Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2023

YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS HIGHER LEVEL PAPER 1

ANALYSIS AND APPROACHES

NOTE: 1. Answers should not be in pencil. Answers in pencil are not acceptable.

2. Answers should be in the given box in section A, if insufficient space students need to indicate where the continuation is.

Section A

***1.** [Maximum mark: 4]

Solve the following equation

$$2^{3x-2} = 5(3^{-x}).$$

Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers.

Solution: $$2^{3x-2} = 5(3^{-x})$$ $$\frac{8^x}{4} = 5(3^{-x})$$ $$8^x \times 3^x = 20$$ $$24^x = 20$$ $$x = \frac{ln20}{ln24}$$	Many students are not able to get this correct as they are not able to manipulate the laws of Logarithm and Indices well. Different kinds of mistakes are made.

2. [Maximum mark: 4]

It is given that $cosec\theta = \frac{3}{2}, \frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $cot\theta$.

Solution: $sin\theta = \frac{2}{3}$ $cos\theta = -\frac{\sqrt{5}}{3}$ Note its 2nd Quadrant $cot\theta = \frac{cos\theta}{sin\theta} = -\frac{\sqrt{5}}{2}$	Many students are not able to get the correct sign, they reject the -ve and accept the positive answer Or just give the positive answer. Some note that the range is 2nd and 3rd quadrant but they did not use the fact sine is positive.

***3.** [Maximum mark: 7]

(a) Show that

$$\sum_{r=1}^{n}(2r-1) = n^2$$

[3]

(b) Hence, find the value of $11 + 13 + 15 + \cdots + 99$. [4]

Solution: a) $\sum_{r=1}^{n}(2r-1) = 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$ $= 2\left(\frac{n}{2}(1+n)\right) - n$ $= n(1+n) - n$ $= n^2$ b) $11 + 13 + 15 + \cdots + 99 = \sum_{r=1}^{50}(2r-1) - \sum_{r=1}^{5}(2r-1)$	(a)Many students are able to use the AP formula to find the sum. Some students use the suggested method. Quite a few uses Induction but did not necessarily score full as their Induction is not fully correctly presented. Technically the question did not ask for Induction. They should not use. (b) "Hence" must use the result in (a) otherwise strictly no marks awarded if the answer is correct. Various mistakes like 45

$= 50^2 - 5^2 = 2475$	instead of 50. 6 instead of 5. $50^2 = 250$.

***4.** [Maximum mark: 7]

The expansion of $(x+h)^7$, where h is a non-zero rational number, can be written as $x^7 + ax^6 + bx^5 + cx^4 + \cdots + h^7$ where a, b, $c \in R$.

Given that a, b and c are three consecutive terms of an arithmetic sequence, find the possible values of h.

Solution: $$(x+h)^7 = x^7 + ax^6 + bx^5 + cx^4 + \cdots + h^7$$ Equating x^6 term, $$C_1^7 \times h = a$$ Equating x^5 term, $$C_2^7 \times h^2 = b$$ Equating x^4 term, $$C_3^7 \times h^3 = c$$ Since a, b and c are three consecutive terms of an arithmetic sequence, $c - b = b - a$ $$2b = a + c$$ $$2(21h^2) = 7h + 35h^3$$ $$7h(6h) - 7h(1 + 5h^2) = 0$$	Most of the students can give the coefficient except some are not able to give the correct values due to no calculator. Quite a few did not reject h=0. Some have obvious mistakes like h is a surd which is impossible as the question stated non-zero rational number. Some has calculation error due to poor factorization.

$$7h(6h-1-5h^2)=0$$ Since $h \neq 0$ (Or reject $h=0$) $$6h-1-5h^2=0$$ $$(5h-1)(h-1)=0$$ $h=\frac{1}{5}$ or $h=1$	

5. [Maximum mark: 9]

(a) Use mathematical induction to prove that for any real number x, $|x|<1$, and any positive integer n, $(1+x)^n \geq 1+nx$. [6]

(b) Hence show that $0.999^4 > 0.996$. [3]

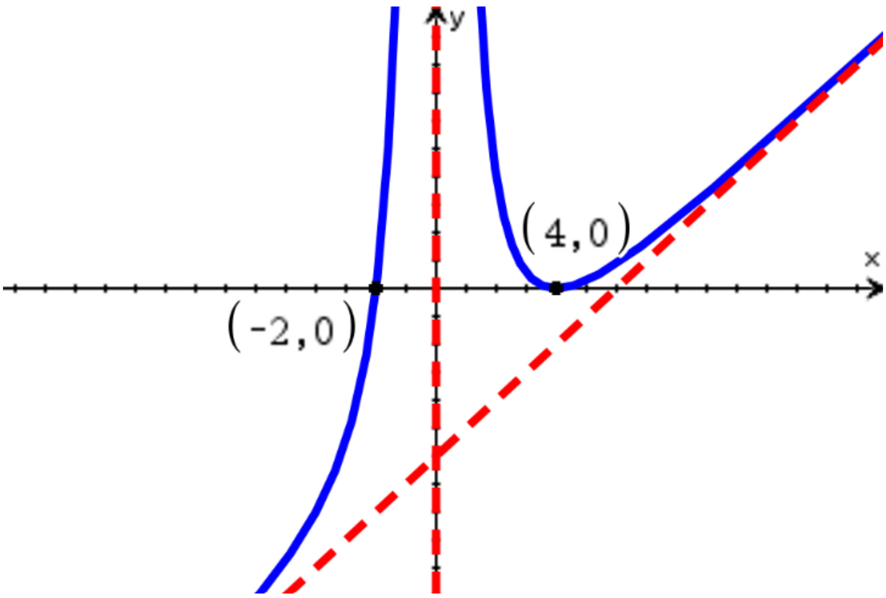
Solution: Let Pn be the proposition such that $(1+x)^n \geq 1+nx$, any positive integer n and any real number x, $\|x\|<1$. LHS = $1+x$ =RHS Hence P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$ $(1+x)^k \geq 1+kx$ To prove P_{k+1} is true. LHS = $(1+x)^{k+1} = (1+x)^k(1+x)$ $\geq (1+kx)(1+x)$ $= 1+(k+1)x+kx^2$ $\geq 1+(k+1)x$ = RHS Since $kx^2 \geq 0$ for real number $\|x\|<1$ and any positive integer k.	Many students do not write their statements properly and hence loose marks. Various mistakes, eg instead of “Some”, students wrote “all”. The closing statements written by them shows that the students do not know the true meaning of the MI process. $\rightarrow$ means approaches so it is not acceptable. $\Rightarrow$ means imply. If not sure should just write in words. Poor presentation of the working for the P_{k+1} case. Students cannot write from L to R properly. Some students try to “force out” the final step of the proof but the link in between does not make sense.

Since P$_1$ is true and P$_k$ is true implies P$_{k+1}$ is true. By MI, $(1+x)^n \geq 1+nx$, any positive integer n and any real number x, $\|x\|$<1.	
Since $(1+x)^n \geq 1+nx$, Let $x = -0.001$ and $n = 4$, Then $(1-0.001)^4 \geq 1+4(-0.001)$ $0.999^4 > 1-0.004$ (remove "=" since $n \neq 0$ *or* 1 *and* $x \neq 0$) $0.999^4 > 0.996$ (shown)	Marker is lenient here as in if the students did not explain or give reasons why = is removed they still can get full marks as the previous part is marked stricter. However if the students did not remove the = sign or some how their writing does not show the results required, they will be penalised.

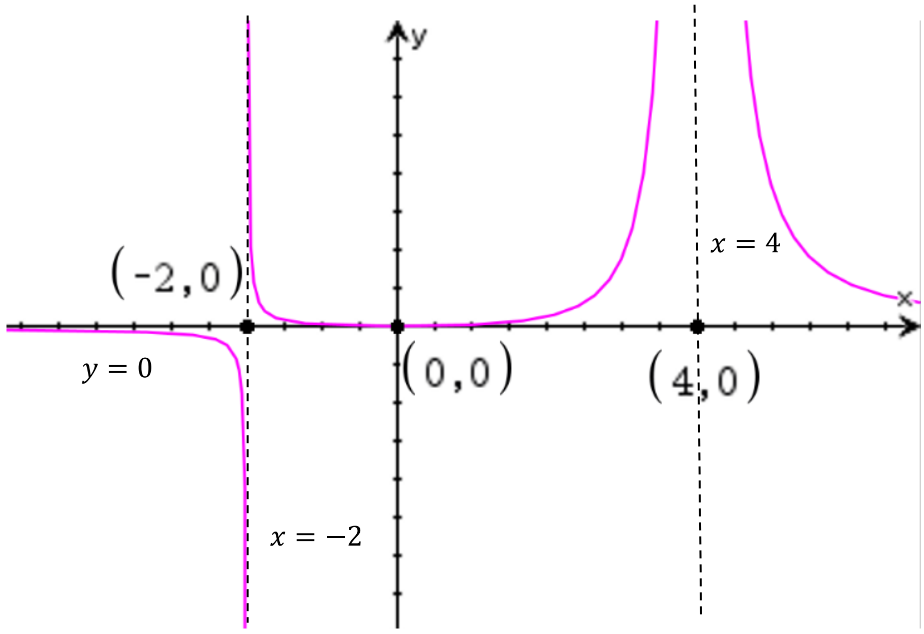
6. [Maximum mark: 6]

Consider the following function:

The graph of $y = f(x)$ is shown in the following diagram. The curve intersects the x-axis at $(-2,0)$ and it has a stationary point at $(4,0)$. The equations of the asymptotes are $x = 0$ and $y = 2(x-6)$.

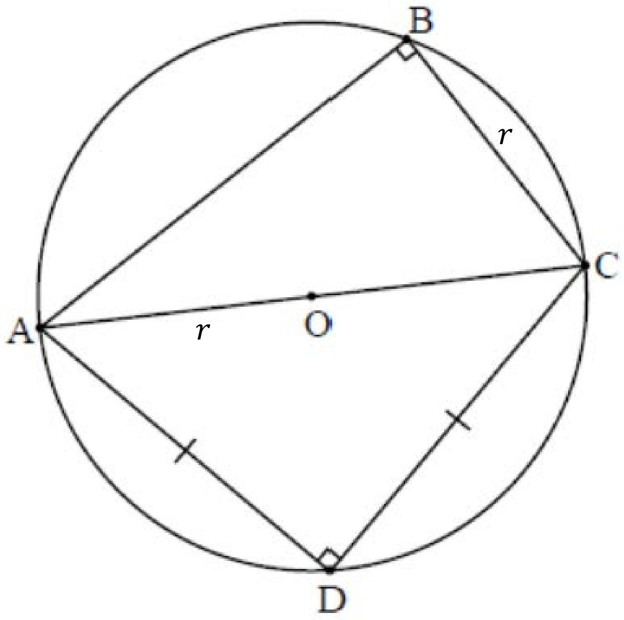


Sketch the curve $y = \frac{1}{f(x)}$, indicating clearly the equations of any asymptotes, coordinates of any intercepts with the axes and the coordinates of any stationary point(s).

	Many students are not able to completely give the full graph and information. Some did not give the equations of the asymptotes. Many did not give y=0. The shapes are not correct and the min and intercepts is not clearly given. Did not label the x and y axes.

7. [Maximum mark: 10]

In the following diagram, the points A, B, C and D are on the circumference of a circle with centre O and radius r. AC is the diameter of the circle. $BC = r$, $AD = CD$ and $A\hat{B}C = A\hat{D}C = 90^{\circ}$.



(a) Show that $\cos(B\hat{A}D) = \frac{\sqrt{3}-1}{2\sqrt{2}}$. [5]

(b) Hence or otherwise, show that the area of triangle $ABD = \frac{3+\sqrt{3}}{4}r^2$. [5]

Solution:	
(a) Since $BC = r$, triangle BOC is an equilateral triangle. $$B\hat{A}O = \frac{60^{\circ}}{2} = 30^{\circ}$$ $$B\hat{A}D = 30^{\circ} + 45^{\circ}$$ $$\cos(B\hat{A}D) = \cos(30^{\circ} + 45^{\circ})$$ $$= cos30^{\circ}cos45^{\circ} - sin30^{\circ}sin45^{\circ}$$ $$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$ $$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$ (b) Area of triangle $ABD = \frac{1}{2} \times AB \times AD \times sinB\hat{A}D$ $$= \frac{1}{2} \times \sqrt{3}r \times \sqrt{2}r \times sin(30^{\circ} + 45^{\circ})$$ $$= \frac{1}{2} \times \sqrt{6}r^2 \times \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\right)$$ $$= \frac{3+\sqrt{3}}{4}r^2$$ Some students try to just "cheat" by stating the last expected answer but the steps before does not actually leads to the given answer. So the question is "do the students understand that their answer is wrong but are trying their luck, hoping the examiner does not see or the students truly are not able to manage their surds thinking that their answer is correct?".	(a) Not many students can do this question, some left it blank. Students does not realise that they can use the addition formulae and the special angles to do this. Some cosine rule twice and equate the length BD. Some still cannot give the values for special angles. (b) Similar method is expected for sin(75°) but many uses the Pythagoras method and could not prove the last step. Some just write down sin(75°)=$\frac{1+\sqrt{3}}{2\sqrt{2}}$ without giving any explanation so penalised. Some find the area of 3 triangles to sum up.

8. [Maximum mark: 6]

Consider the following functions:

$$f(x) = x^2 + ax + b$$

$$g(x) = ax^2 + 2bx - 2 \ ,$$

where a and b are positive integers.

It is given that the solution to $f(x) \geq g(x)$ is $-5 \leq x \leq 1$, find the values of a and b.

Solution: $$f(x) \geq g(x)$$ $$x^2 + ax + b \geq ax^2 + 2bx - 2$$ $$(a-1)x^2 + (2b-a)x - (2+b) \leq 0$$ Since $x = 1$ and $x = -5$ are roots, by Factor Theorem, $$(a-1)(1)^2 + (2b-a) - (2+b) = 0$$ $$b = 3$$ And $$(a-1)(-5)^2 + (2b-a)(-5) - (2+b) = 0$$ $$25a - 25 - 30 + 5a - 5 = 0$$ $$a = 2$$	The qn was well attempted however several students used less efficient methods of solving, such as using sum and product of roots or solving for the roots of the equation using the quadratic equation. Some gave a range of values for a and b which does not satisfy the values given in the qn for the roots.

Section B

***9.** [Maximum mark: 19]

Consider the function $f(x) = \frac{1-2x}{x+1}$, $x \neq -1$.

(a) Sketch the graph of $f(x)$ indicating clearly the equations of any asymptotes and coordinates of any intercepts with the axes. [3]

(b) Given that $g(x) = 3f(x+1)$, find the range and domain of $g(x)$. [2]

(c) Find the inverse function f^{-1}, stating its domain. [4]

(d) Find x where $f(x) = f^{-1}(x)$, giving your answers in the form of $\frac{a \pm \sqrt{b}}{2}$, where a and b are integers to be determined. [3]

(e) Sketch the graph of $f(|x|)$. [3]

(f) On a different set of axes, sketch the graph of $y = [f(x)]^2$, indicating clearly the equations of any asymptotes, coordinates of any intercepts with the axes and any stationary point(s). [4]

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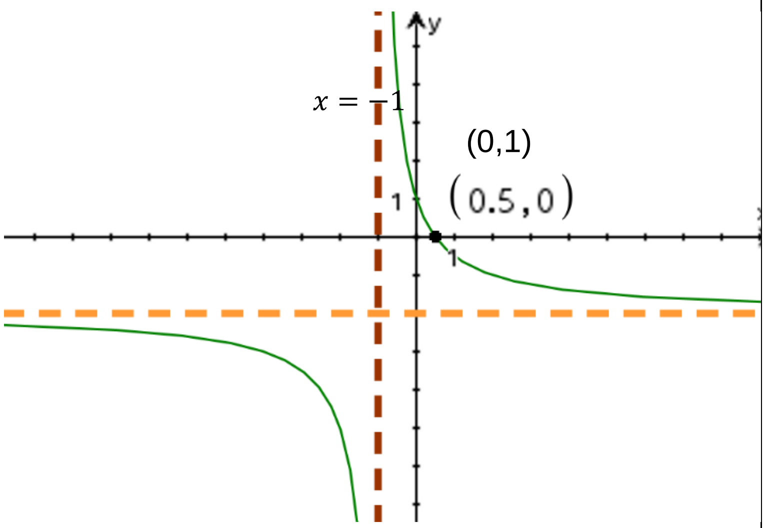
Solution:



a)

$x=-1$ and $y=-2$

(0.5, 0), (0,1)

Correct shape

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Well attempted. Some students didn't read the question correctly and missed out the intercepts.

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b) Range of $g(x)$: $y \in R, y \neq -6$

Domain of $g(x)$: $x \in R, x \neq -2$

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Some were unable to perform simple transformation to solve the question.

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c) Let $y=\frac{1-2x}{x+1}$

$$y(x+1)=1-2x$$

$$x=\frac{1-y}{y+2}$$

$$f^{-1}(x)=\frac{1-x}{x+2}$$

Domain of f^{-1}: $x \in R, x \neq -2$

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Well attempted.

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d) Solving $f(x)=x$

$$\frac{1-2x}{x+1}=x$$

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Most students solved using a less efficient method of $f(x)=f^{-1}(x)$

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$$x^2 + 3x - 1 = 0$$

$$x = \frac{-3 \pm \sqrt{13}}{2}$$

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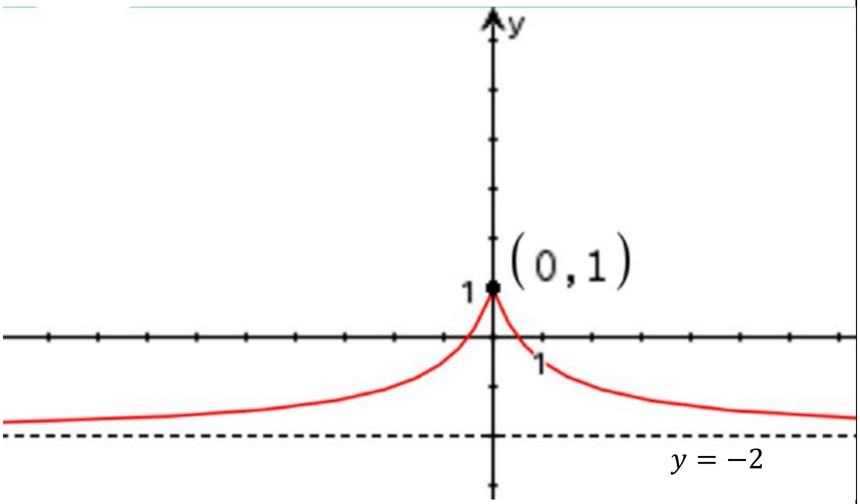
e)



Correct shape (symmetrical about x-axis)

y-intercept at (0, 1)

Asymptote at y = -2

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Most were unable to sketch this. A version of $|f(x)|$ was used instead.

Note that the graph of $f(|x|)$ is always symmetrical about the x-axis.

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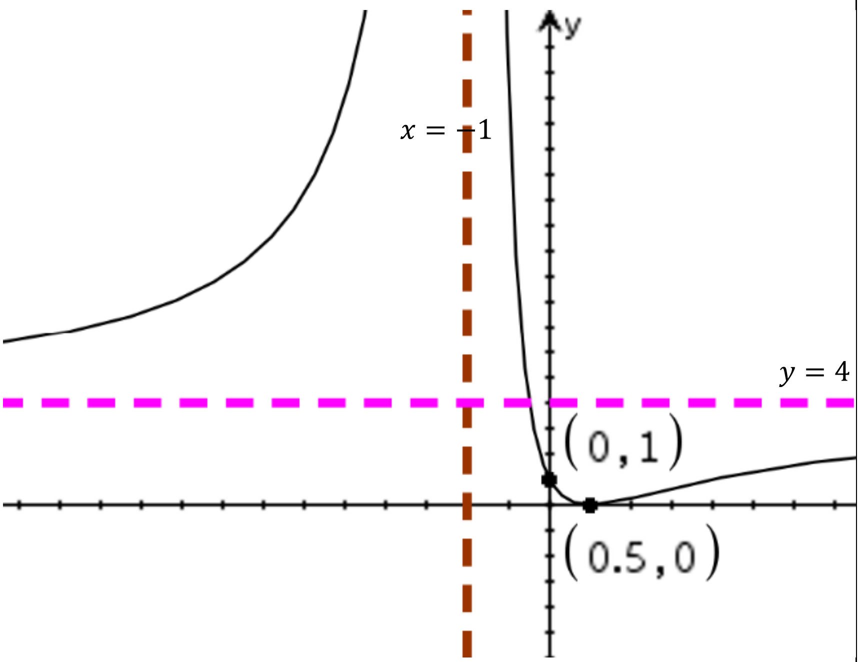

Asymptotes: x = -1 , y = 4

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Students commonly wrote the wrong horizontal asymptote, $y = 2$.

Some were unable to sketch the correct shape of the functions despite calculating the intercepts and asymptotes correctly.

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Min point (0.5, 0) Y-intecept (0, 1) Correct shape and must be above the x-axis.	

10. [Maximum mark: 20]

Three points ***A***, ***B*** and ***C*** in three-dimensional space are defined by position vectors $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a} - 2\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.

(a) Given that ***OAB*** is an equilateral triangle, show that $2\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^{\mathbf{2}}$. [2]

(b) By considering the vectors $\overrightarrow{\mathbf{AB}}$ and $\overrightarrow{\mathbf{BC}}$, give a geometric interpretation of the relationship between points ***A***, ***B*** and ***C***. [3]

(c) The point ***A*** has coordinates $(2,0,t)$, where t is a real and positive constant. Given that the angle between $\overrightarrow{\mathbf{OA}}$ and the x-axis is $arccos\frac{1}{\sqrt{3}}$, show that $t = 2\sqrt{2}$. [3]

(d) Consider the lines l_1 and l_2 defined by

$$l_1: \boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} h \\ 2 \\ -2 \end{pmatrix}, \qquad \mu \in R$$

$l_2: \frac{6-x}{3} = \frac{y-k}{4} = z + 1,$ where h and k are constants.

Given the lines are perpendicular, show that the value of $h = 2$. [2]

i) If $k = 3$, find the foot of the perpendicular from the point $\boldsymbol{D}(6,0,-2)$ to l_2. [5]

ii) If $k \neq 7$, use proof by contradiction to show that the lines are skew. [5]

Solution: a) Since ***OAB*** is an equilateral triangle, $A\hat{O}B = 60^{\circ}$ $$cosA\hat{O}B = cos60^{\circ} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{\|a\|\|\boldsymbol{b}\|}}$$ $$\frac{1}{2} = \frac{\overrightarrow{\mathbf{AB}}}{\mathbf{\|a\|^2}}$$ $$2\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|^{\mathbf{2}}$$	a) Common error: assuming since $\|a\| = \|b\|$, then $a = b$.
b) $\overrightarrow{\mathbf{AB}} = \mathbf{b} - \mathbf{a}$ $$\overrightarrow{\mathbf{BC}} = \mathbf{c} - \mathbf{b} = 3\mathbf{a} - 2\mathbf{b} - \mathbf{b} = 3\mathbf{a} - 3\mathbf{b}$$ Since $\overrightarrow{\mathbf{BC}} = -3\overrightarrow{\mathbf{AB}}$, with common point $\mathbf{B}$,	b) Some only observed the parallel property of the vectors.

points ***A***, ***B*** and ***C*** are collinear.	
c) Vector along the x-axis = $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $$cos\left(arccos\frac{1}{\sqrt{3}}\right)=\frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\\t\end{pmatrix}}{\left\|\begin{pmatrix}1\\0\\0\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\0\\t\end{pmatrix}\right\|}$$ $$\frac{1}{\sqrt{3}}=\frac{2}{\sqrt{2^2+t^2}}$$ $$4+t^2=4\times 3$$ $$t=2\sqrt{2}$$	c) Some were able to use $cos\theta=\frac{adj}{hyp}$ and observed from ***A*** that $adj=2$ to solve.
d) $l_1\colon \boldsymbol{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\mu\begin{pmatrix}h\\2\\-2\end{pmatrix}\ ,\ \mu\in R$ $$l_2\colon \boldsymbol{r}=\begin{pmatrix}6\\k\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\1\end{pmatrix}\ ,\qquad \lambda\in R$$ Since the lines are perpendicular, $$\begin{pmatrix}h\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$$ $$-3h+8-2=0$$ $$h=2$$	d) Well attempted.
i) $k=3$, $$l_2\colon \boldsymbol{r}=\begin{pmatrix}6\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\1\end{pmatrix}\ ,\qquad \lambda\in R$$ Let $\overrightarrow{OF}=\begin{pmatrix}6-3\lambda\\3+4\lambda\\-1+\lambda\end{pmatrix}$ $$\overrightarrow{DF}=\begin{pmatrix}-3\lambda\\3+4\lambda\\1+\lambda\end{pmatrix}$$ Since $\overrightarrow{DF}$ is perpendicular to l_2	i) Generally well attempted but some were very careless in the calculations of λ. Students did not express the foot of the perpendicular as coordinates.

$$\overrightarrow{DF}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$$ $$\begin{pmatrix}-3\lambda\\3+4\lambda\\1+\lambda\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$$ $$9\lambda+12+16\lambda+1+\lambda=0$$ $$\lambda=-\frac{1}{2}$$ $F\left(\frac{15}{2},1,-\frac{7}{2}\right)$	Some students did not realise that the lines are perpendicular but skew (for $k=3$) hence they will get the wrong coordinates if they solve using intersection of lines.
ii) Assume for $k=7$, the lines are skew. $$\begin{pmatrix}1\\2\\4\end{pmatrix}+\mu\begin{pmatrix}h\\2\\-2\end{pmatrix}=\begin{pmatrix}6\\7\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\1\end{pmatrix}$$ From (1), $1+2\mu=6-3\lambda$ From (3), $4-2\mu=-1+\lambda$ Solving, $\lambda=0$, $\mu=\frac{5}{2}$ Sub into (2), $2+2\times\frac{5}{2}=7+(0)4$ $7=7$ (Verify that the values of $\lambda=0$, $\mu=\frac{5}{2}$ are consistent.) Since the lines intersects at a point, they are not skew. This contradicts the assumption that they are skew lines when $k=7$.	ii) Poorly attempted. The assumption was not clearly stated at the start of the question, with most either leaving out the condition for k or the relationship between the lines. Some students implied that not skew is the same as intersecting without stating that the lines are not parallel. Some didn't consider the values for k in their proof.

11. [Maximum mark: 18]

The complex numbers u and w satisfy the equations

$$2u^{*}=u+1-3i$$

$$uw=7+i$$

(a) Find u and w, giving your answers in the form $a + ib$, where a and b are real numbers. [5]

(b) Given that $u = 1 + i$ and $w = 4 - 3i$ are the roots of the equation

$$(x^2 + hx + k)(x^2 + mx + p) = 0,$$

find the values of the real numbers h, k, m and p. [3]

Let $z = cos\theta + isin\theta$.

(c) Prove that $z^n + (z^*)^n = 2cosn\theta$, $n \in Z^+$. [2]

(d) Write down the binomial expansion of $(z + z^*)^3$ in terms of z. [3]

(e) Hence show that $cos3\theta = 4cos^3\theta - 3cos\theta$ and find the exact value of $cos135^\circ$. [5]

Solution:

(a) $2u^* = u + 1 - 3i$

Let $u = x + iy$,

$$2u^* = u + 1 - 3i$$

$$2(x - iy) = x + iy + 1 - 3i$$

Equating real and imaginary parts,

$$x = 1, y = 1$$

$$u = 1 + i$$

$$uw = 7 + i$$

$$w = \frac{7 + i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$w = 4 - 3i$$

b) Since the polynomial has real coefficients, the roots are

$u = 1 + i,\ u^* = 1 - i,\ w = 4 - 3i$ and $w^* = 4 + 3i.$

Hence, $(x^2 + ax + b) = \big(x - (1 + i)\big)\big(x - (1 - i)\big)$

$$= x^2 - 2x + 2$$

$$(x^2 + cx + d) = \big(x - (4 - 3i)\big)\big(x - (4 + 3i)\big)$$

$$= x^2 - 8x + 25$$

c) LHS= $z^n + (z^*)^n = (cos\theta + isin\theta)^n + (cos\theta - isin\theta)^n$

$$= cosn\theta + isinn\theta + cosn\theta - isinn\theta$$

$= 2cosn\theta$ =RHS (Proven)

Generally poorly attempted by students who had run out of time for the paper.
Good attempt on (a) and (b).
Some students attempted to make use of the sum and product of roots formula for (b). However, they equated the sum of roots to $\frac{\boldsymbol{b}}{\boldsymbol{a}}$ instead of $-\frac{\boldsymbol{b}}{\boldsymbol{a}}$.

d) $(z+z^*)^3 = z^3 + 3z^2z^* + 3zz^{2*} + z^{3*}$ $= z^3 + 3z^2\frac{1}{z} + 3z\frac{1}{z^2} + \frac{1}{z^3}$ $= z^3 + \frac{1}{z^3} + 3z + 3\frac{1}{z}$ e) $(z+z^*)^3 = z^3 + \frac{1}{z^3} + 3z + 3\frac{1}{z}$ $(2cos\theta)^3 = 2cos3\theta + 6cos\theta$ $8cos^3\theta = 2cos3\theta + 6cos\theta$ Since $cos135^\circ = cos(3\times 45^\circ)$, let $\theta = 45^\circ$ $cos135^\circ = 4cos^3 45^\circ - 3cos45^\circ$ $= 4\left(\frac{1}{\sqrt{2}}\right)^3 - 3\frac{1}{\sqrt{2}}$ $= -\frac{1}{\sqrt{2}}$	Common mistake for (d) was to ignore the command term "Hence" and thus showing a proof that does not involve the result from (c)