Anglo-Chinese School

(Independent)

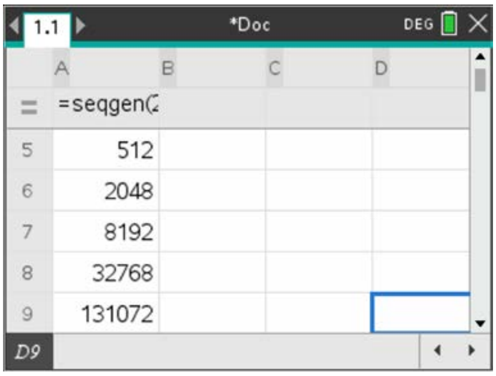
FINAL EXAMINATION 2023

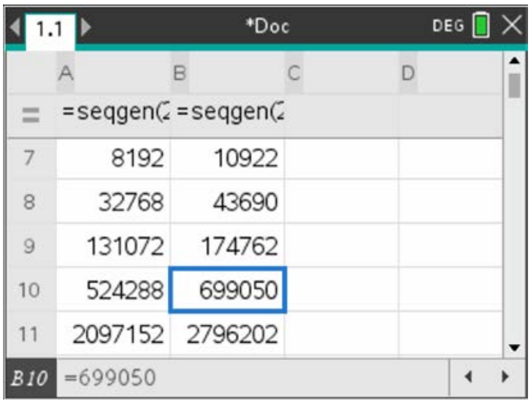
YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS HIGHER LEVEL PAPER 2

ANALYSIS AND APPROACHES

<u>SECTION A</u>

Qn	Solution	Marks
1*.		[7 marks]
(a)	4	
(b)	$u_n = 2(4)^{n-1}$ For $u_n < 10000$ $2(4)^{n-1} < 10000$ $4^{n-1} < 5000$ $n - 1 < \frac{\lg(5000)}{\lg(4)}$ $n < \dfrac{\lg(5000)}{\lg 4} + 1 = 7.1438$ The largest term is $u_7 = 2(4^6) = 8192$ **Alternative:** From GDC, we generate a sequence with general term $u_n = 2(4)^{n-1}$  The largest term smaller than 10000 is $u_7 = 8192$ (as $u_8 = 32768 > 10000$)	

Qn	Solution	Marks
	Generally well attempted. However many candidates did not find the term U_7 but give the answer s n=7 .	
(c)	$S_n = \frac{2(4^n-1)}{4-1} = \frac{2}{3}(4^n-1)$ $S_n < 100000$ $\frac{2}{3}(4^n-1) < 100000$ $4^n - 1 < 150000$ $4^n < 150001$ $n < \frac{\lg(150001)}{\lg 4} = 8.59$ The largest possible value of n is 8. **Alternative:** $S_n = \frac{2(4^n-1)}{4-1} = \frac{2}{3}(4^n-1)$ From GDC, we generate a sequence with general term $S_n = \frac{2}{3}(4^n-1)$  The largest possible value of n is 8 (as $S_9 = 174762 > 100000$)	
	Well attempted.	

Qn	Solution	Marks
2*.		[7 marks]

Qn	Solution	Marks
(a)	$\log_{\sqrt{a}} m = \frac{\log_a m}{\log_a \sqrt{a}}$ $= \frac{\log_a m}{\frac{1}{2}\log_a a}$ $= 2\log_a m$ $= \log_a m^2$	
	Well attempted. However, a number of students just write $\log_{\sqrt{a}} m = \log_{(\sqrt{a})^2} m^2$ without showing the proof and no mark is awarded for this.	
(b)	$\log_{\sqrt{2}} x + 3\log_x 2 = 5$ $\log_2 x^2 + \frac{3}{\log_2 x} = 5$ Let $y = \log_2 x$ $2y + \frac{3}{y} = 5$ $2y^2 - 5y + 3 = 0$ $(2y-3)(y-1) = 0$ $y = \frac{3}{2}$ or $y = 1$ Therefore $x = 2^{\frac{3}{2}}$ or $x = 2$	
	Very well done	

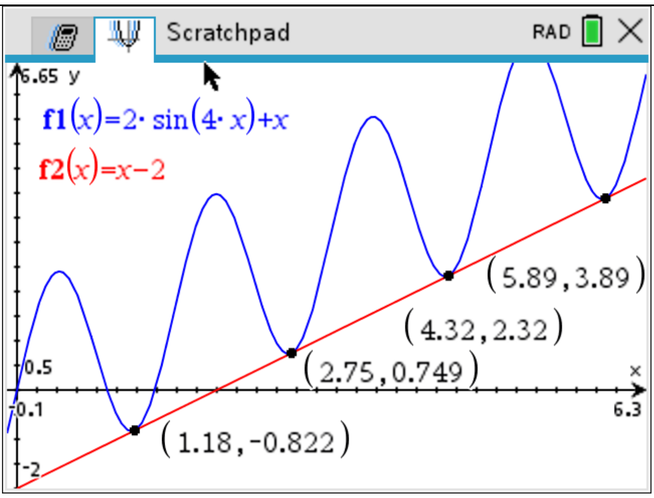
Qn	Solution	Marks
3*.		[8 marks]
(a)	$\frac{\lambda+2}{2} = 4.5$ $\lambda = 9 - 2 = 7$	
	Generally well done	
(b)	Let $x = 0, y = -4$ $-4 = a(-7)(-2)$ $a = -\frac{4}{14} = -\frac{2}{7}$	
	Generally well done	

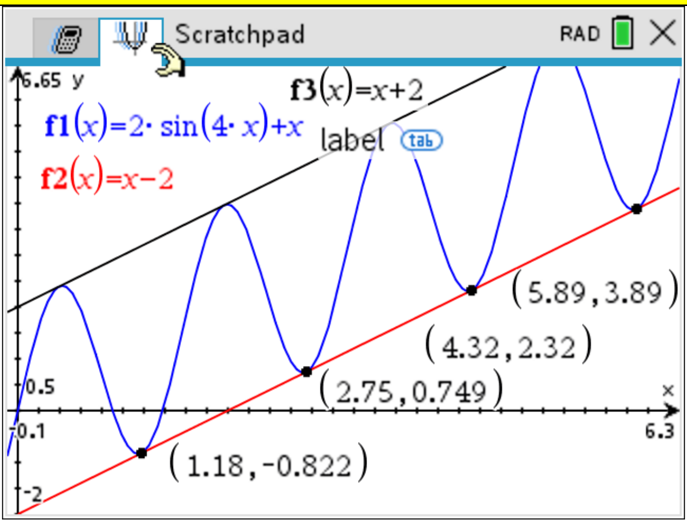
Qn	Solution	Marks
(c)	$kx + 9 = -\frac{2}{7}(x-7)(x-2)$ $-7kx - 63 = 2(x^2 - 9x + 14)$ $2x^2 - 18x + 7kx + 28 + 63 = 0$ $2x^2 + (7k-18)x + 91 = 0$ Since the line is a tangent to the curve, the discriminant is 0 $(7k-18)^2 - 4(2)(91) = 0$ $(7k-18)^2 = 728$ $7k - 18 = \pm\sqrt{728}$ $k = \frac{18\pm\sqrt{728}}{7} = 6.43$ or -1.28	
	Generally well done	

Qn	Solution	Marks
4.		[4 marks]
	$\alpha+\beta=\frac{3}{2},\ \alpha\beta=-3$ $2\alpha+2\beta=-p \Rightarrow 2(\alpha+\beta)=-p$ $2\left(\frac{3}{2}\right)=-p$ $p=-3$ $2\alpha(2\beta)=q \Rightarrow 4\alpha\beta=q$ $4(-3)=q$ $q=-12$ **Alternative:** $y=2x$ $Subst\ x=\frac{y}{2}$ $2\left(\frac{y}{2}\right)^2-3\left(\frac{y}{2}\right)-5=1$ $y^2-3y-12=0$ $p=-3,\ q=-12$	

Qn	Solution	Marks
	Generally well done but a number of students made the careless mistake of giving the answer of *p* as 3 instead of -3. A number of students did not use the sum and product of roots to solve for p and q. They solve the equation to find the roots and use it to find the new equation to obtain the values of p and q.	

Qn	Solution	Marks
5.		[4 marks]
(a)	$\arcsin\left(\frac{1}{x}\right) \equiv \text{arccosec}(x)$ let $\arcsin\left(\frac{1}{x}\right) = \alpha$ $\sin\alpha = \frac{1}{x}$ $x = \frac{1}{\sin\alpha}$ $x = \text{cosec}\,\alpha$ $\alpha = \text{arccosec}(x)$. **Alternative:** Accept constructed triangle as working	
	Very poorly done. Most students cannot provide the proof	
(b)	$\sin(\text{arccosec}(2y)) = e^{-1}$. $\sin\left(\arcsin\left(\frac{1}{2y}\right)\right) = e^{-1}$ $\frac{1}{2y} = \frac{1}{e}$ $y = \frac{e}{2}$. $\text{arccosec}(2y) = \arcsin\left(\frac{1}{e}\right)$	
	Generally well done, but many students who could not do part (a) did not bother to do part (b).	

Qn	Solution	Marks
6.		[9 marks]
(a)	 t values as shown in graph. **Alternative:** $2\sin(4t)+t=t-2$ $2\sin(4t)=-2$ $\sin(4t)=-1$ $4t=\frac{3\pi}{2},\frac{7\pi}{2},\frac{11\pi}{2},\frac{15\pi}{2}$ $t=\frac{3\pi}{8},\frac{7\pi}{8},\frac{11\pi}{8},\frac{15\pi}{8}$	
	Generally well done, but a number of students did not sketch the graph as part of the working.	

Qn	Solution	Marks
(b)	 Using GDC, $h(t)=t+2$ is the upper limit Hence $-2\le k\le 2$ **Alternative:** $2\sin(4t)+t=t+k$ $\sin(4t)=\frac{k}{2}$ $-1\le\frac{k}{2}\le 1$ $-2\le k\le 2.$	
	Well done but a number of students give the upper limit as 1.98… instead of 2	
(c)	Most intersections is when line is center of the range of k. k = 0, most intersections = 9. **Alternative:** $\sin(4t)=0$ $\frac{k}{2}=0$ $k=0$ most intersections = 9.	
	Well done	

Qn	Solution	Marks
7.		[7 marks]
(a)	$z-1+i=x+iy-1+i$ $=(x-1)+i(y+1)$ $\tan\dfrac{\pi}{5}=\left(\dfrac{y+1}{x-1}\right)$ $y+1=(x-1)\tan\dfrac{\pi}{5}$ $y=x\tan\dfrac{\pi}{5}-\tan\dfrac{\pi}{5}-1$ $y=\left(\tan\dfrac{\pi}{5}\right)x+\left(-1-\tan\dfrac{\pi}{5}\right)$ $m=0.727 \qquad \text{c}=-1.73$	
	Very poorly done. Many students do not understand the question at all.	
(b)	Gradient is tan of the required angle: $\arg(z-1+i)=\dfrac{\pi}{5}+\dfrac{\pi}{3}=\dfrac{8\pi}{15}.$	
	Poorly done.	

Qn	Solution	Marks
8.		[9 marks]
(a)	$\begin{pmatrix}-2\\1\\-2\end{pmatrix}\times\begin{pmatrix}-1\\-3\\1\end{pmatrix}$ $=\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\3\\-1\end{pmatrix}$ $=\begin{pmatrix}1-6\\-(-2-2)\\6+1\end{pmatrix}$ $=\begin{pmatrix}-5\\4\\7\end{pmatrix}$ $r=\begin{pmatrix}-2\\1\\-2\end{pmatrix}+\mu\begin{pmatrix}-5\\4\\7\end{pmatrix},\ \mu\in\mathbb{R}.$	

Qn	Solution	Marks
	Many careless mistakes with the cross product. Some students used a longer method involving dot product and system of equations to work out the cross vector and wasted time.	
(b)	$\left\|\begin{pmatrix}-2\\1\\-2\end{pmatrix}\right\| = \sqrt{4+1+4} = \sqrt{9} = 3$ $\min\|a+b\| = 10-3 = 7$ $\max\|a+b\| = 10+3 = 13$ $7 \le \|a+b\| \le 13$	
	Badly done. Did not realize the maximum length is created by a vector in the same direction as the original and the minimum length is created by a vector in the Opposite direction.	
(c)	$\min \text{ when } b = -ka$ $\|b\| = k\|a\| = 10$ $k(3) = 10$ $k = \frac{10}{3}$ $b = -\frac{10}{3}\begin{pmatrix}-2\\1\\-2\end{pmatrix} \qquad \left(or \ \frac{10}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}\right)$	
	Badly done. Similar comment to above. Note this is an IB exam style question.	

Section B

Qn	Solution	Marks
9		[25 marks]
9(a)	$f(-x)$ $= 2^{-x} + \frac{1}{2^{-x}}$ $= 2^{x} + \frac{1}{2^{x}} = f(x) \qquad \therefore even$	
	A significant number of students thought that even function meant a function divisible by 2. Even function does not imply even 'number'.	
9(b)	$f(1) = 2^1 + 2^{-1} = 2.5$ $f(-1) = 2.5$	

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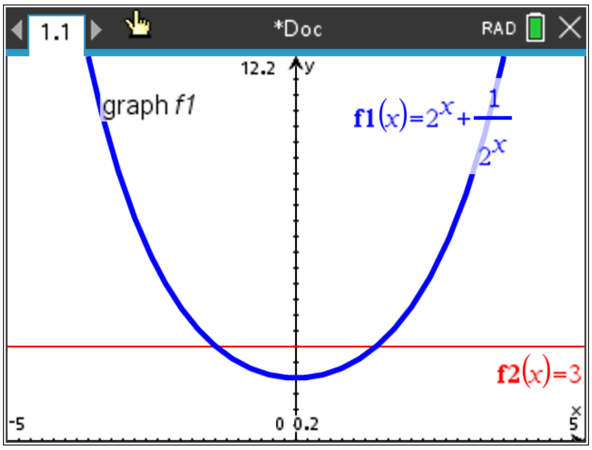
$x = 1, -1$ for $y = 2.5$.

$\therefore$ not one-one and hence no inverse.

Alternative:



Using horizontal line test, a horizontal y = 3 intersects the graph twice, hence the function is not one-one and inverse does not exist.

Or using symmetry of the even function to deduce x has multiple values for some values of y.

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Poorly done. It is not enough to state no inverse. Must indicate the reason for no inverse by either providing a counter example (one value) or using horizontal line test and show graphically that there are 2 values of x that correspond to one value of y. note also the function just need to be one to many at one point. The entire graph need not be always many to one.

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9(c)

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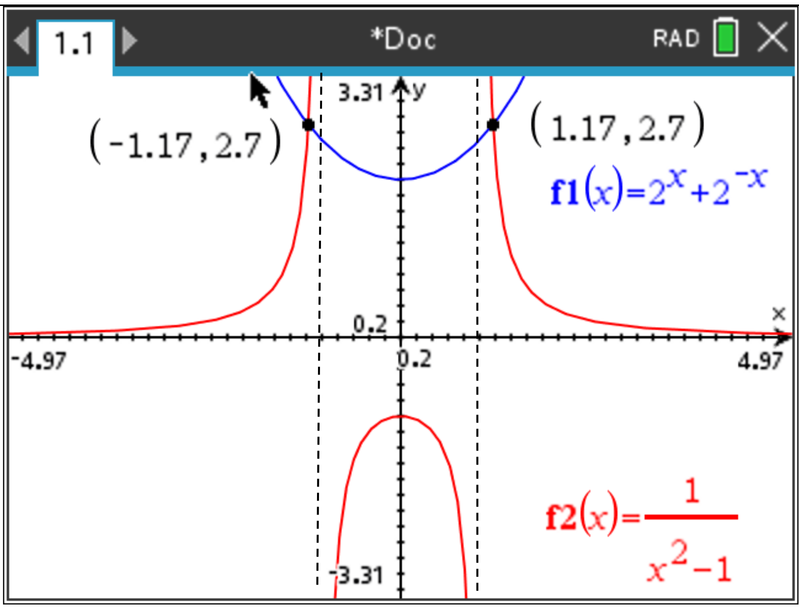

$x < -1.17$ or $-1 < x < 1$ or $x > 1.17$

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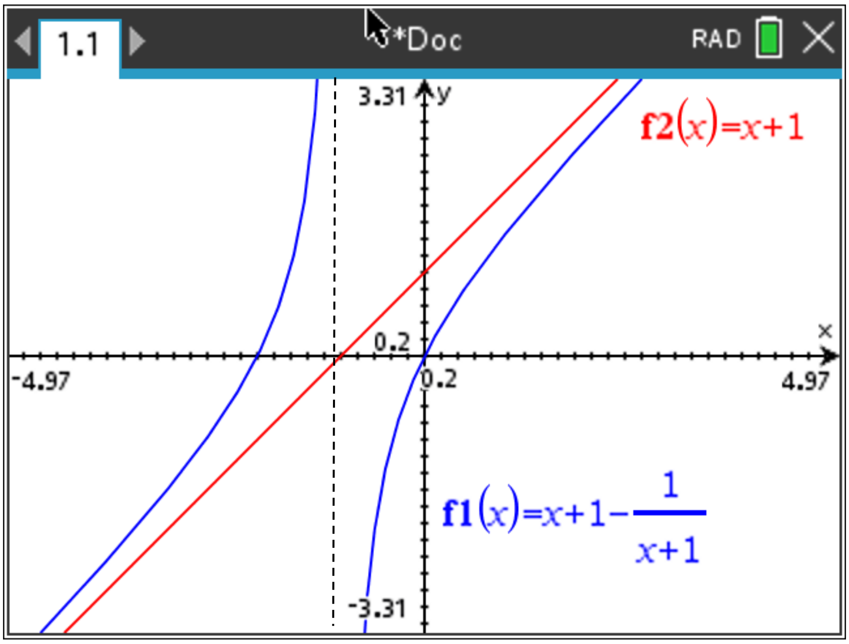
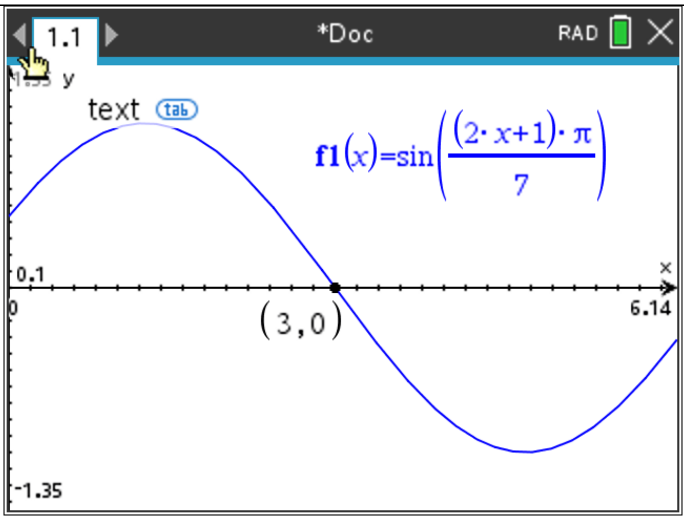
	Badly done. Many students did not appreciate the effects of the 2 vertical asymptotes and obtained wrong ranges of values. Even if the graph was sketched accurately, many did not give the complete range of values which has 3 sections.	
9(d)	$g\left(g\left(x\right)\right)=\dfrac{1}{\left(g\left(x\right)\right)^2-1}$ $=\dfrac{1}{\left(\dfrac{1}{x^2-1}\right)^2-1}$ $=\dfrac{\left(x^2-1\right)^2}{1-\left(x^2-1\right)^2}$ $\left(or\ \dfrac{x^4-2x^2+1}{-x^4+2x^2},\ \dfrac{1}{-x^4+2x^2}-1 \right)$	
	Badly done. Many did not realize there is a square in the denominator when substitute into g(x). A few misinterpret it as the square function of g(x). Several made all kinds of careless mistakes in simplifying the expression.	
9(e)	$\dfrac{1}{x^2-1}\to\dfrac{1}{2x^2-2}\to\dfrac{1}{-2x^2+2}\to\dfrac{1}{-2x^2+2}+5.$ $f\left(x\right)\to 0.5f\left(x\right)\to -0.5f\left(x\right)\to -0.5f\left(x\right)+5$ 1. Scale by 0.5 units parallel to the y-axis (note 1 and 2 are interchangeable) 2. Reflection about the x-axis 3. Translation by 5 units in the positive y-direction. **Alternative:** $\dfrac{1}{x^2-1}\to\dfrac{1}{x^2-1}-10\to\dfrac{1}{-x^2+1}+10\to\dfrac{1}{-2x^2+2}+5.$ 1. Translation by 10 units in the negative y-direction. 2. Reflection about the x-axis (note 2 and 3 are interchangeable) 3. Scale by 0.5 units parallel to the y-axis	
	Badly done. There are too many incorrect descriptions, phrasings, and sequences to list here. Students are highly advised to revised this topic again even if they may have obtained some marks for this part.	

9(f)	$y=\dfrac{1}{x^2-1}$ $x^2=\dfrac{1}{y}+1$ $x=\pm\sqrt{1+\dfrac{1}{y}} \quad (x<-1)$ $g^{-1}(x)=-\sqrt{1+\dfrac{1}{x}}$ $g^{-1}(2)=-\sqrt{1+\dfrac{1}{2}}=-\sqrt{\dfrac{3}{2}}. \quad \left(or \ -\dfrac{\sqrt{6}}{2}\right)$	
	Some did not take into account the restriction that the answer needs to be a negative number.	
9(g)	$h(x)=\dfrac{x^2+2x}{x+1}$ $=\dfrac{x(x+1)+(x+1)-1}{x+1}=x+1-\dfrac{1}{x+1}$  	
	Badly done. Some either forgot to sketch the diagonal asymptote or the vertical asymptote and obtained wrong answers.	
9(h)	Based on the graph, different gradient means the line with rotate about the center at (-1, 0). For at least one intersection, $m<1$.	

	See previous comment. Those who sketched wrongly tried to use analytical method but did not manipulate the inequalities accurately. Note that the analytical method is NOT RECOMMENDED as it is quite long.	

	Solution	**Marks**
10.		[11 marks]
10(a)	$\lvert zw \rvert = 2(8) = 16$	
	Good. A few added the modulus instead of multiply.	
10(b)	$\arg(zw) = \arg z + \arg w$ $= \dfrac{\pi}{7} + \dfrac{2k\pi}{7}$ $= \dfrac{\pi}{7}(2k+1)$	
	Good. A few made careless mistakes in simplifying the expression. Note that you have to give the angle and not the complex number itself.	
10(c)	$\dfrac{(2k+1)\pi}{7} = 0, \pi, 2\pi, \ldots$ $\dfrac{(2k+1)\pi}{7} = \pi$ $\dfrac{(2k+1)}{7} = 1$ $2k+1 = 7$ $k = 3$ **Alternative:** $\text{plot } \sin\left(\dfrac{(2x+1)\pi}{7}\right) = 0$	

	 $k=3.$		
	Badly done. Many could not solve the trigonometric expression correctly. Some did not consider the requirement for the k to be a positive number. A rare few used the graphical method which is the best for this question.		
10(d)	$p^4=8\times 2\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)$ $p^4=16\left(\cos\left(\frac{\pi}{7}+2k\pi\right)+i\sin\left(\frac{\pi}{7}+2k\pi\right)\right)$ $p=2\left(\cos\left(\frac{\pi}{28}+\frac{2k\pi}{4}\right)+i\sin\left(\frac{\pi}{28}+\frac{2k\pi}{4}\right)\right)$ $p=2\left(\cos\left(\frac{\pi}{28}+\frac{k\pi}{2}\right)+i\sin\left(\frac{\pi}{28}+\frac{k\pi}{2}\right)\right)$ $p=2\left(\cos\left(\frac{\pi+14k\pi}{28}\right)+i\sin\left(\frac{\pi+14k\pi}{28}\right)\right)$, where $k=-2,-1,0,1.$ $p=2e^{i\left(\frac{-27\pi}{28}\right)},2e^{i\left(\frac{-13\pi}{28}\right)},2e^{i\left(\frac{\pi}{28}\right)},2e^{i\left(\frac{15\pi}{28}\right)}.$		
	Ok. Some did not consider that the argument needs to be kept within the range as stated in the question. Some made careless mistakes in taking fourth root of 16.		

Qn	Solution	Marks
11		[19 marks]
11(a)	$\operatorname{rref}\left(\begin{bmatrix} 6 & 3 & -1 & -1 \\ 2 & -7 & 5 & 1 \\ 12 & 6 & -2 & -10 \end{bmatrix}\right)$ $\quad \begin{bmatrix} 1 & 0 & \frac{1}{6} & 0 \\ 0 & 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Contradiction. No solutions. Hence, the 3 planes do not intersect. **Alternative:** Scratchpad RAD $\operatorname{linSolve}\left(\begin{cases} 6\cdot x+3\cdot y-z=-1 \\ 2\cdot x-7\cdot y+5\cdot z=1 \\ 12\cdot x+6\cdot y-2\cdot z=-10 \end{cases},\{x,y,z\}\right)$ "No solution found" No solutions. Hence, the 3 planes do not intersect. **Alternative:** Since the equation of π_3 can be written as $6x+3y-z=-5$, and the equation of π_1 is $6x+3y-z=-1$, π_1 and π_3 are distinct and parallel. So the three planes do not have any common point.	
	Many students didn't present their answers appropriately. The answer should show explicitly the reduced row echelon form obtained from GDC or the system of linear equations input to GDC. Some students put a meaningless arrow or an incorrect equal sign between the original matrix and its RREF, not showing understanding of the method. To deduce the conclusion from GDC's RREF output, students need to interpret how the RREF implies that there is <u>no solutions</u> to the system of linear equations, which then translates to an implication in geometry for the three planes to have no common point.	
11(b)	$\prod_1$ and $\prod_3$ are parallel planes, and $\prod_2$ is plane intersecting with the 2 planes separately. Note: can accept diagram to represent the answer. Award 1 mark for indicating which planes are parallel. Award 1 mark for showing that the normal vectors indicate planes are parallel. Do not accept just "planes do not intersect"	
	This is poorly done in general. The descriptions should be presented as specific as possible in a direct manner. For example, instead of "two planes are parallel	

	and the third is not", students need to write "π_1 and π_3 are parallel and π_2 intersects them".	
11(c)	$\text{rref}\left(\begin{bmatrix}6&3&-1&-1\\2&-7&5&1\\0&0&0&0\end{bmatrix}\right) \qquad \begin{bmatrix}1&0&\frac{1}{6}&\frac{-1}{12}\\0&1&\frac{-2}{3}&\frac{-1}{6}\\0&0&0&0\end{bmatrix}$ $x+\frac{1}{6}z=\frac{-1}{12}\Rightarrow x=-\frac{1}{12}-\frac{1}{6}z$ $y-\frac{2}{3}z=\frac{-1}{6}\Rightarrow y=-\frac{1}{6}+\frac{2}{3}z$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}+\mu\begin{pmatrix}-1/6\\2/3\\1\end{pmatrix}=\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\6\end{pmatrix},\ \lambda\in\mathbb{R}.$ **Alternative 1:** $\text{linSolve}\left(\begin{cases}6\cdot x+3\cdot y-z=-1\\2\cdot x-7\cdot y+5\cdot z=1\\1=1\end{cases},\{x,y,z\}\right)$ $\left\{\frac{-c1}{6}-\frac{1}{12},\frac{2\cdot c1}{3}-\frac{1}{6},c1\right\}$ $x=-\frac{1}{12}-\frac{1}{6}\mu$ $y=-\frac{1}{6}+\frac{2}{3}\mu$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}+\mu\begin{pmatrix}-1/6\\2/3\\1\end{pmatrix}=\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\6\end{pmatrix},\ \lambda\in\mathbb{R}.$ **Alternative 2:**	

	$\begin{pmatrix}6\\3\\-1\end{pmatrix}\times\begin{pmatrix}2\\-7\\5\end{pmatrix}$ $=\begin{pmatrix}8\\-32\\-48\end{pmatrix}$ $=-8\begin{pmatrix}-1\\4\\6\end{pmatrix}$ Solve for point z = 0 $\left(\frac{-1}{12},\frac{-1}{6},0\right)$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\6\end{pmatrix},\ \lambda\in\mathbb{R}.$	
	This part is well done in general. One common mistake is that after obtaining $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}+\mu\begin{pmatrix}-1/6\\2/3\\1\end{pmatrix}$ from interpreting GDC output, many students correctly used a scalar multiple of $\begin{pmatrix}-1/6\\2/3\\1\end{pmatrix}$, that is, $\begin{pmatrix}-1\\4\\6\end{pmatrix}$, as the direction vector, but erroneously used a scalar multiple of $\begin{pmatrix}-1/12\\-1/6\\0\end{pmatrix}$ as the position of a point on the line as well, showing that they do not understand the difference between a direction vector and a position vector.	

11(d)	$\prod_1 : r.\begin{pmatrix}6\\3\\-1\end{pmatrix} = -1$ $\prod_3 : r.\begin{pmatrix}-12\\-6\\2\end{pmatrix} = 10 \qquad \Rightarrow \prod_3 : r.\begin{pmatrix}6\\3\\-1\end{pmatrix} = -5$ $\lvert D_1 - D_3 \rvert = \left\lvert \frac{-1}{\sqrt{6^2+3^2+1^2}} - \frac{-5}{\sqrt{6^2+3^2+1^2}} \right\rvert$ $= \left\lvert \frac{4}{\sqrt{46}} \right\rvert$ $= 0.590$ **Alternative:** $\text{Distance} = \left\lvert \frac{12\left(-\frac{1}{12}\right)+6\left(-\frac{1}{6}\right)+10}{\sqrt{12^2+6^2+2^2}} \right\rvert$ $= \left\lvert \frac{8}{\sqrt{184}} \right\rvert$ $= \left\lvert \frac{4}{\sqrt{46}} \right\rvert$ $= 0.590$	
	This part is poorly done in general. Many students did not realise that the distance required is the same as the distance between the planes. Some of those who took an alternative approach to find the foot of perpendicular from L to the plane first often could not set up the equation of the line from L to the foot correctly. And after much effort to find the position vector of the foot of perpendicular, some mistook the modulus of the position vector as the distance required.	

11(e)	$\Pi_3 : r.\begin{pmatrix}-12\\-6\\2\end{pmatrix}=10$ 1) $\overrightarrow{OF}=\mu\begin{pmatrix}-12\\-6\\2\end{pmatrix}$ 2) $\overrightarrow{OF}.\begin{pmatrix}-12\\-6\\2\end{pmatrix}=10$ $\Rightarrow \mu\begin{pmatrix}-12\\-6\\2\end{pmatrix}.\begin{pmatrix}-12\\-6\\2\end{pmatrix}=10$ $(144+36+4)\mu=10$ $184\mu=10$ $\mu=0.0543$ $\overrightarrow{OF}=0.0543\begin{pmatrix}-12\\-6\\2\end{pmatrix}$ $F\left(\frac{-15}{23},\frac{-15}{46},\frac{5}{46}\right)$ or $(-0.652,-0.326,0.109)$	
	Many students could apply the correct method to find the foot of perpendicular. Some failed to present the answer in coordinates as required in the question.	
11(f)	$\overrightarrow{OF}=\frac{1}{2}\overrightarrow{OP}$ $\overrightarrow{OP}=2(0.0543)\begin{pmatrix}-12\\-6\\2\end{pmatrix}$ $\begin{bmatrix}-12\\-6\\2\end{bmatrix}\cdot 2\cdot 0.054347826086957 \quad \begin{bmatrix}-1.30435\\-0.652174\\0.217391\end{bmatrix}$ Coordinates of P = (-1.30, -0.652, 0.217) or $\left(\frac{-30}{23},\frac{-15}{23},\frac{5}{23}\right)$	
	Many students could not present the solution appropriately to show that the answer is derived from the previous part. Students need to know that in answering a “Hence” question they must show clearly how the answers from the earlier parts can be applied to find the answer in this part.	