9. Oscillations

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Learning Objectives

Note: this entire chapter is H2 only.

9.1 Simple harmonic motion

(a) Describe simple examples of free oscillations, where particles periodically return to an equilibrium position without gaining energy from or losing energy to the environment.

(b) Investigate the motion of an oscillator using experimental and graphical methods.

(c) Show an understanding of and use the terms amplitude, period, frequency, angular frequency, phase and phase difference, and express the period in terms of both frequency and angular frequency.

(d) Show an understanding that $a = -\omega^2 x$ is the defining equation of simple harmonic motion, where acceleration is (directly) proportional to displacement from an equilibrium position and acceleration is always directed towards the equilibrium position.

(e) Recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.

(f) Recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\omega\sqrt{{x_0}^2 - x^2}$.

(g) Describe, with graphical illustrations, the relationships between displacement, velocity, and acceleration during simple harmonic motion.

9.2 Energy in oscillations

(h) Describe the interchange between kinetic and potential energy during simple harmonic motion.

(i) Describe practical examples of damped oscillations, with particular reference to the effects of the degree of damping (light/under, critical, heavy/over), and to the importance of critical damping in applications such as a car suspension system.

(j) Describe graphically how the amplitude of a forced oscillation changes with driving frequency, resulting in maximum amplitude at resonance when the driving frequency is close to or at the natural frequency [1] of the system.

(k) Show a qualitative understanding of the effects of damping on the frequency response and sharpness of the resonance.

(l) Describe practical examples of forced oscillations and resonance, and show an appreciation that there are some circumstances in which resonance is useful, and other circumstances in which resonance should be avoided.

Important Note: For this topic, your calculator needs to be in **<u>radian mode</u>**.

Otherwise, there will be calculation errors.

[1] Natural frequency refers to the frequency when the system is in free oscillation.

9.1 SIMPLE HARMONIC MOTION

9.1.1 Free Oscillation and Simple Harmonic Motion

Self-study resources		
SLS lesson on SHM (1/5)	**https://for.edu.sg/shm1**	

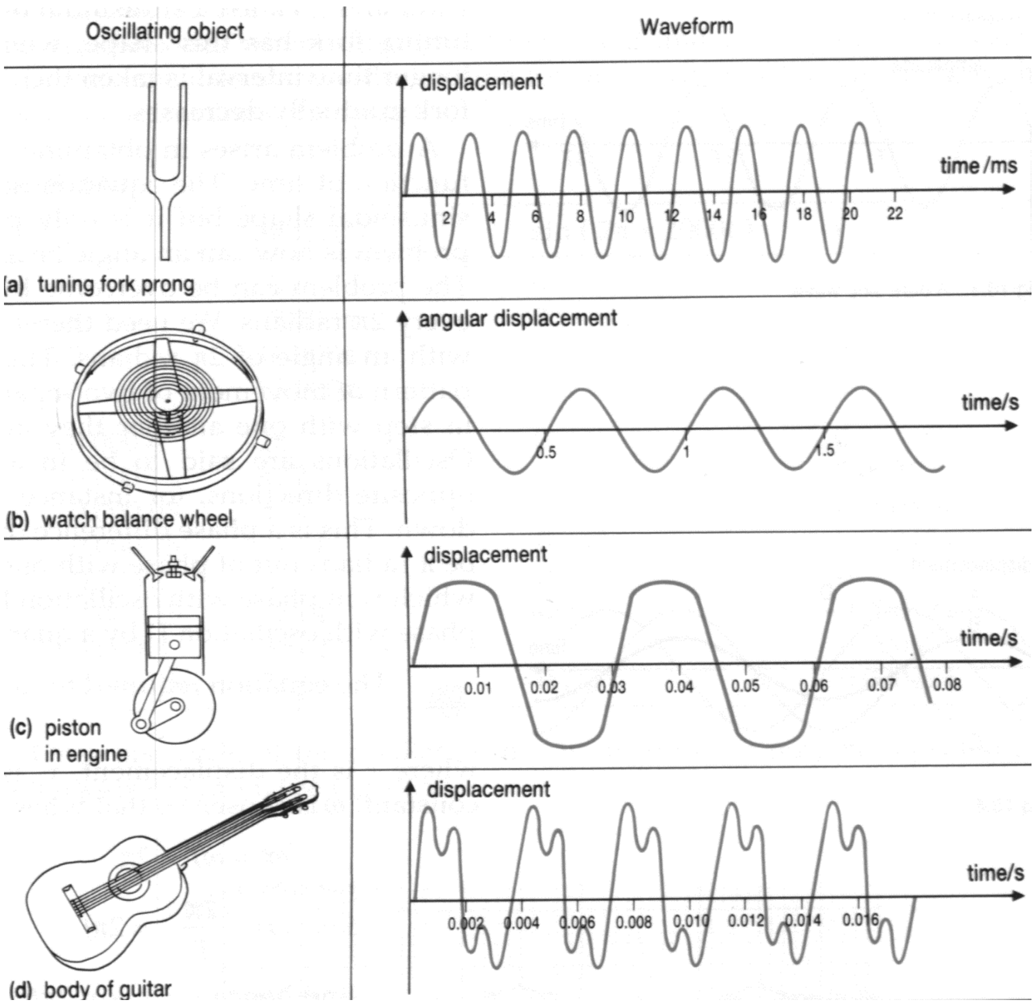
A system is said to be undergoing a free oscillation when the only external force acting on it is a **restoring force**.

A restoring force is a resultant force that acts in opposite direction to the displacement of the object from its equilibrium position (where the resultant force is zero).

A free oscillation is usually set into motion by displacing the system from its equilibrium position and then releasing it.

Simple harmonic motion is the simplest type of free oscillation.

Examples of oscillations



Self-study resources		
Other examples	https://youtu.be/VKtEzKcg6_s	

9.1.2 The Meaning of Some Terms used in SHM

Term	Symbol	Meaning	SI unit	Vector / Scalar	Relevant Equation
Period	T	time taken for one complete oscillation	s	scalar	
Frequency	f	number of oscillations per unit time 1 Hz = 1 cycle (or oscillation) per second	Hz (hertz)	scalar	$f = \frac{1}{T}$
Angular Frequency	ω	product of frequency and 2π rad can also be understood as the rate of change of phase	rad s^{-1}	scalar	$\omega = 2\pi f$ $\omega = \frac{2\pi}{T}$
Equilibrium Position	position at which no net force acts on an oscillating object				
Displacement	x	distance from a fixed point in a specified direction measured with respect from the equilibrium position in the context of oscillatory motion	m	vector	
Amplitude	x_0	magnitude of the maximum displacement of an oscillation	m	scalar	$x = x_0 \sin(\omega t + \phi)$
Phase	θ	denotes the <u>state of the oscillatory cycle</u> at a particular time t	rad	scalar	$\theta = \omega t + \phi$
Initial Phase	ϕ	state of the oscillatory cycle at $t = 0$	rad	scalar	
Phase Difference	$\Delta\theta$	used to compare the difference in phase between two oscillators or two instances in time for one oscillator a measure of how much an oscillation is out of step with another	rad	scalar	
in phase	Two oscillations that are exactly in step with one another are said to be <u>in phase</u> with one another The <u>phase difference</u> these two oscillations is any even integer multiples of π rad, including 0 rad.				
in antiphase	Two oscillations that are <u>always moving in opposite directions</u> are said to be <u>in antiphase</u> with one another. They can also be understood as being exactly half a cycle out of step with one another. The <u>phase difference</u> between these two oscillations is any odd integer multiples of π rad.				

9.1.3 Simple Harmonic Motion (SHM)

Definition

The motion of an object is simple harmonic if its acceleration is directly proportional to its displacement from its equilibrium position and its acceleration is always opposite in direction to its displacement (this also means that its acceleration is always directed towards the equilibrium position).

MUST MEMORISE!

Note:

To check if a motion is simple harmonic, we can use **Newton's 2nd law** to find the relationship between acceleration a and displacement x.

If the motion is simple harmonic, acceleration is directly proportional to – displacement (i.e. a ∝ –x).

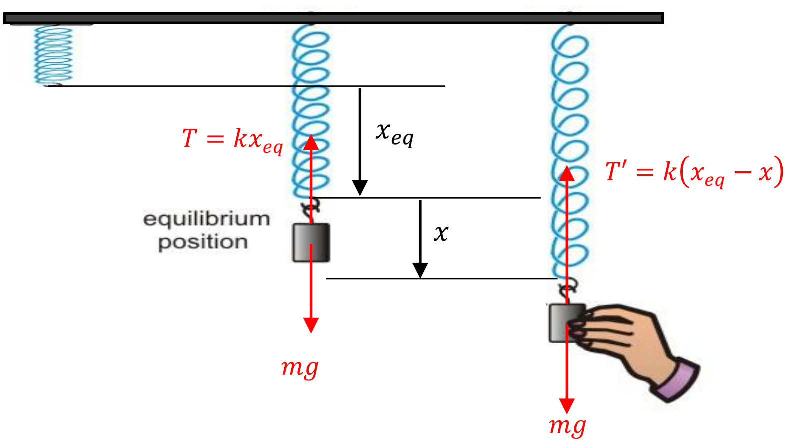
In such cases, we usually write:

$$a = -\omega^2 x$$

where ω^2 is the constant of proportionality.

Example 1

The figure below shows a mass m attached to an ideal spring of spring constant k. When the mass is at rest, the extension of the spring is x_{eq}.



The mass is then pulled downwards by a small displacement x where $x < x_{eq}$ and released.

(a) Show that the subsequent motion of the mass is simple harmonic.

(b) Determine an expression for the period and frequency of oscillation.

(c) Let $t = 0$ when the mass is released. State an equation to describe the variation with time t of the displacement x of the mass.

Solution

(a) Taking vectors upwards as positive,
When mass is at the equilibrium position, from Newton's 2nd law: $T - mg = 0$

$$T = kx_{eq} = mg \quad \text{------------} \quad (1)$$

When mass is displaced from the equilibrium position, from Newton's 2nd law: $mg - T' = ma$
Note: we take mg – T' so that we account for the restoring direction of acceleration relative to displacement

$mg - k(x_{eq} + x) = ma$
$mg - kx_{eq} - kx = ma$
$-kx = ma$ since from (1), $kx_{eq} = mg$
$a = -\frac{k}{m}x$ ------------ (2)

Since k and m are constants, equation (2) shows that the acceleration a of m is directly proportional to its displacement x from its equilibrium position and is directed in the opposite direction to its displacement x due to the negative sign. Hence the motion of the mass m is simple harmonic.

(b) Comparing equation (2) with the defining equation for s.h.m. ie. $a = -\omega^2 x$,

we have: $\omega^2 = \frac{k}{m}$ $\Rightarrow$ $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$

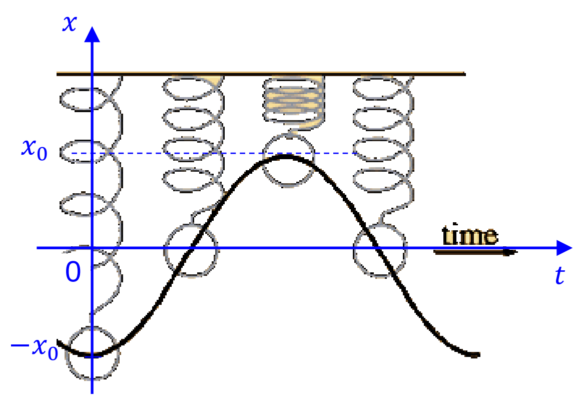
Period of oscillation, $T = 2\pi\sqrt{\frac{m}{k}}$

Frequency of oscillation, $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

(c) Imagine the motion of the mass after it is released:

Taking displacement above the equilibrium position as positive:

$$x = -x_0 \cos(\omega t) = -x_0 \cos\left(\sqrt{\frac{k}{m}}t\right)$$



Note: If you differentiate the above equation with respect to time t twice, you will get

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}\left[-x_0 \cos\left(\sqrt{\frac{k}{m}}t\right)\right] = -\frac{k}{m}x$$

showing that the equation stated is a solution to $a = -\frac{k}{m}x$

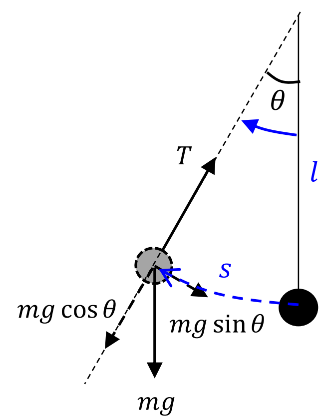
Note: $x = -x_0$ when $t = 0$

Example 2 **(For illustration only. No need to memorise.)**

Show that the motion of a simple pendulum of length l is simple harmonic if the displacement is small and hence derive an expression for the period T of the pendulum in terms of l and any other constants.

Solution

If θ is sufficiently small, the arc s can be approximated to a horizontal line of displacement s.



The approximately horizontal restoring force is $-mg\sin\theta \approx -mg\theta$ for small angular displacement θ. The negative sign indicates that the direction of the restoring force is opposite to the angular displacement θ.

Hence, from Newton's 2nd law:

$$-mg\theta = ma \qquad \text{where } \theta \text{ in radians} \approx \frac{s}{l}$$

$$-mg\frac{s}{l} = ma$$

$$a = -\left(\frac{g}{l}\right)s \qquad \text{------------} \qquad (1)$$

Equation (1) shows that the acceleration of the bob is directly proportional to its displacement since
g and l are constants and a is directed in a direction opposite to s from the negative sign. Hence the motion of the bob is simple harmonic.

To find the period, we have to compare equation (1) with the defining equation for s.h.m., $a = -\omega^2 x$, to get:

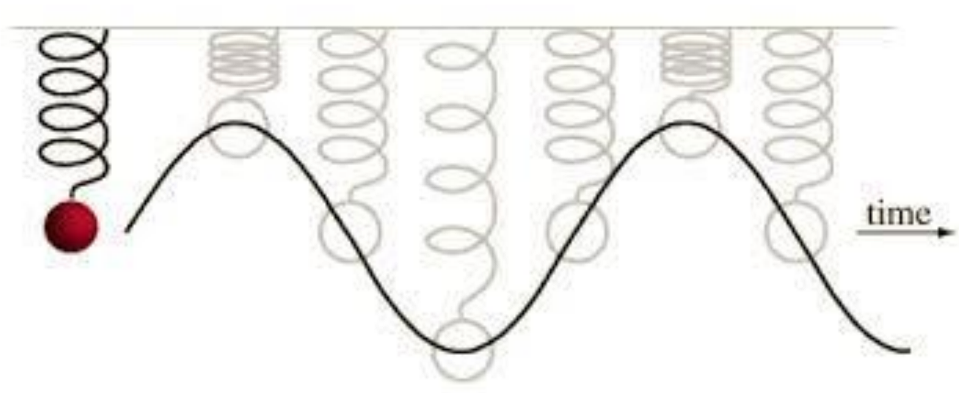
$$\omega^2 = \frac{g}{l} \qquad \Rightarrow \qquad \omega = \sqrt{\frac{g}{l}} = \frac{2\pi}{T} \qquad \Rightarrow \qquad \text{Period, } T = 2\pi\sqrt{\frac{l}{g}}$$

9.1.4 Equations of Motion in SHM

Self-study resources		
SLS lesson on SHM (2/5)	**https://for.edu.sg/shm2**	

A mass attached to the end of a long spring can be displaced to oscillate with a harmonic motion. The motion is maintained by a restoring force and is sinusoidal in time, with a single frequency.

As the object oscillates up and down, its displacement when mapped with time constitutes a sine or cosine curve.



$$x = x_0 sin\, \omega t$$

is used to describe the subsequent motion of an SHM system, if timing is started when the object is at the equilibrium position.

$$x = x_0\, cos\, \omega t$$

is used when the timing is started while the oscillating mass is at the amplitude position, i.e., when $t = 0$, $x = x_0$.

Using the relationship $v = \frac{dx}{dt}$, we have

$$v = \omega x_0 \cos(\omega t) \text{ or } v = -\omega x_0 \sin(\omega t)$$

Differentiating velocity with respect to time, and substituting the original equation for x, we will obtain the relationship

$$a = -\omega^2 x$$

Without doing a full derivation, we can also find velocity in terms of displacement using this equation:

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

Note: x_0 is the amplitude,

ωx_0 is the maximum magnitude of velocity and,
$\omega^2 x_0$ is the maximum magnitude of acceleration.

Example 3

The displacement x (in metres) of an object in simple harmonic motion is given by

$$x = 2.0\ \sin 4.0t$$

(a) State the amplitude of the oscillations
(b) State the angular frequency of the oscillations
(c) Calculate the magnitude of the maximum acceleration of the object.
(d) Derive an expression for the velocity of the object in terms of the time t
(e) Calculate the maximum velocity of the object and the next time it occurs after $t = 0$ s.
(f) Calculate the displacement at which the velocity of the object is half its maximum value.
(g) Derive an expression for the acceleration of the object in terms of the time

Solution

(a) $x_0 = 2.0\ \mathrm{m}$
(b) $\omega = 4.0\ \mathrm{rad\ s^{-1}}$
(c) $a_0 = \omega^2 x_0 = 4.0^2 \times 2.0 = 32\ \mathrm{m\ s^{-2}}$
(d) $v = \frac{dx}{dt} = 2.0 \times 4.0 \cos 4.0t = 8.0 \cos 4.0t$ (in m $\mathrm{s^{-1}}$)
(e)
$8.0\ \mathrm{m\ s^{-1}}$ (when $t = 0$ s)

At $t = 0$ s, the object is at the equilibrium position. The velocity is maximum at the equilibrium position.
The next time the maximum velocity occurs after $t = 0$ s is the next time the object is at the equilibrium position which is half a period later. These two maximum velocities are however in opposite directions.

$\omega = \frac{2\pi}{T}$
$T = \frac{2\pi}{4.0} = \frac{\pi}{2}$

The next time the object is at maximum velocity after $t = 0$ s is $t = \frac{\pi}{4} = 0.79$ s.

Note: You can substitute $t = 0.79$ s to see that the velocity is also 8.0 m $\mathrm{s^{-1}}$ in the opposite direction.

(f) $v = \pm\omega\sqrt{x_0^2 - x^2}$
$4.0 = \pm 4.0\sqrt{2.0^2 - x^2}$
$x = \pm 1.7\ \mathrm{m}$

(g) $a = \frac{dv}{dt} = 8.0 \times 4.0(-\sin 4.0t) = -32 \sin 4.0t$

(Observe that the maximum acceleration is consistent with the value calculated in (c).)

Exercise 1: Free Oscillations

You should take about 30 minutes to complete this exercise

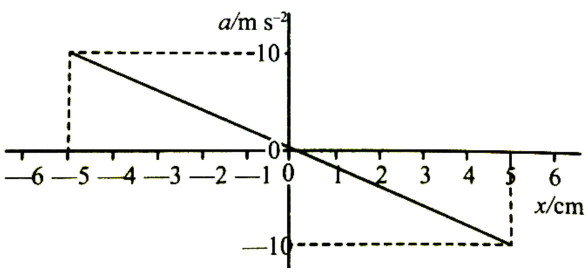
E1. The displacement x (in metres) of a body in oscillatory motion is given by the equation:
$x = 3.5 \sin 4.0\, t$

a. Determine the amplitude of the motion.
b. Determine the angular frequency.
c. Determine the displacement 0.20 s after oscillation is begun.
d. Find the expression of velocity in terms of time and,
e. Hence determine the velocity at time t = 0.50 s.

E2. [N06/1/13] A simple harmonic oscillator has a time period of 10 seconds. Which equation relates its acceleration **a** and displacement **x**?

A $a = -(20/2\pi)^2x$ **B** $a = -(20\pi)^2x$
C $a = -(20/2\pi)x$ **D** $a = -(2\pi/10)^2x$

E3. [N08/1/15] The defining equation for a particle moving in simple harmonic motion a=-2x where a is the acceleration of the particle, x is the displacement and is the angular frequency. The graph on the right shows how a varies with x for a particle moving in simple harmonic motion.



Which is the amplitude and period of the motion?

	Amplitude/cm	period/s
A	5.0	0.44
B	5.0	14
C	10	0.44
D	10	14

E4. A body performs simple harmonic motion with period of 0.063 s.
The maximum speed is 3.0 ms^{-1}.
What are the values of the amplitude x_0 and the angular frequency ω?

	x_0/m	ω/$rads^{-1}$
A	0.030	100
B	0.19	16
C	5.3	16
D	33	100

E5. An oscillating system performs s.h.m. with an amplitude of 2.0 cm. If the period of the motion is 5.0 s, calculate

(a) the magnitude of the velocity of the particle at 0.50 cm from equilibrium position.

(b) the maximum acceleration of the particle.

E6. An object moving with simple harmonic motion has an amplitude of 0.020 m and a frequency of 20 Hz. Calculate

a. the period of oscillation.
b. the velocity at the equilibrium point and at the position of maximum negative displacement.

E7. The motion of a piston in a certain car engine is approximately simple harmonic with amplitude
40 mm. The frequency of oscillation is 120 Hz. Determine
a. the maximum acceleration,
b. the maximum speed, of the piston.

E8. When the mass *M* on the spring is 0.040 kg, the vertical displacement y of the mass varies with time *t* according to the relation
$y = a \cos \omega t$
where a = 0.010 m and ω = 20 rad s^{-1}.
Determine
(a) the amplitude of the vibration,
(b) the period *T* of the vibration.

Answer key

E1 a) 3.5m, b) 4.0 rad s^{-1}, c) 2.5 m, d) v = 14 cos 4.0t, e) -5.8 m s^{-1} E2 D E3 A

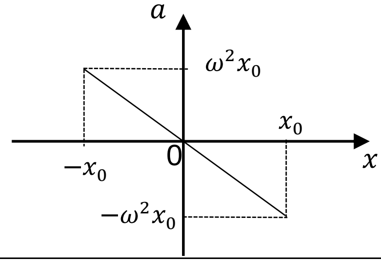
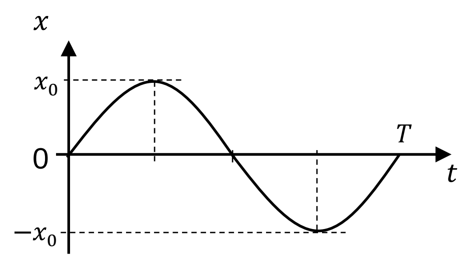
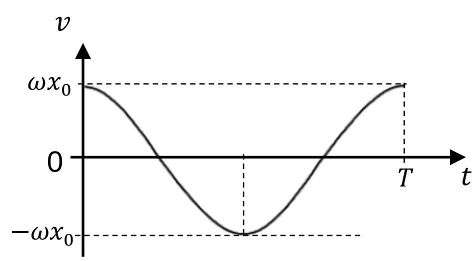
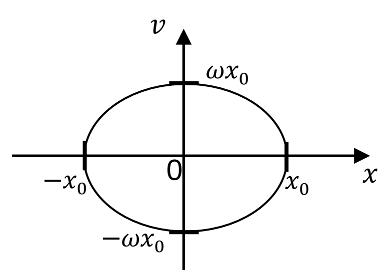
E4 A E5 a) 2.43 cm s^{-1}, b) 3.16 cm s^{-2} E6 a) 0.050s, b) 2.51 m s^{-1}, 0 m s^{-}

E7 a) 2.27 x 10^4 ms^{-2}, b) 30.1 ms^{-1} E8 a) 0.010 m, b) 0.314 s

9.1.5 Graphing the equations of motion

Self-study resources		
SLS lesson on SHM (3/5)	**https://for.edu.sg/shm3**	

You also need to know how to sketch the graphs for each equation of motion.

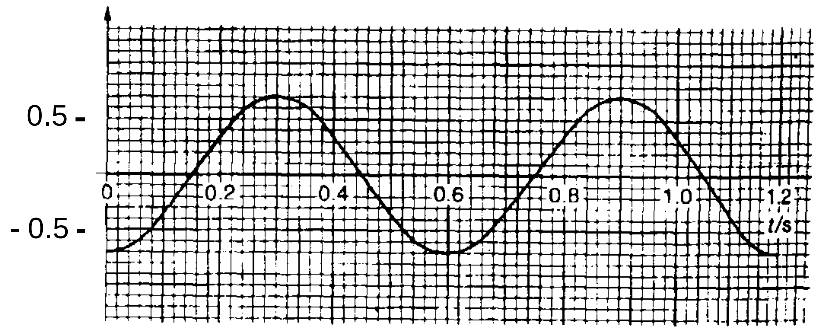
Equation	Graph
$a = -\omega^2 x$	
$x = x_0 \sin \omega t$	
$v = \frac{dx}{dt} = \omega x_0 \cos \omega t$ OR $v = v_0 \cos \omega t$ where $v_0 = \omega x_0$	
$v = \pm\omega\sqrt{{x_0}^2 - x^2}$	

Explain the physical significance of the " $\pm$ " sign in the formula $v = \pm\omega\sqrt{{x_0}^2 - x^2}$

At any displacement x, the oscillator can be moving up or down (to or fro), hence the $\pm v$

The relationship between acceleration, velocity and displacement can be studied by looking at the equations and graphs above.

Example 4



The graph above shows the displacement – time graph of an object oscillating in simple harmonic motion.

(a) State the amplitude of the oscillations.
(b) Calculate the angular frequency of the oscillations.
(c) Calculate the maximum velocity of the oscillations and circle the points on the graph at which the maximum velocity occurs.
(d) Calculate the maximum acceleration and state the time at which the acceleration is maximum.

Solution

(a)

0.70 m

(b)

$T = 0.60 \text{ s}$

$\omega = \frac{2\pi}{T} = 10.5 \text{ rad s}^{-1}$

(c)

$v_0 = \omega x_0 = 10.5 \times 0.70 = 7.3 \text{ m s}^{-1}$ (at the equilibrium positions)

(d)

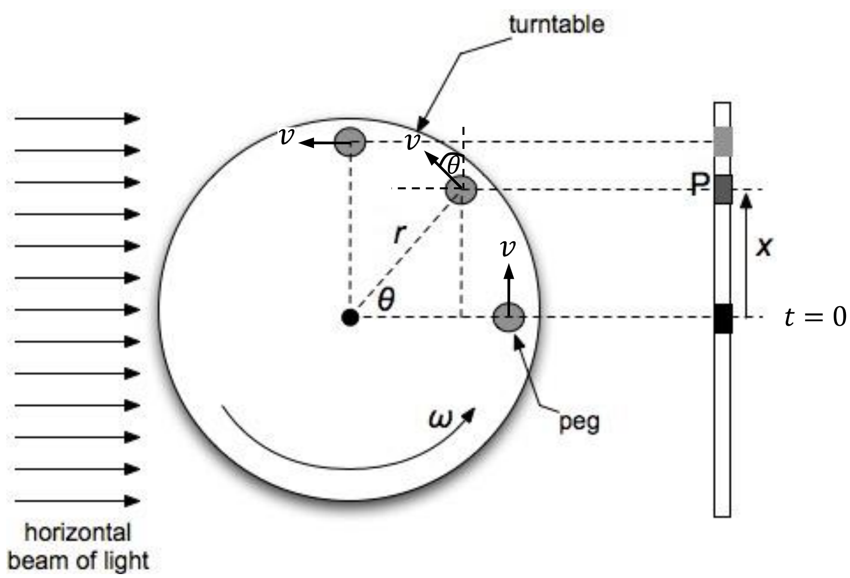
$a_0 = \omega^2 x_0 = 10.5^2 \times 0.70 = 77\ m\ s^{-2}$

At the amplitude positions where $t = \ 0 \text{ s}, 0.3 \text{ s}, 0.6 \text{ s}, 0.9 \text{ s}, 1.2 \text{ s}$

9.1.6 Relation between uniform circular motion and simple harmonic motion

Self-study resources		
Simple Harmonic Motion animation relating simple harmonic motion to uniform circular motion.	https://ophysics.com/w4.html	

The figure below shows the top view of a peg fixed on a turntable that is turning at uniform angular velocity ω.



<u>Displacement from reference point</u>

Displacement x of the shadow is measured from the reference point when $t = 0$, as shown in Fig. 1.8.

At time t, $\theta = \omega t$

So, $x = r\sin\theta = r\sin(\omega t)$

The shadow is at maximum displacement x_0 from the reference point when $\sin(\omega t) = 1 \Rightarrow x_0 = r$

Since the motion of the shadow obeys $x = x_0 \sin(\omega t)$, the shadow motion is simple harmonic.

circular Motion	oscillations of shadow
radius of circular path *r*	amplitude x_0
angular velocity ω	angular frequency ω

Velocity

Linear velocity v of the peg is $v = \omega r$

At time t, the velocity of the shadow v_{shadow} is the component of velocity v parallel to the screen

$$v_{\text{shadow}} = v\cos(\omega t)$$
$$= \omega r \cos(\omega t)$$

This is the same form as the velocity–time relation (eq. 3), $v = \omega x_0 \cos(\omega t)$.

Difference between circular motion and simple harmonic motion

There are many similarities between circular motion and oscillations. However, in oscillations, there is no physical angular displacement. The angle ωt in the equations in section 1.2 is interpreted as the fraction of the cycle completed and is known as the **phase**. The interpretation of phase is:

- In time Δt, the phase $= 2\pi \times \frac{\Delta t}{T}$
- A body completes one cycle in one period will have a phase $= 2\pi \times \frac{T}{T} = 2\pi$

We can draw parallels between the terms used in circular motion and oscillations.

	Circular Motion	**Oscillations**
θ	angular displacement	phase (fraction of the cycle completed)
period T	time to complete one revolution	time to complete one cycle
frequency $f = \frac{1}{T}$	number of revolutions in 1 s	number of cycles in 1 s
$\omega = \frac{2\pi}{T} = 2\pi f$	angular velocity	angular frequency

9.2 ENERGY IN OSCILLATIONS

Self-study resources		
SLS lesson on SHM (4/5)	**https://for.edu.sg/shm4**	

9.2.1 Energy in Simple Harmonic Motion

In oscillating systems, we study how energy moves between different stores as the system moves through its cycle. The concept of **energy stores** gives us a more intuitive framework to understand what happens during oscillation.

1. Types of Energy Transfers in Oscillating Systems

Depending on the system, the energy transfers between kinetic store (KE) and various forms of potential energy stores (PE):

- **Simple Pendulum**
    - Energy transfers between **kinetic store** and **gravitational potential store**.
    - At the lowest point, energy is fully in the kinetic store; at the highest points, fully in the gravitational store.
- **Mass–Spring System (Horizontal)**
    - Energy transfers between **kinetic store** and **elastic potential store**.
    - There is no change in gravitational store (assuming horizontal plane).
- **Mass–Spring System (Vertical)**
    - Energy transfers among **kinetic store**, **gravitational potential store**, and **elastic potential store**.
    - The gravitational store PE adds complexity, as the spring's equilibrium position is now shifted by the weight of the mass.

💡 *Important:* In all cases, the resultant force is the mechanism by which mechanical energy is transferred between stores.

9.2.2 Expressions for Energy

For all simple harmonic systems, the **total mechanical energy** remains constant in the absence of damping:

$$E_T = \frac{1}{2}m\omega^2{x_0}^2$$

- Depends only on amplitude x_0 of motion.
- This is also the maximum potential energy (when KE = 0), or the maximum kinetic energy (when PE = 0).

Consider the following equation of motion for a mass m in SHM: $v = \pm\omega\sqrt{{x_0}^2 - x^2}$

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\pm\omega\sqrt{{x_0}^2 - x^2}\right]^2$$

$$E_k = \frac{1}{2}m\omega^2{x_0}^2 - \frac{1}{2}m\omega^2x^2$$

$$E_k = \frac{1}{2}m\omega^2({x_0}^2 - x^2)$$

From the relationship $E_T = E_k + E_p$, we can then show that potential energy (at point x) is

$$E_p = \frac{1}{2}m\omega^2x^2$$

- where x is the displacement from equilibrium.
- This refers to the **total** potential energy which, depending on the context, often consists of **both/either** gravitational and/or elastic potential energy.

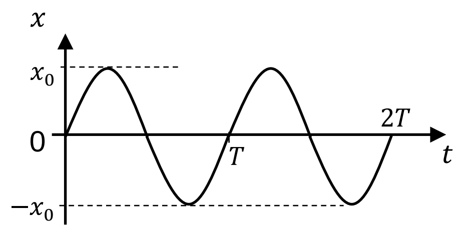
Example 5

Consider a spring-mass system of mass m and spring constant k, oscillating vertically in simple harmonic motion with amplitude x_0 as shown below.

At $t = 0$, the mass is at the equilibrium position and moving upwards.

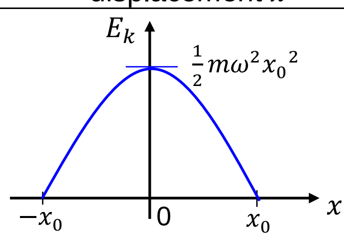
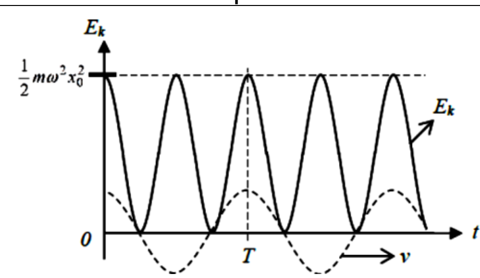
(a) State an equation for the displacement x of the mass as a function of time t and sketch the graph of x against t for two periods.

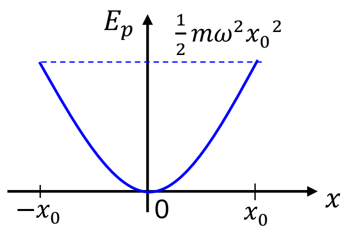
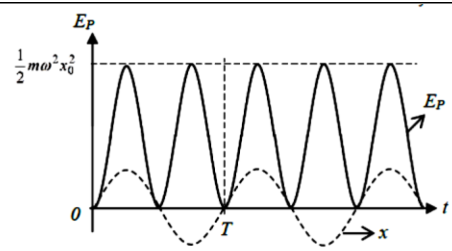
$x = x_0 \sin \omega t$



(b) Sketch the following graphs and state the equation relating the two variables represented on each of the graphs.

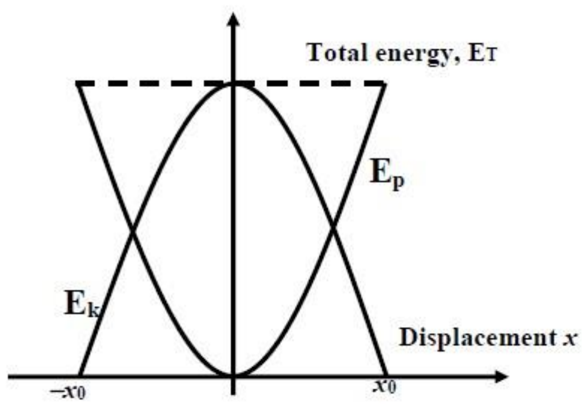
Self-study resources	
Simple Harmonic Motion oscillation with graphs	https://ophysics.com/w1.html

kinetic energy E_k against displacement x	kinetic energy E_k against time t for 2 periods
	 Note: Please label the time axis carefully. The graph drawn is actually t = 2.25T at the end.
$E_k = \frac{1}{2}m\omega^2{x_0}^2 - \frac{1}{2}m\omega^2x^2$	$E_k = \frac{1}{2}m\omega^2{x_0}^2\cos^2\omega t$

<table>
<tr><td>potential energy E_p against displacement x</td><td>potential energy E_p against time t for 2 periods</td></tr>
<tr><td>
</td><td>

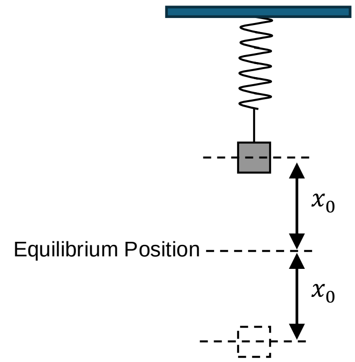
Note: Please label the time axis carefully.
The graph drawn is actually t = 2.5T at the end.</td></tr>
<tr><td>$E_p = \frac{1}{2}m\omega^2x^2$</td><td>$E_p = \frac{1}{2}m\omega^2{x_0}^2\sin^2\omega t$</td></tr>
</table>

Note that the total energy graph can be obtained by adding the graphs of the kinetic and potential energy graphs. The total energy which is constant will be a horizontal straight line when plotted against either the displacement or the time axis.



Example 6

Consider the same spring-mass system mentioned in the previous example.



Suppose that, at $x = +x_0$, the extension of the spring is zero.

(a) State an equation relating the forces acting on mass, m at the equilibrium position. Express your answer in terms of the spring constant k, x_0, m and the free fall acceleration, g.

(b) Hence, show that the elastic potential energy stored in the spring when the mass is at the equilibrium position is given by

$$\text{elastic potential energy} = \frac{1}{2}mgx_0$$

(c) Complete the table below. Express your answer in terms of mgx_0 where applicable.

	mass at the highest point of oscillation	mass at the equilibrium position	mass at the lowest point of oscillation
kinetic energy			
gravitational potential energy			0
elastic potential energy	0	$\frac{1}{2}mgx_0$	
total energy			

(d) Using your answers to (c), sketch, in the figure below, the variation with displacement x,

(i) the kinetic energy,

(ii) the gravitational potential energy,

(iii) the elastic potential energy, and

(iv) the total energy.

(e) The total energy of oscillation is $E_T = \frac{1}{2}m\omega^2{x_0}^2$ where the angular frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$. Using your answer in (a), express E_T in terms of m, g and x_0.

(f) Comment on your expression for E_T obtained in (e) in relation to your answers in (c).

Solution

(a) $mg = kx_0$

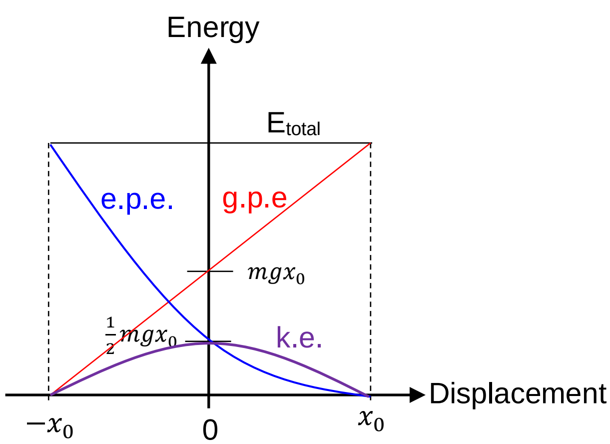
(b) At the equilibrium position, the extension of the spring is $x = x_0$

The elastic potential energy $= \frac{1}{2}kx^2 = \frac{1}{2}kx_0{}^2 = \frac{1}{2}(kx_0)x_0 = \frac{1}{2}mgx_0$ since $mg = kx_0$ (shown)

(c)

	mass at the highest point of oscillation	mass at the equilibrium position	mass at the lowest point of oscillation
kinetic energy	0	$\frac{1}{2}mgx_0$	0
gravitational potential energy	$2mgx_0$	mgx_0	0
elastic potential energy	0	$\frac{1}{2}mgx_0$	$2mgx_0$
total energy	$2mgx_0$	$2mgx_0$	$2mgx_0$

(d)



(e) From $\omega = \sqrt{\frac{k}{m}}$, we have $m\omega^2 = k$ hence $E_T = \frac{1}{2}m\omega^2 x_0{}^2 = \frac{1}{2}kx_0{}^2 = \frac{1}{2}(kx_0)x_0 = \frac{1}{2}mgx_0$

(f) E_T in (e) is the total energy associated with the oscillation and excludes any non-zero minimum potential energy of the system. The total energy in (c) however takes into account the minimum potential energy which is the gravitational and elastic potential energy of the system when the mass is at the equilibrium position. It can be observed that during the oscillation, the kinetic energy changes from 0 to $\frac{1}{2}mgx_0$ while the total potential energy (g.p.e + e.p.e) changes from $2mgx_0$ to $\frac{3}{2}mgx_0$ with $\frac{1}{2}mgx_0$ being the total energy associated with the oscillation.

9.3 DAMPED OSCILLATIONS

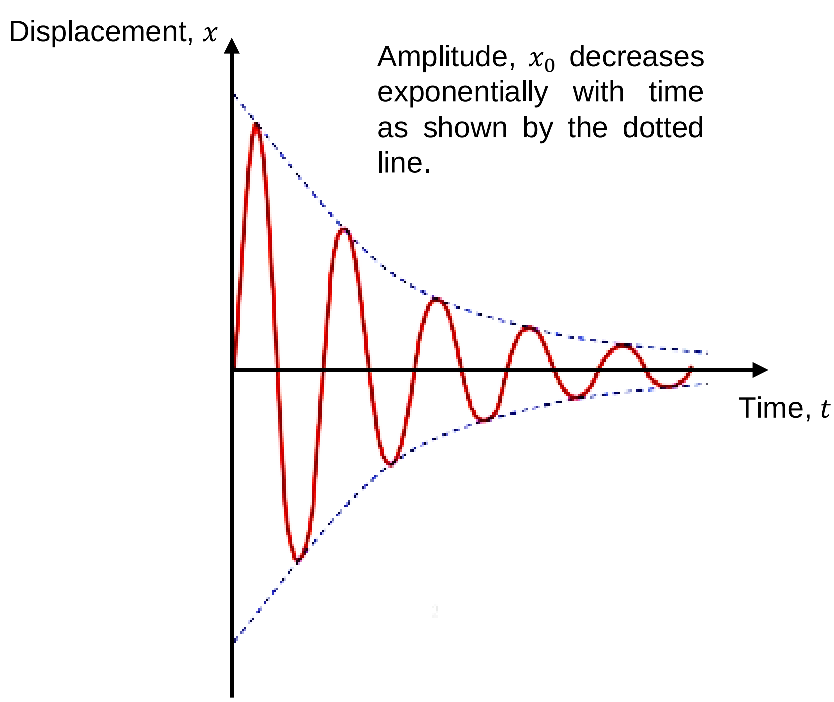
Self-study resources		
SLS lesson on SHM (5/5)	**https://for.edu.sg/shm5**	

For simple harmonic motion**,** the amplitude of oscillation remains constant. However, in real-life systems, oscillations usually experience a gradual decrease in amplitude due to energy loss. This loss occurs because of resistive forces such as air resistance or friction. An oscillation that comes with a decrease in amplitude over time due to resistive forces is called a damped oscillation.

The degree of damping depends on the strength of resistive force acting on the oscillator and it determines how fast the amplitude decreases over time. Damping can be categorised into 3 broad degrees:

Light Damping (Underdamped)

A system that is lightly damped shows definite oscillations after being displaced, but the amplitude of oscillation decays exponentially with time.



There are different degrees of light damping affecting how quickly the amplitude decreases with time.

Characteristics of lightly damped oscillation

(1) The system continues to oscillate.

(2) Amplitude of the oscillations decreases exponentially over time.

(3) Period of oscillation T_{damped} is constant but is **slightly** longer than the period T of the system if it is not damped.

What you need to be able to do at A-level for light damping situations

(1) **Quantitative calculations.** Take the period of the oscillating system when it is lightly damped to be **equal** to the period of the system when it is not damped.

(2) **Graph drawing** (refer to Fig. 2.6)

- The period of the damped oscillation is **equal** to the undamped oscillation for ease of drawing (note: attempting to draw a longer period oscillation without ensuring that the period for each cycle is constant will get marks deducted).
- Use an exponential envelop, symmetrical about the time axis, to guide you to sketch the decreasing amplitude.

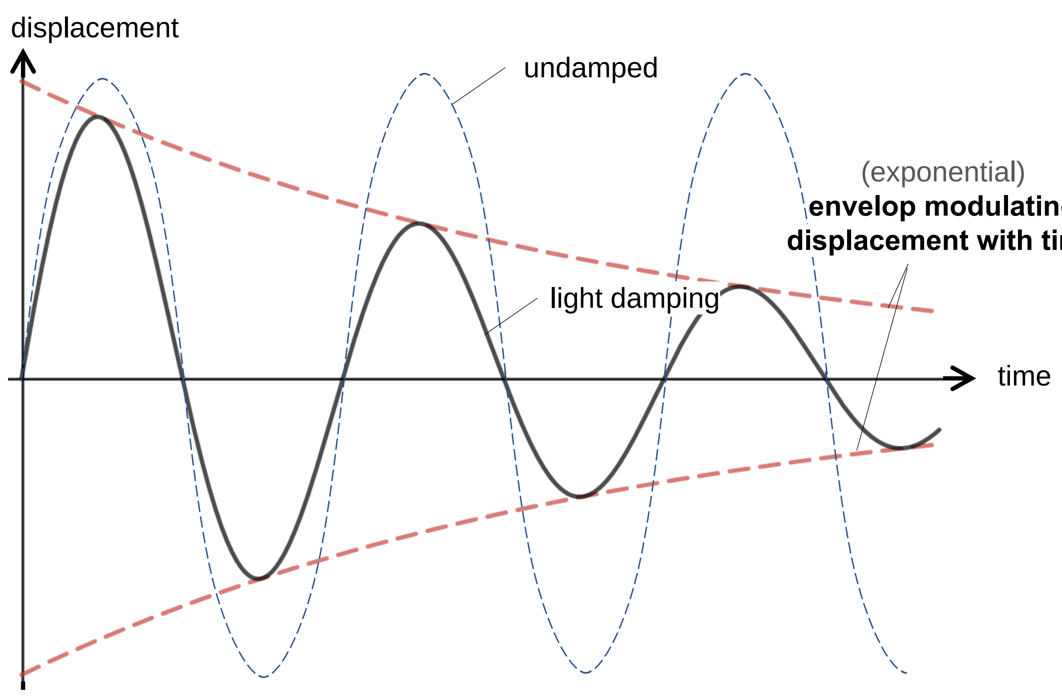


Fig. 2.6

Critical Damping

A system that is critically damped shows no oscillation after being displaced and returns to its equilibrium position in the shortest time possible.

Critical damping happens when the resistive force is **just** sufficient to prevent oscillation after being displaced, yet not so great that the system takes a long time to the equilibrium position.

There is only one degree of critical damping.

Heavy Damping (Overdamped)

A system that is heavily damped also shows no oscillation after being displaced but it takes a longer time to return to its equilibrium position that a system that is critically damped.

There are different degrees of heavy damping affecting how slowly the system returns to its equilibrium position.

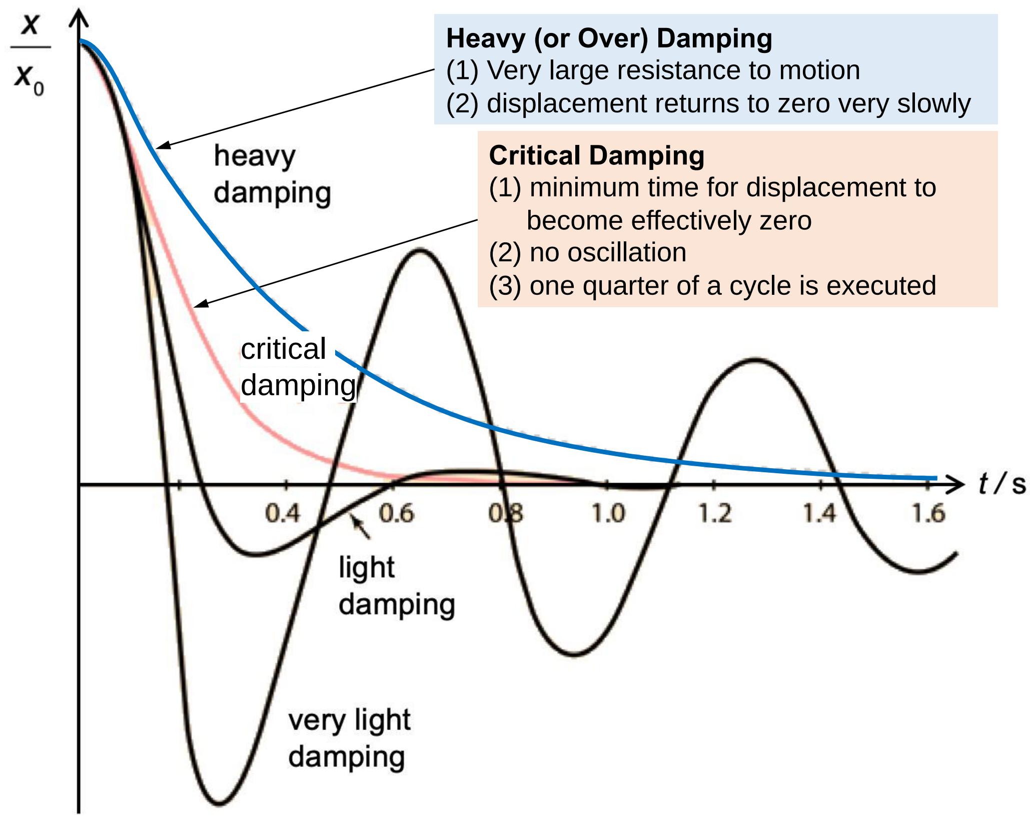


Fig. 2.7

Note: You must be able to sketch the displacement-time graphs for the different amounts of damping.

Examples of Damping

1. **Balance Wheel of a Watch (**video: https://youtu.be/Hkb5IlX9svA)

   The balance wheel in a mechanical watch will stop oscillating unless energy is supplied continuously by the watch spring. Frictional forces cause energy loss, reducing the total mechanical energy of the oscillating system over time.

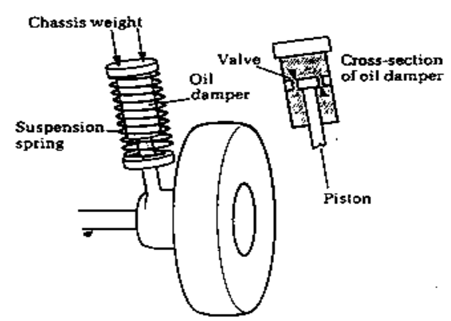
2. **Internal Friction in Springs**

   In real springs, internal forces can be **dissipative** (e.g. due to internal friction in the material), preventing the spring from being perfectly elastic. This causes damping and gradual energy loss during oscillations.

3. **Moving Coil Meters** (simulation: https://for.edu.sg/damping-moving-coil)

   Critical damping is important in devices like **moving coil meters**, which measure current or voltage. If the needle is **underdamped**, it oscillates before settling, making quick readings difficult. A **critically damped needle** moves to the correct reading in the **shortest time without overshooting**, allowing for quick and accurate measurement.

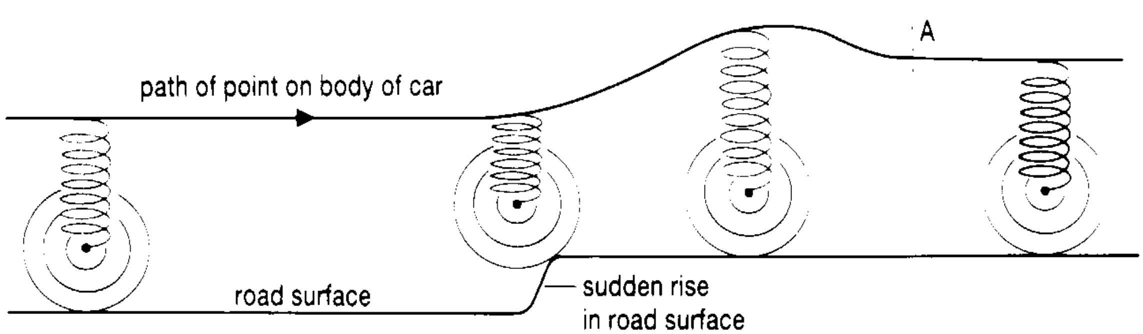
4. **The critical damping in car suspension system** (video: https://youtu.be/j0vhTg82Pz0)**:**



- The **suspension system** in a car uses springs and shock absorbers to smooth out bumps.
- If damping is **too little** (underdamped), the car becomes **bouncy**, leading to an uncomfortable ride.
- If damping is **too much** (overdamped), the car may return too slowly to equilibrium, making it sluggish.
- **Slightly under-critical damping** provides a **comfortable and responsive** ride.

A well-damped car suspension returns to equilibrium **quickly enough** to absorb the next bump - ensuring both comfort and safety.

The following figure shows that by the time the car has reached A, the shock absorbing system is ready for another bump.



The spring of a car's suspension is ***critically damped*** so that when a car goes over a bump the passengers in the car quickly and smoothly regain equilibrium. This results in a more comfortable ride.

Self-study resources		
Video on different types of damping	https://youtu.be/sP1DzhT8Vzo	

You don’t have to be concerned with the constants that characterise the different types of damping, only the degrees of damping.

9.4 FORCED OSCILLATIONS AND RESONANCE

9.4.1 Forced Oscillation

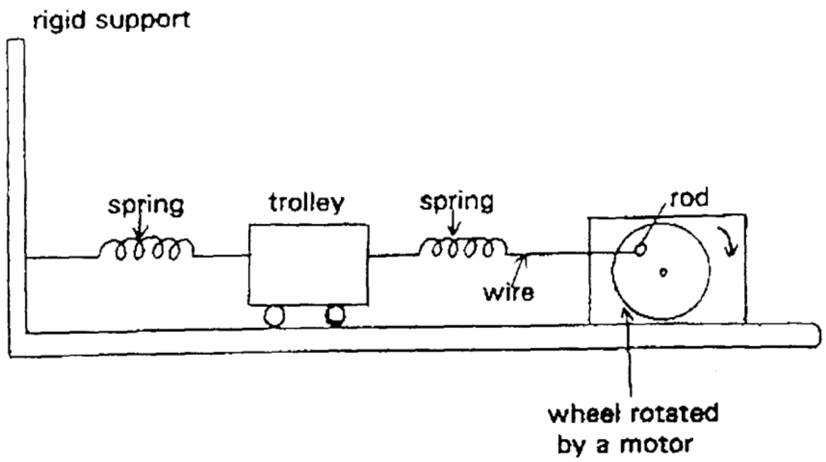
Imagine pushing a child on a swing. If each push is timed suitably, the child swings higher and higher. The pushes are a simple example of a periodic force, an external force applied at regular intervals. The periodic force provides a means of supplying energy to the system. This periodic force is called a *driving force* and its frequency is called the *driving frequency*.

A system made to oscillate by a *driving force* will keep in step with the force and oscillate with a frequency equal to the *driving frequency*. Such an oscillation is called a *forced oscillation*.

Examples of forced oscillations include:

- engine vibrations (*driving force*) making bus windows oscillate (*forced oscillation*)
- the spinning drum (*driving force*) causing vibrations in a washing machine (*forced oscillation*)

The diagram below shows a way of producing forced oscillations in the laboratory using a trolley-and-spring oscillator driven by a motor. Under steady conditions the amplitude of the forced oscillation is fixed for a fixed driving frequency.



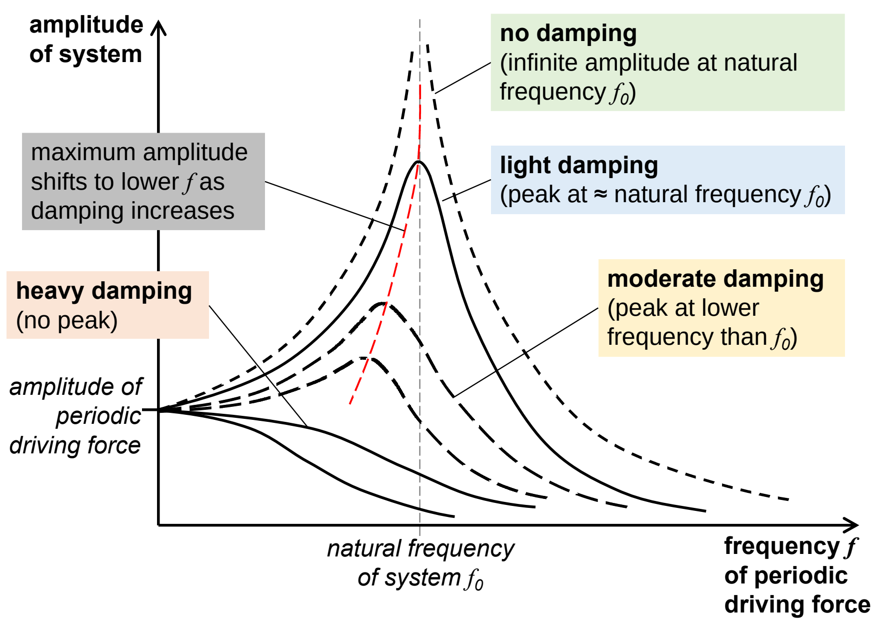
The amplitude of oscillation depends on

(1) how close the frequency of the driving force is to the natural frequency of the oscillator

(2) the extent to which the oscillator is damped

9.4.2 Frequency Response

The frequency response graph below shows how the amplitude of the oscillation at steady state varies with the *driving frequency*. The family of curves shown are for different degrees of damping for a particular driven system.



The *natural frequency* of the driven system refers to the frequency it would oscillate at when subject to a *free oscillation*, i.e. when it is displaced and released. The *natural frequency* of the driven system with zero damping is denoted by f_0.

The oscillator will oscillate with maximum amplitude when the *driving frequency* matches the *natural frequency* of the driven system (this effect is called *resonance*). Recall that damping causes the period of oscillation to increase (frequency to decrease), hence the frequencies at which the peaks occur decreases with increased degree of damping.

The total energy of an oscillation $\left(E_T = \frac{1}{2}m\omega^2 {x_0}^2\right)$ depends on the amplitude of the oscillation. The variation of the amplitude of oscillation with the *driving frequency* implies that for an oscillating system, the amount of energy transferred from the *driving force* to the driven system depends on how close the *driving frequency* to the *natural frequency* of the driven system. Maximum energy transfer occurs when the *driving frequency* is equal to the *natural frequency* of the driven system.

This concept is important in the transmission and reception of electromagnetic waves.

What you need to be able to do at A-level for frequency response graphs

You must be able to draw the frequency response graphs for different amount of damping.

When the amount of damping is increased, the following is observed:

(1) **Resonance (peaks) shift to lower frequencies.**

In section 2.3(A) we mentioned that the period of oscillation with damping is slightly longer than the period with no damping, so resonance occurs at lower frequency.

(2) **Maximum amplitude of resonance is lower.**

Rate of energy transferred becomes equal to the rate of energy lost to the environment faster due to larger damping and therefore the system accumulates less total energy and so smaller amplitude.

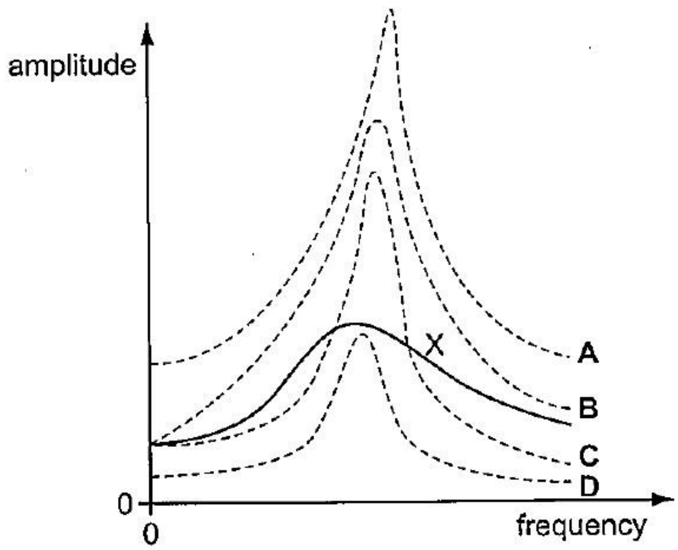
(3) **Response curve becomes broader.**

Self-study resources		
Video on resonance	https://youtu.be/ATRzwhk7UvY	

Example 7

The solid line X on the graph shows how the amplitude of an oscillating system varies with the frequency at which it is driven. The driving oscillator has a constant amplitude at all frequencies.

Which dashed line shows the response of the system under conditions where there is less damping?



Solution

Answer: B

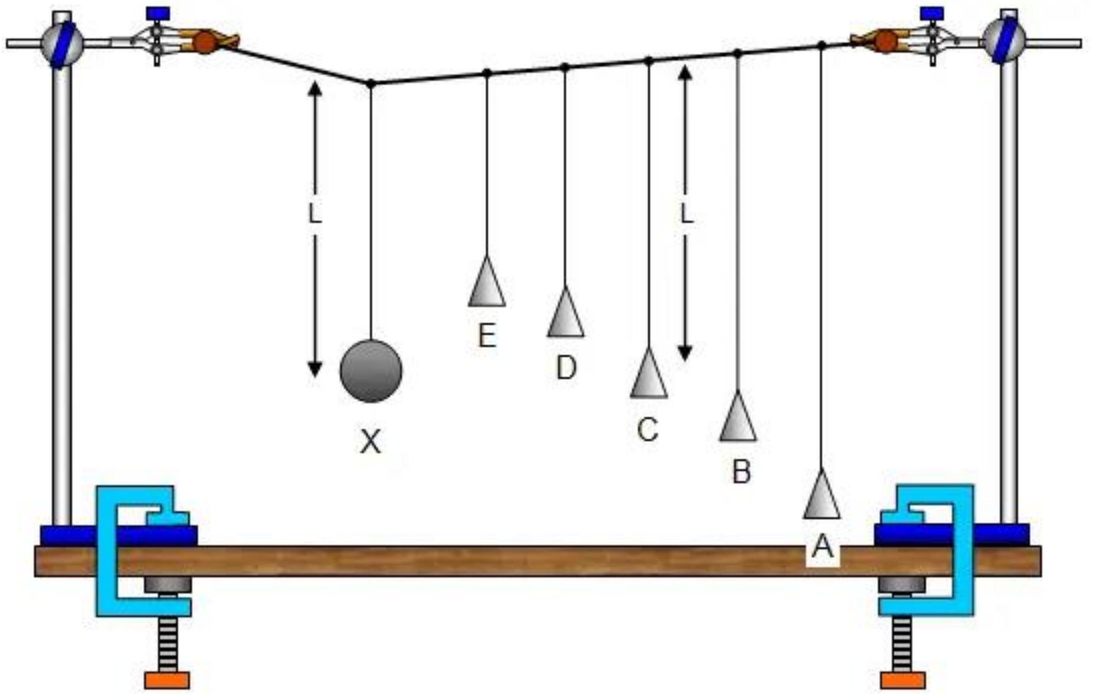
With less damping, the amplitude is <u>higher for all frequencies</u> except when the driving frequency is zero where the amplitude is unchanged. The resonant or natural frequency will also be higher and the sharpness of the resonance increases (narrower peak).

Note: A zero driving frequency implies that the driving force is applied once and never again. Hence at zero frequency, the oscillating system simply oscillates freely with an amplitude equal to that it has been given by the driving force. This amplitude will be the same no matter the degree of damping.

9.4.3 Barton's Pendulums

Barton's pendulums, shown in the figure below, can be used to illustrate forced/coupled oscillations and resonance.

In this setup, several pendulums (A,B, C, D and E) are suspended from a common string support. Each has a **light bob** and **different string lengths**. A **heavier pendulum X**, with the same length as pendulum C, is also suspended from the same string support.



When pendulum X is set into motion, it acts as a **driving source** and causes the other pendulums to oscillate. This is an example of **forced oscillations**, where an external source drives the motion of other systems.

Because pendulum C has the **same natural frequency** as X (since their lengths are equal), it resonates with X. As a result, **pendulum C will oscillate with the largest amplitude —** this is a demonstration of **resonance**.

9.4.4 Other Examples of Resonance

(A) Situations where resonance is useful

1. **Tuning of radio receiver** (video: https://youtu.be/8d2XLoL90ec)

   The tuning of radio and TV receivers is based on electrical resonance. Each station transmits signals at a specific frequency, and the receiver circuit is tuned such that its natural frequency matches the frequency of the radiowave so that resonance occurs for clear reception.

2. **Microwaves oven** (video: https://youtu.be/MqsDPmnPKEk)

   Molecules of food item vibrates due to the driving force produced by oscillating electric field of the microwaves (refer to electromagnetic spectrum). The natural frequency of water molecules is similar to the microwave frequency which causes resonance (larger amplitude), so the temperature of the water increases.

3. **Magnetic resonance imaging or MRI** (video: https://youtu.be/nFkBhUYynUw)

   Radio waves cause nuclei/proton to resonate.

4. **Musical instruments**

   All musical instruments rely on **mechanical resonance** to amplify sound. Different notes are produced when different parts of the instrument (e.g. strings, air columns) resonate at specific frequencies. We will study this in the topic of superposition.

(B) Situations where resonance is not useful

1. **Wind or walkers causes bridge to resonate** (video: https://youtu.be/XggxeuFDaDU)

   The Tacoma Narrows Bridge in Washington State collapsed due to **resonant vibrations**. The wind created vortices at a frequency that matched a natural frequency of the bridge. This caused large oscillations and eventually structural failure.

   Learn more about the physics and engineering behind the collapse here: https://practical.engineering/blog/2019/3/9/why-the-tacoma-narrows-bridge-collapsed

2. **Shattering of glass** (video: https://youtu.be/bJj4Wjjf0WI)

   An opera singer may shatter a wineglass by singing a note at a frequency that matches one of the natural frequencies of vibration of the glass. This sets up a large-amplitude vibration (resonance) along the rim, causing the glass to break.

3. **Earthquakes shaking buildings** and solution (video: https://youtu.be/pMr1MzSv044)

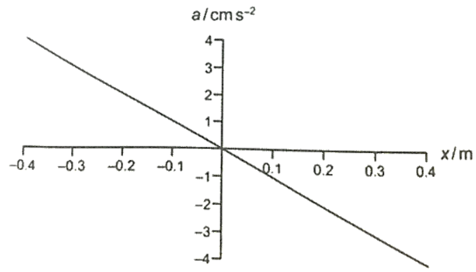
4. Vibrations from engine / road causes **interior fixtures in cars** (e.g. panels/mirrors) to resonate.

Exercise 2: Application

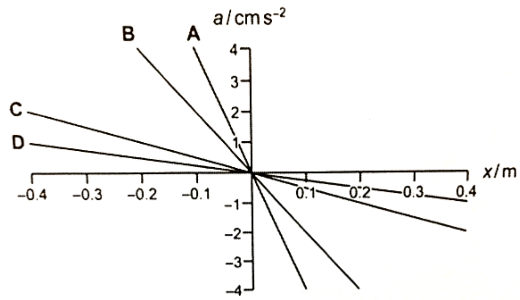
You should take about 10 minutes to complete this exercise.

Graphs

E9. [N11/1/17] The graph below shows the variation of acceleration **a** with displacement **x** for an object undergoing simple harmonic motion.



The simple harmonic motion system is altered so that it has a period of oscillation twice that of before. Which line could be produced?

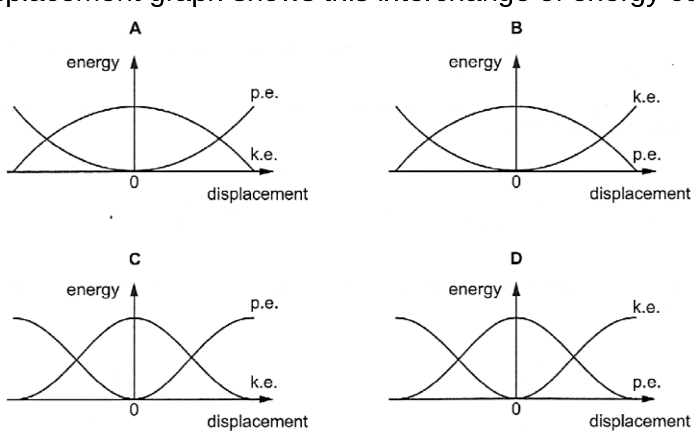


Energy of Free Oscillations

E10. [N10/1/17] A mass of 8.0 g oscillate in simple harmonic motion with an amplitude of 5.0 mm at a frequency of 40 Hz. What is the total energy of this simple harmonic oscillator?

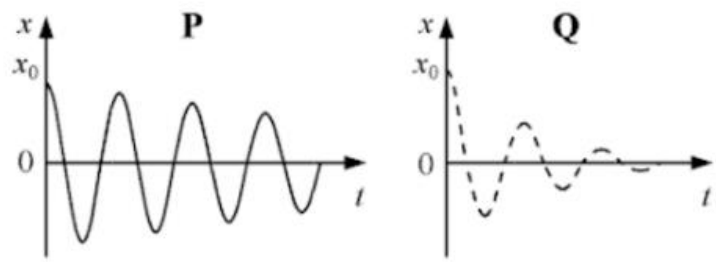
A 0.16 mJ **B** 6.3 mJ **C** 13 mJ **D** 640 mJ

E11. [N16/1/16] For an object undergoing simple harmonic motion, there is a continuous interchange of potential energy (p.e.) and kinetic energy (k.e.).
Which energy-displacement graph shows this interchange of energy correctly?



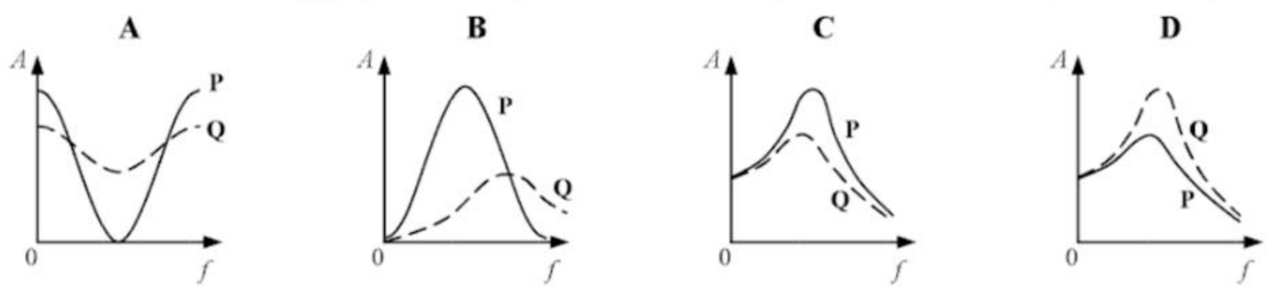
Damping and Resonance

E12. Given the same initial displacement and then released, the variation with time *t* of the displacements *x* of two objects **P** and **Q** are as shown in the given graphs.

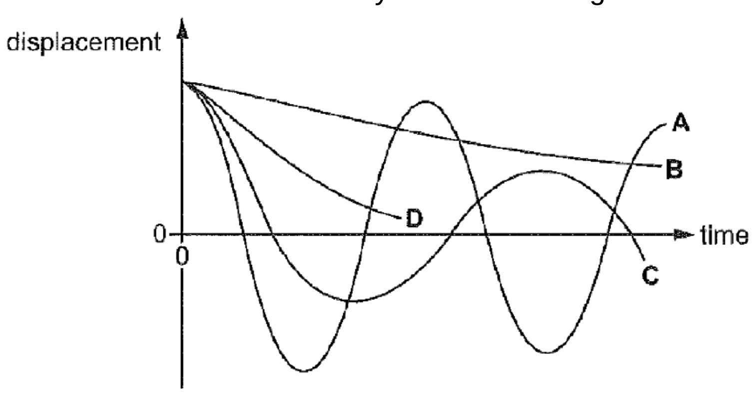


The two objects are then driven by sinusoidal driving forces of the same constant amplitude and of variable frequency *f*.

Which of the following graphs correctly shows how the amplitudes *A* of **P** and **Q** varies with *f*?



E13. The diagram shows four displacement-time curves for oscillations with various degrees of damping.
Which curve would ideally characterise a good car suspension system?



Answer Key

E9 D E10 B E11 A E12 C E13 D

APPENDIX A: <u>MATHEMATICAL INTUITION</u> (OPTIONAL BUT GOOD TO HAVE)

In this chapter, you will use the sine or cosine (sinusoidal) graph frequently and hence, you need to know how to sketch the sine graph given an equation of the form $x = x_0 \sin\theta$ as shown in Fig. 1.1.

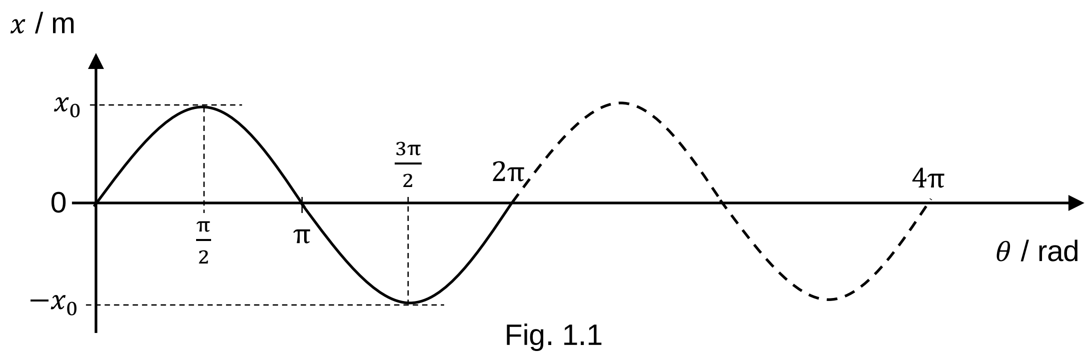


Fig. 1.1

Notice that the sine graph repeats itself after every 2π rad or 360°. The periodic pattern is shown in dotted line in Fig. 1.1.

We can make use of the periodic property of the sinusoidal graph to guess that at least some oscillations follow a sinusoidal equation (we will see later that this is true for a set of oscillations called simple harmonic motion) but replacing θ with ωt, such that a displacement x against time t graph can be plotted as in Fig. 1.2.

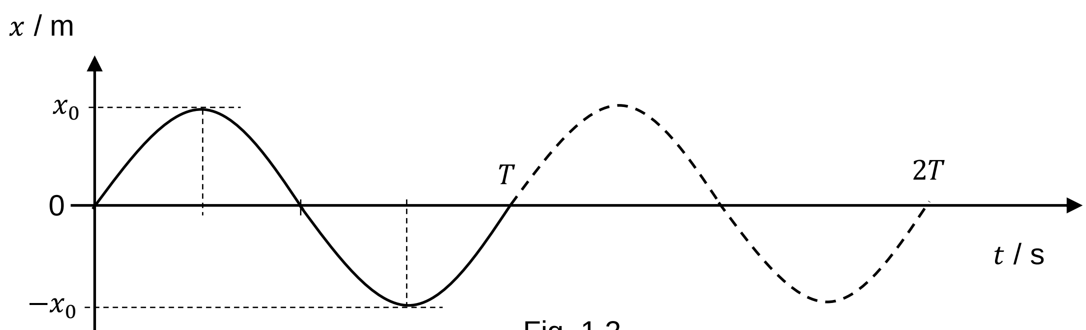


Fig. 1.2

We know that this graph also repeats itself after every T seconds where T represents the period of oscillation. We can use this fact to convert θ in rad to t in s, or vice versa, using direct proportion as follows:

$$2\pi \rightarrow T$$
$$\theta \rightarrow t$$

By proportion: $\frac{\theta}{2\pi} = \frac{t}{T}$ hence, $\theta = \left(\frac{2\pi}{T}\right)t = \omega t$ where $\omega = \frac{2\pi}{T}$

Replacing θ with ωt in the equation $x = x_0 \sin\theta$, we get $x = x_0 \sin\omega t$

It is also good to know the following relations:

$\frac{d}{dt}(\sin\omega t) = \omega\cos\omega t$ i.e. differentiating a sine function gives a positive cosine function

$\frac{d}{dt}(\cos\omega t) = -\omega\sin\omega t$ i.e. differentiating a cosine function gives a negative sine function