}{2}\sqrt{1+4+25}=\frac{1}{2}\sqrt{30}$ Therefore, area of the quadrilateral $ABCD$ $=\frac{3}{2}\sqrt{30}$ units^2	Again, a handful of students assume $\overrightarrow{AB}=\begin{pmatrix}1\\2\\5\end{pmatrix}$ based on the answer in (b).

12.		[16 marks]
(a)	Let $z=x+iy$ $\lvert z\rvert=\lvert z-\alpha i\rvert$ $\lvert x+iy\rvert=\lvert x+(y-\alpha)i\rvert$ $\sqrt{x^2+y^2}=\sqrt{x^2+(y-\alpha)^2}$ $x^2+y^2=x^2+y^2+2y\alpha-\alpha^2$ $\alpha(2y-\alpha)=0$ Since $\alpha>0$, then $y=\dfrac{\alpha}{2}$.	A simple question but many students were unable to do it.
(b)		Generally well done.
(c)(i)	Let $z=a+ib$ $\lvert z\rvert=5\Rightarrow a^2+b^2=25$ Area of triangle $=\dfrac{1}{2}\times a\times 2b$ $\dfrac{1}{2}\times 2a\times b=12$ $ab=12\Rightarrow a=\dfrac{12}{b}$ Therefore $\dfrac{144}{b^2}+b^2=25$ $(b^2)^2-25b^2+144=0$ $(b^2-16)(b^2-9)=0$ $b^2=16\Rightarrow b=4,-4(\text{NA}\because b>0)$ or $b^2=9\Rightarrow b=3,-3(\text{NA}\because b>0)$ If $b=4$, then $a=3$ (NA $\because a>b$) If $b=3$, then $a=4$ (NA) Therefore $z_1=4+3i$	Many students assume the answers for a and b since their squares is 5 (by using Pythagoras' Theorem). This is not accepted because there exist other values for a and b as well. They need to provide another equation and attempt to solve these equations simultaneously.
(ii)	$z_2=-4+3i$	Generally well done.

(d)	$\arg(z_1-1)=\arg(3+3i)$ $=\arctan(1)$ $=\dfrac{\pi}{4}$ $\arg(z_2+i)=\arg(-4+4i)$ $=\pi-\dfrac{\pi}{4}$ $=\dfrac{3\pi}{4}$	Generally well done.
(e)	$\arg\left[\dfrac{(z_1-1)^3(z_2+i)^k}{-2i}\right]=3\arg(z_1-1)+k\arg(z_2+i)-\left(-\dfrac{\pi}{2}\right)$ by DMT Then we have $\dfrac{3\pi}{4}+\dfrac{3k\pi}{4}+\dfrac{\pi}{2}=\dfrac{3\pi}{2}+2n\pi$ where $n\in\mathbb{Z}$ $\dfrac{3k}{4}=\dfrac{1}{4}+2n$ $k=\dfrac{1+8n}{3}$ Least value of k when $n=1$. Therefore, the least value of $k=3$.	Only a small number managed to get this question right. Students need to justify why k=3 is the least value for k and not just simply solving the equation.