A - Functions

1 - Notations

A - Set

A. $\in$ / $\notin$

- {Element} **is** / **is not** an element of {set}
- 2 $\in$ / $\notin$ A : The real number 2 **is** / **is not** an element of the set A

B. $\mathbb{R}$ / $\mathbb{R}^+$ / $\mathbb{R}^-$

- The set of **all** / **positive** / **negative** real numbers, such as fractions, decimals, and integers, etc…

C. $\mathbb{Z}$ / $\mathbb{Z}^+$ / $\mathbb{Z}^-$

- The set of **all** / **positive** / **negative** integers
- $\mathbb{Z}$ / $\mathbb{Z}^+$ / $\mathbb{Z}^-$ = {-3,-2,-1,0,1,2,3,...} / {1,2,3,...} / {...,-3,-2,-1}
- 0 $\notin$ $\mathbb{Z}^+$ / $\mathbb{Z}^-$

D. $\subseteq$

a. {set_A} is a subset of {set_B}
b. Everything from set A is inside of set B
c. $x = \{1, 2, 3\}$
d. $y = \{1, 2, 3, 4, 5\}$
e. Since y contains all of the elements from x, x is a subset of y

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B - Interval

- An alternative way of showing a set that's bounded by a range of values

- **Type 1 : Bounded**
  - Bracket : Rounded ()
  - Is exclusive of the intervals' values
  - $(a, b) = \{SET\ NOTATION: a < b\}$
  - Example : $(0, 9) = \{x \in R : 0 < 9\}$
  - The set notation $x \in R$ is bounded by the intervals 0 and 9, excluding 0 and 9 themselves

- **Type 2 : Unbounded**
  - Bracket : Square []
  - Is inclusive of the intervals' values
  - $[a, b] = \{SET\ NOTATION:\ a \leq x \leq b\}$
  - Example : $[0, 9] = \{x \in R:\ 0 \leq x \leq 9\}$
  - The set notation $x \in R$ is bounded by the intervals 0 and 9, including 0 and 9 themselves

- **Type 3 : Hybrid**
  - Bracket : Both (] [)
  - Is both inclusive and exclusive of the intervals' values
  - $(a, b] \ = \ \{SET\ NOTATION : \ a \ < \ x \ \leq \ b\}$
  - $[a, b) \ = \ \{SET\ NOTATION : \ a \ \leq \ x \ < \ b\}$
  - Example : $(0, 9] \ = \ \{x \in R : \ 0 \ < \ x \ \leq \ b\}$
  - The set notation $x \in R$ is bounded by the intervals 0 and 9, excluding 0 and including 9
  - Example : $[0, 9) \ = \ \{x \in R : \ 0 \ \leq \ x \ < \ b\}$
  - The set notation $x \in R$ is bounded by the intervals 0 and 9, including 0 and excluding 9

2 - Types

A - Generic

- ➢ A mapping of exactly 1 value from a set of inputs to exactly 1 value from a set of outputs
- ➢ For example, the equation $f(x) = 2x$ will map 2 to 4 , 4 to 8, etc…
- It's defined by its rule and domain

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1. Visuals
   - $f : input \mapsto \{output\}\ for\ \{domain\}$
     - Example : $f : x \mapsto 2x\ for\ x \geq 0$
     - The output $2x$ is the rule of the function f
   - $f(x) = \{output\}\ for\ \{domain\}$
     - Example : $f(x) = 2x\ for\ x \geq 0$
   - $y = \{output\}\ for\ \{domain\}$
     - Example : $y = 2x\ for\ x \geq 0$

   ★ The domain is not compulsory to define a function

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2. Domain & Rule
   - Domain
     - Shown as $D_{f(x)}$
     - The set of inputs
     - Graphically it's the x-axis
     - Determines how the graph will look like horizontally
     - Makes the function unique
   - Rule
     - Shown as $R_{f(x)}$
     - The set of outputs
     - Graphically it's the y-axis
     - Determines how the graph will look like vertically
     - Dependent on its function's domain

★ Functions with the **same rule but different domains** are **considered different functions**
   - A function x is a **restriction** of function y if $D_x \in D_y$

B - Piecewise

- It's a function which is defined by multiple sub-functions, whereby each sub-function only applies to a specific interval of the function's domain

$$f(x) = \begin{cases} g(x), & a < x < b \\ h(x), & w < x < z \end{cases}$$

$f(x) = g(x)$ when x is in (a, b) and $f(x) = h(x)$ when x is in (w, z)

C - One-To-One

- It's a function whereby exactly 1 value from the range corresponds to exactly 1 value from the domain
- Horizontal Line Test
  - To test if the given function is one-to-one
  - Pass
    - If any horizontal line $y = k$, $(k \in \mathbb{R})$ cuts the graph $y = f(x)$ **at most once**, then the function $y = f(x)$ **is** a one-to-one function
  - Fail
    - If any horizontal line $y = k$, $(k \in \mathbb{R})$ cuts the graph $y = f(x)$ **more than once**, then the function $y = f(x)$ **is not** a one-to-one function

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D - Inverse

- It's a function $g(x)$ that reverses the domain and range of another function $f(x)$
- Written as $f^{-1}(x)$
- Hence, $D_{f^{-1}(x)} = R_{f(x)}$ and $R_{f^{-1}(x)} = D_{f(x)}$

★ In order for a function to have an inverse counterpart
   1. It must be a one-to-one function

- To create a function's inverse
   1. Make the variable from the RHS the subject
      1. $f(x) = 3x + 7$
      2. $y = 3x + 7$
      3. $x = \frac{y-7}{3}$
      4. $y = \frac{x-7}{3}$
      5. $f^{-1}(x) = \frac{x-7}{3}$

★ $f^{-1}(x)$ is a reflection of $f(x)$ in the line $f(x) = x$. Hence
   1. They will intersect there
   2. $f(a, b) = f^{-1}(b, a)$
   2. If $f(x)$ has a vertical asymptote $x = a$, $f^{-1}(x)$ will have a horizontal asymptote $y = a$
   3. If $f(x)$ has a horizontal asymptote $y = a$, $f^{-1}(x)$ will have a vertical asymptote $x = a$

E - Composite

- It's a function that contains another function within it (Nested Function)

$$fg(x) = f(\,g(x)\,)$$

$The\ composite\ function\ fg\ /\ f \circ g\ has\ the\ inner\ function\ g(x)\ inside\ the\ outer\ function\ f(x)$

1. For a composite function $f(g(x))$ to exist
   - ★ $R_g \subseteq D_f$
   - ★ $D_{fg} = D_g$
1. To find R_{fg}
   - ○ **Method 1**
     1. Sketch the graph of $y = fg(x)$ based on D_{fg}
     2. Obtain the range from the graph

   2. Method 2
     1. Find R_g
     2. Sketch $y = f(x)\ for\ R_g$
     3. R_{fg} will be the corresponding set of values for y

- ★ $gf \neq fg$
  - ➢ The order matters
- ★ $f^{-1}(f(x)) = x\ for\ x \in D_{f(x)}$
  - ➢ This is the identity function $y = x\ for\ x \in D_{f(x)}$
- ★ $ff^{-1}(x) = x\ for\ x \in D_{f^{-1}(x)}$
  - ➢ This is the identity function $y = x\ for\ x \in D_{f^{-1}(x)}$

B - Graphs & Transformations

1 - Graph Properties

A - Features

1 - Axes - Intercept

- $x - intercept$: Value of x when $y = 0$
- $y - intercept$: Value of y when $x = 0$

★ Not all graphs will intercept the axes

2- Stationary Points

- Points on the graph when the gradient $m = 0$
- Type 1 : Turning Point (Max / Min)
- Type 2 : Stationary Point Of Inflexion

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3 - Asymptotes

- A straight line that a curve approaches as either one of its variables $x/y \rightarrow \infty$
- The function will never intercept this line

★ **Type 1** : Vertical
  ➢ $x = a$ is a vertical asymptote of the curve $y = f(x)$ if $y \rightarrow \infty$ as $x \rightarrow a$

★ **Type 2** : Horizontal
  ➢ $y = a$ is a horizontal asymptote of the curve $y = f(x)$ if $y \rightarrow a$ as $x \rightarrow \infty$

★ **Type 3** : Oblique
  ➢ $y = mx + c,\ m \neq 0$

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4 - Axes Of Symmetry

- A straight line that divides a graph into 2 equal halves that are reflections of each other, such as
  ★ Vertical : $x = a$
  ★ Horizontal : $y = b$
  ★ Oblique : $y = mx + c$

B - Types

1 - Linear

- ➢ It's a straight line
- ➢ $y = mx + c,\ m = gradient\ \&\ c = y - intercept$
- ➢ 2 distinct linear lines are parallel if $m_1 = m_2$
- ➢ 2 distinct linear lines are perpendicular to each other if $m_1 m_2 = -1$

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2 - Quadratic

- ➢ It's a curve with a degree of 2
- ➢ $y = ax^2 + bx + c,\ \{a, b, c\} \in \mathbb{R},\ a \neq 0$
  - ■ If $a > 0$, the graph has a minimum point
  - ■ If $a < 0$, the graph has a maximum point

- ★ To find the root, use the quadratic formula
  - ■ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- ★ To determine the roots' nature, use the discriminant $b^2 - 4ac$
  - ■ If $b^2 - 4ac > 0$, the function has 2 real distinct roots
  - ■ If $b^2 - 4ac = 0$, the function has 2 real equal roots
  - ■ If $b^2 - 4ac < 0$, the function has 0 real root

- ★ In the completed square form $y = a(x - h)^2 + k$
  - ■ Axis Of Symmetry : $x = h$
  - ■ If $a > 0$, (h, k) is a minimum point
  - ■ If $a < 0$, (h, k) is a maximum point

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3 - Cubic

- ➢ It's a curve with a degree of 3
- ➢ $y = ax^3 + bx^2 + cx + d,\ \{a, b, c, d\} \in \mathbb{R},\ a \neq 0$

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4 - Exponential

- ➢ $y = e^x$
- ★ Horizontal Asymptote : $y = 0$
- ★ Vertical Asymptote : NA

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5 - Logarithmic

- ➢ $y = ln(x)$
- ★ Horizontal Asymptote : NA
- ★ Vertical Asymptote : $x = 0$

6 - Rational

- $y = \frac{f(x)}{g(x)}$, $where\ f(x)\ and\ g(x)\ are\ polynomials$
- $deg()$ refer to the polynomial's highest power

★ Horizontal Asymptote

- If $deg(f(x)) > deg(g(x))$
  - No Horizontal Asymptote

B. If $deg(f(x)) = deg(g(x))$

- $y = \frac{f(x)'s\ leading\ coefficient}{g(x)'s\ leading\ coefficient}$

C. If $deg(f(x)) < deg(g(x))$

- $y = 0$

★ Vertical Asymptote

D. $g(x) = 0$

E. Solve for x

F. Each solution gives a vertical asymptote each

★ Oblique Asymptote

G. Only possible if $deg(f(x) = deg(g(x)) + 1$

H. Do long division

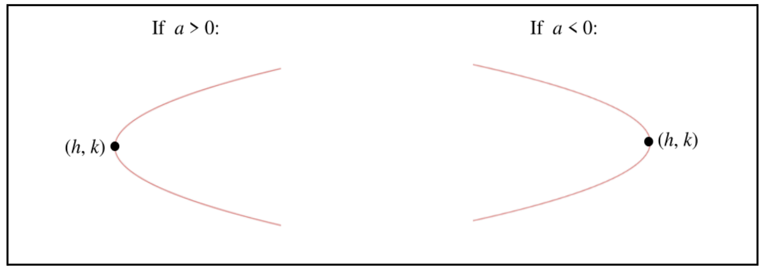
I. The oblique asymptote is the expression that remains as the rest becomes 0 when $x \to \infty$

- $\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$, $\frac{R(x)}{g(x)} = 0\ as\ x \to \infty$
- $y = Q(x)$

2 - Conic Sections

- Conic sections are curves formed by the intersection of a plane with a double-napped cone.
- The type of curve obtained depends on the angle and position of the intersecting plane relative to the cone
- The four main types of conic sections are the parabola, hyperbola, circle, and ellipse

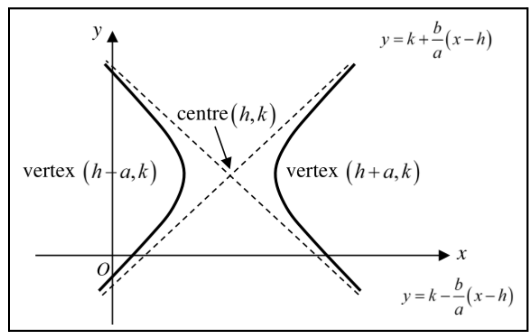
A - Parabola



- $(y - k)^2 = a(x - h),\ a \neq 0$
  - ★ Center : *None*
  - ★ Vertices : (h, k)
  - ★ Axis Of Symmetry : $y = k$
  - ★ Tangents : $x = h$ @
  - ★ Asymptotes : *None*
  - ★ Restrictions : If $a > 0,\ (x - h) \geq 0.\ Hence\ x \geq h$
    : If $a < 0,\ (x - h) \leq 0.\ Hence\ x \leq h$

B - Hyperbola

1. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,\ a > 0,\ b > 0$
   - ★ Center : (h, k)
   - ★ Vertices : $(h \pm a, k)$
   - ★ Axes Of Symmetry : $y = k$
     : $x = h$
   - ★ Tangents : *None*
   - ★ Asymptotes : $y = k \pm \frac{b}{a}(x - h)$
   - ★ Restrictions : $x \geq h + a$
     : $x \leq h - a$



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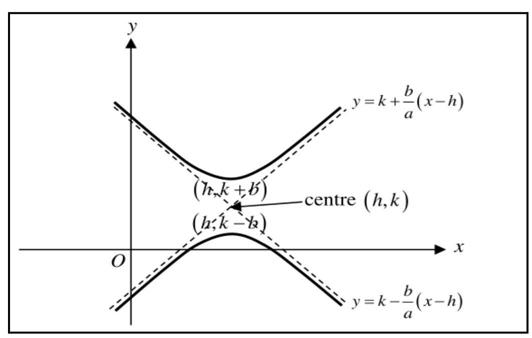
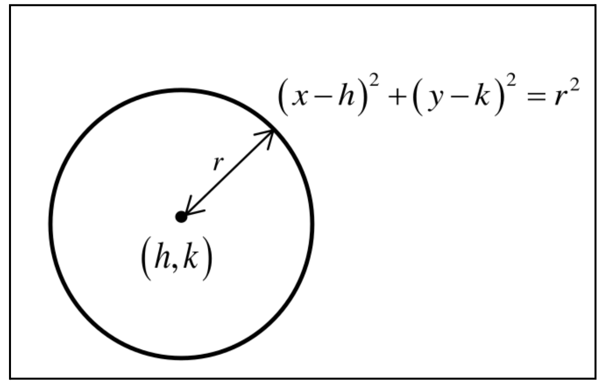
2. $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,\ a > 0,\ b > 0$
   - ★ Center : (h, k)
   - ★ Vertices : $(h, k \pm b)$
   - ★ Axis Of Symmetry : $y = k$
     : $x = h$
   - ★ Tangents : *None*
   - ★ Asymptotes : $y = k \pm \frac{b}{a}(x - h)$
   - ★ Restrictions : $y \geq k + b$
     : $y \leq k - b$



C - Circle

$(x - h)^2 + (y - k)^2 = r^2$, $where\ r\ is\ the\ circle's\ radius$

$ax^2 + ay^2 + cx + d = 0$

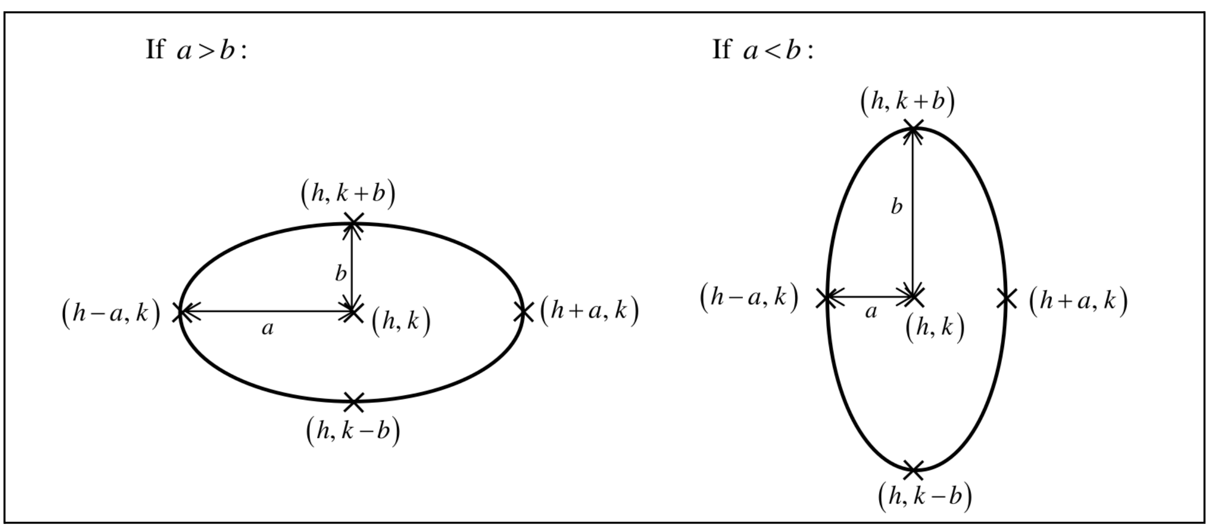


- ★ Center : (h, k)
- ★ Vertices : $(h \pm a, k)$
  : $(h, k \pm b)$
- ★ Axis Of Symmetry : $x = h$
  : $y = k$
- ★ Tangents : $None$
- ★ Asymptotes : $None$
- ★ Restrictions : $None$

D - Ellipse

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ a > 0,\ b > 0$

- ★ Center : (h, k)
- ★ Vertices : $(h \pm a, k)$
  : $(h, k \pm b)$
- ★ Axes Of Symmetry : $x = h$
  : $y = k$
- ★ Tangents : $None$
- ★ Asymptotes : $None$
- ★ Restrictions : $None$



- ★ When $a = b \Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1 \Rightarrow (x - h)^2 + (y - k)^2 = a^2$, $where\ a\ is\ the\ radius$
  - ➢ Hence, a circle is a special case of the ellipse when the axes a and b are equivalent to each other

3 - Parametric Curves

- ➢ It's a way of representing a function $y = f(x)$ by introducing a 3rd variable called a parameter t. Both x and y are expressed as functions of t

$$x = x(t),\ y = y(t)$$

- ★ The graph of a parametric curve is called the **locus**

- ★ $x = x(t)$: This means $x\ is\ a\ function\ of\ t$
  - ○ Giving an input t will produce the corresponding output for the $x - coordinate$

- ★ $y = y(t)$: This means $y\ is\ a\ function\ of\ t$
  - ○ Giving an input t will produce the corresponding output for the $y - coordinate$

▼Example▼

$$y = x^2$$

$$x(t) = t,\ y(t) = t^2$$

1. To convert $y = x^2$ into 2 parametric equations, introduce a parameter t and create 2 functions $x(t)$ and $y(t)$. They must output the original x and y values

2. The output of y is due to its input x. Since y is obtained by squaring x
   - ➢ For $y(t)$ to output the same original $y - values$, its input must also be squared
   - ➢ Therefore, it should be t^2
   - ➢ $x(t) = t$ [Form 1]
   - ➢ $x = t$ [Form 2]

3. The output of x is the same as the input
   - ➢ Hence, for $x(t)$ to output the same original $x - values$, its input must be the same as the output
   - ➢ Therefore, it should be t
   - ➢ $y(t) = t^2$ [Form 1]
   - ➢ $y = t$ [Form 2]

To convert a pair of cartesian equations back into its cartesian form, the **parameter** t must be **removed**

1. Solve one equation for t

$$y = t^2$$
$$t = \pm \sqrt{y}$$

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2. Substitute the solution for t into the second equation

$$x = \pm \sqrt{y}$$

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3. Simplify

$$x^2 = y$$
$$y = x^2$$

4 - Transformations

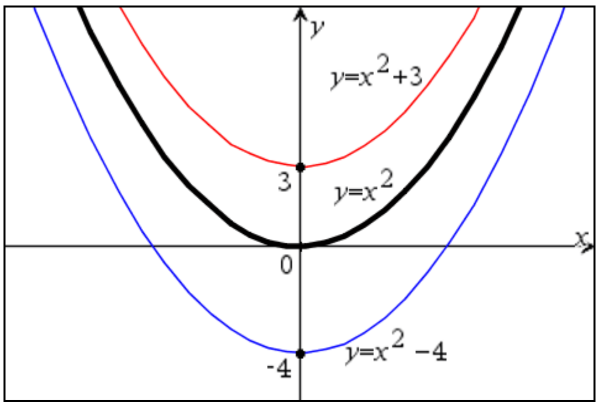
- ➢ It is a modification of a graph's physical properties

A - Translation

- ➢ Mutates the graph's position
- ➢ Does not mutate the graph's shape & size

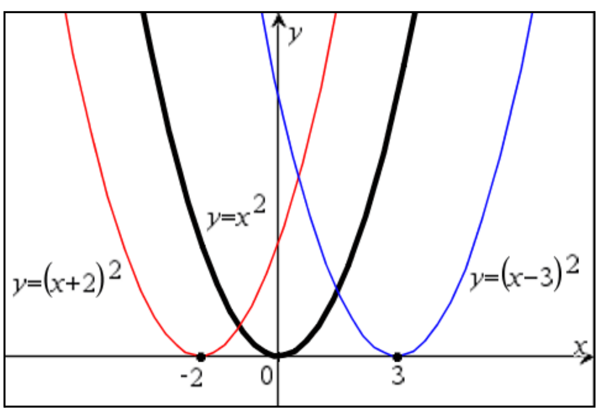
Type 1 - Vertical

- $y = f(x) \pm a$
- The graph of $y = f(x) \pm a$ is a translation of the graph $y = f(x)$ by a units in the **positive** / **negative** $y - axis$
- $x - coordinate$ remains unchanged
  - $(x, y) \Rightarrow (x, y \pm a)$



Type 2 - Horizontal

- $y = f(x \pm a)$
- The graph of $y = f(x \pm a)$ is a translation of the graph $y = f(x)$ by a units in the **negative** / **positive** $x - axis$
- $y - coordinate$ remains unchanged
  - $(x, y) \Rightarrow (x \mp a, y)$

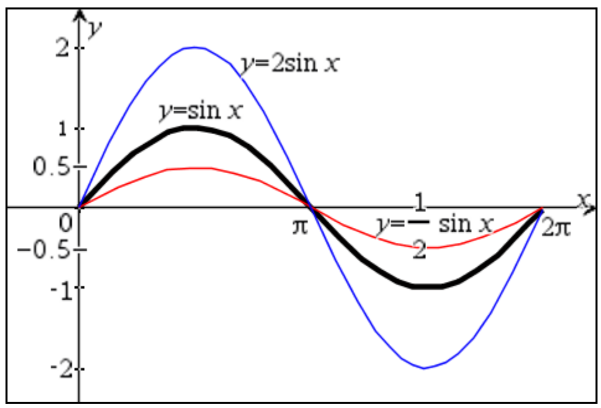


B - Scaling

- Mutates the graph's size & position
- Does not mutate the graph's shape

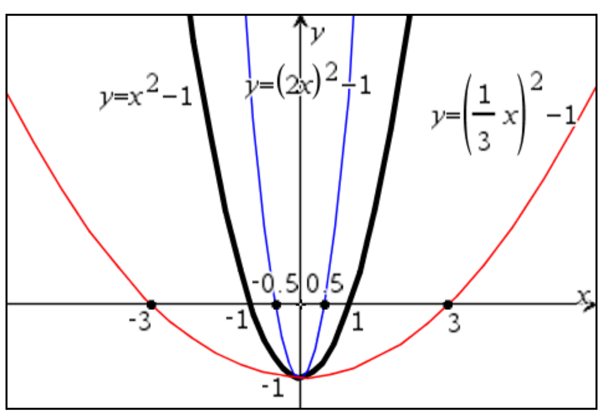
Type 1 - Vertical

- $y = af(x)$, $a > 0$
- The graph of $y = af(x)$, $a > 0$ is a scaling of the graph $y = f(x)$ with a scale factor of a parallel to the $y - axis$
- $x - coordinate$ remains unchanged
    - $(x, y) \Rightarrow (x, ay)$



Type 2 - Horizontal

- $y = f(ax)$, $a > 0$
- The graph of $y = f(ax)$, $a > 0$ is a scaling of the graph $y = f(x)$ with a scale factor of $\frac{1}{a}$ parallel to the $x - axis$
- $y - coordinate\ remains\ unchanged$
    - $(x, y) \Rightarrow (\frac{x}{a}, y)$

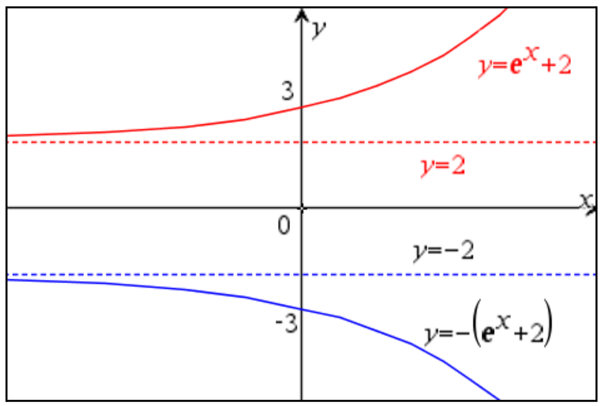


C - Reflection

- ➢ Mutates the graph's position
- ➢ Does not mutate the graph's shape & size

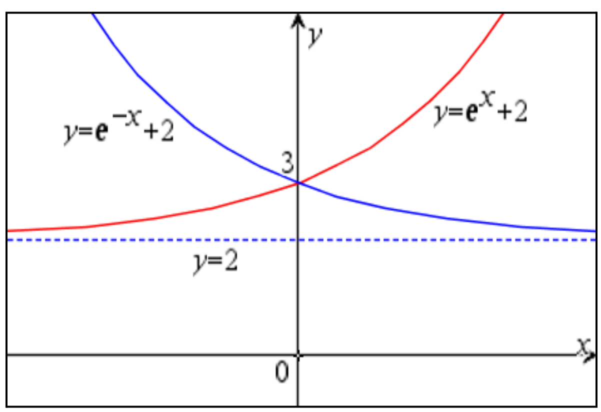
Type 1 - Vertical

- $y = -f(x)$
- The graph of $y = -f(x)$ is a reflection of the graph $y = f(x)$ in the $x - axis$
- $x - coordinate$ remains unchanged
    - $(x, y) \Rightarrow (x, -y)$



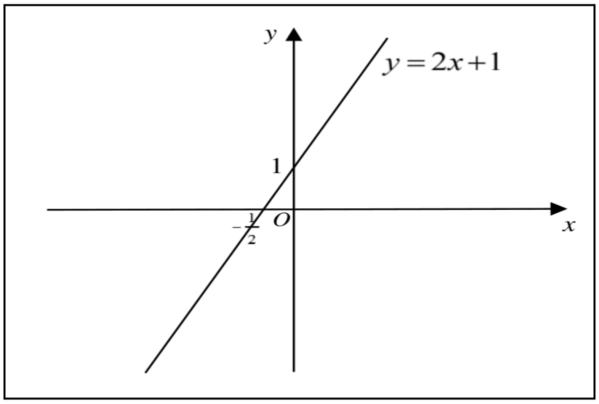
Type 2 - Horizontal

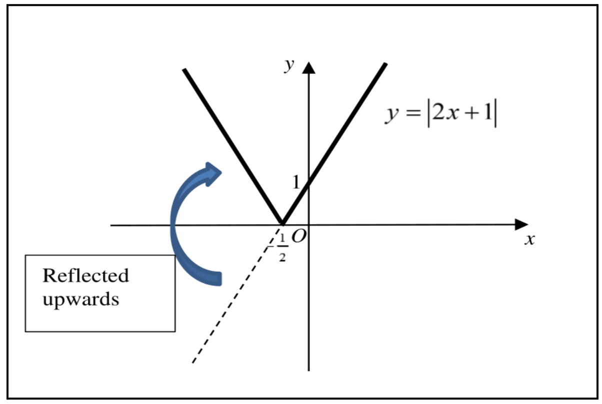
- $y = f(-x)$
- The graph of $y = f(-x)$ is a reflection of the graph $y = f(x)$ in the $y - axis$
- $y - coordinate$ remains unchanged
    - $(x, y) \Rightarrow (-x, y)$



Type 3 - Outer Modulus

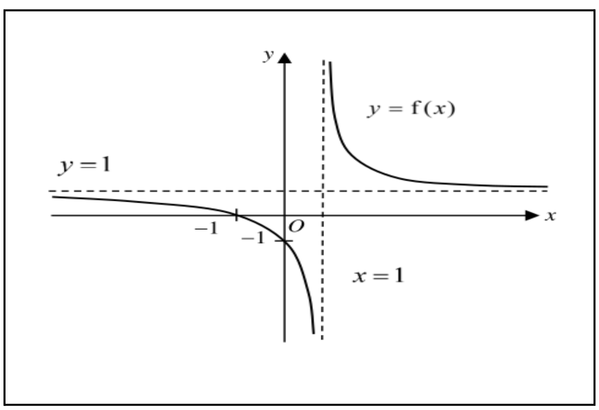
- $y = |f(x)|$
- The graph of $y = |f(x)|$ is a reflection of all the negative portions of the graph of $y = f(x)$ in the $x - axis$, while the positive portions remain the same
    - The modulus sign turns all negative $y - coordinates$ positive

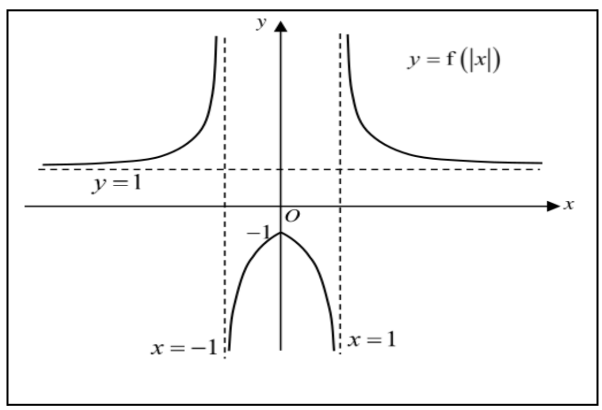




Type 4 - Inner Modulus

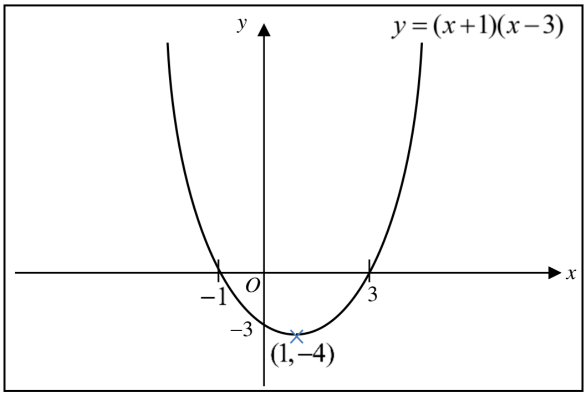
- $y = f(|x|)$
- The graph of $y = f(|x|)$ is a reflection of all the positive portion of the graph $y = f(x)$ onto the negative $y - axis$, while the pre-existing negative portions disappear and the pre-existing positive portions remain
    - The modulus sign turns all negative $x - coordinates$ positive

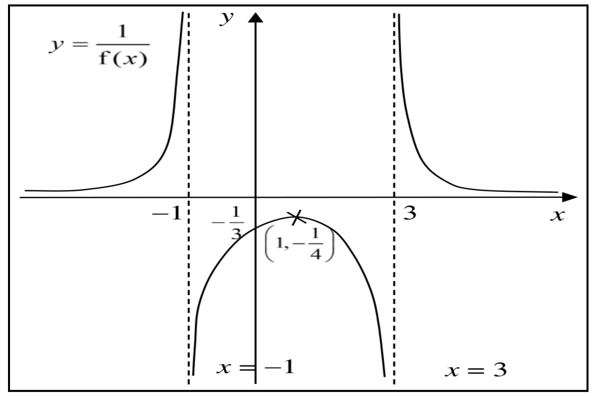




5 - Reciprocal Graph

- $y = \frac{1}{f(x)}$, $y \neq 0$
- $y - coordinate$ gets **inversed**
- $x - coordinate$ remains **unchanged**
- $(x, y) \Rightarrow (x, \frac{1}{y})$



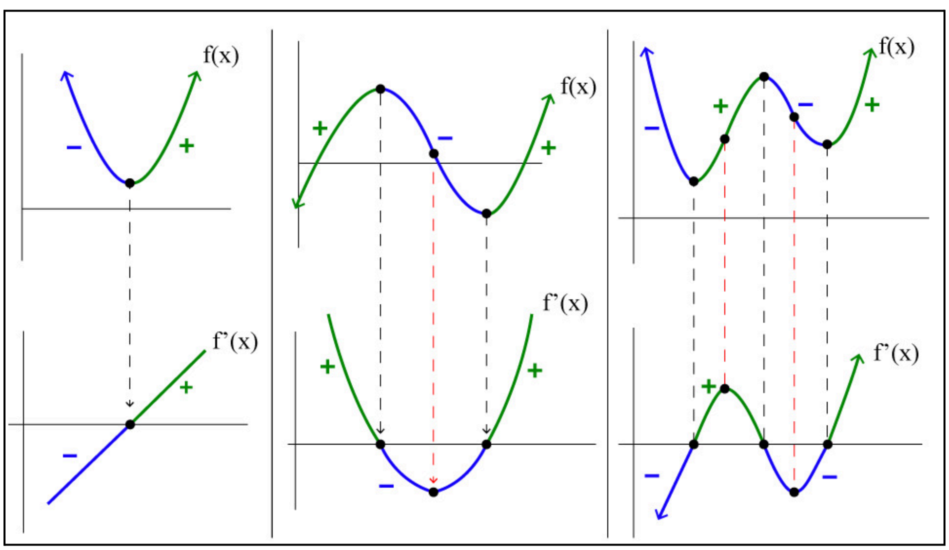


$Differences$	$y = f(x)$	$y = \frac{1}{f(x)}$
1	$x - intercept\ (a, 0)$	$Vertical\ Asymptote\ x = a$
2	$Vertical\ Asymptote\ x = b$	$x - intercept\ (b, 0)$
3	$Horizontal\ Asymptote\ y = k$ $k \neq 0$ $Horizontal\ Asymptote\ y = 0$ $(a)\ f(x) \to 0^+$ $(b)\ f(x) \to 0^-$	$Horizontal - Asymptote\ y = \frac{1}{k}$ $k \neq 0$ $(a)\ \frac{1}{f(x)} \to \infty^+$ $(b)\ \frac{1}{f(x)} \to \infty^-$
4	$Oblique\ Asymptote\ y = mx + c$	$Horizontal\ Asymptote\ y = 0$
5	$y - intercept\ (0, m)$	$y - intercept\ (0\ ,\ \frac{1}{m})$ $m \neq 0$
6	$Maximum\ Point\ (x_1\ ,\ y_1)$	$Minimum\ Point\ (x_1\ , \frac{1}{y_1})\ ,\ y \neq 0$
7	$Minimum\ Point\ (x_2\ ,\ y_2)$	$Maximum\ Point\ (x_2\ , \frac{1}{y_2})\ ,\ y \neq 0$
8	$f(x) \to \infty^+$ $f(x) \to \infty^-$	$\frac{1}{f(x)} \to \infty^+$ $\frac{1}{f(x)} \to \infty^-$
9	$f(x)\ increasing\ gradient$	$\frac{1}{f(x)}\ decreasing\ gradient$
10	$f(x)\ decreasing\ gradient$	$\frac{1}{f(x)}\ increasing\ gradient$
11	$f(x) > 0$ $f(x) < 0$	$\frac{1}{f(x)} > 0$ $\frac{1}{f(x)} < 0$

6 - Derivative Graph

- $y = f'(x)$
- Represents the **gradient** of $f(x)$

$Differences$	$y = f(x)$	$y = f'(x)$
$x - intercepts$	$(a, 0)$	$Disappear$
$y - intercepts$	$(0, f(0))$	$(0, f'(0))$
$Turning\ Points$	$f'(x) = 0$	$x - intercepts$
$Inflection\ Points$	$f''(x) = 0$ $With\ Concavity\ Change$	$Turning\ Points$
$Vertical\ Asymptotes$	$x = a$	$x = a$
$Horizontal\ Asymptotes$	$y = b$	$y = 0$
$Oblique\ Asymptotes$	$y = mx + c$	$y = m$
$Increasing\ Intervals$	$f'(x) > 0$	$f''(x) > 0$
$Decreasing\ Intervals$	$f'(x) < 0$	$f''(x) < 0$



7 - Sequence Of Transformations

- Give instructions on how to transform a graph in the following order of precedence
    1. Horizontal Translations
    2. Horizontal Scaling & Reflections
    3. Vertical Scaling & Reflections
    4. Vertical Translations

Format

1. **Translation**
    - ➢ Translate in the positive / negative $x/y - axis$ by {} units
2. **Scaling**
    - ➢ Scale with a scale factor of {} parallel to the $x/y - axis$
3. **Reflection**
    - ➢ Reflect in the $x/y - axis$

Example

State the sequence of transformations from $y = f(x)$ to $y = 2f(-5 - 3x) - 4$

1. **Translate** in the positive $x - axis$ by 5 units
2. **Scale** with a scale factor of $\frac{1}{3}$ parallel to the $x - axis$
3. **Reflect** in the $y - axis$
4. **Scale** with a scale factor of 2 parallel to the $y - axis$
5. **Translate** in the negative $y - axis$ by 4 units

C - Equations & Inequalities

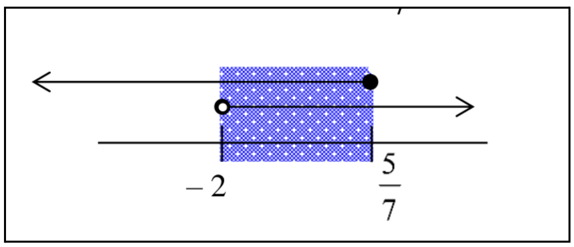
1 - Linear

1. Make the variable the subject
2. Reverse the comparison sign if multiplying / dividing by a negative number

$$2x + 4 > 3x - 8$$
$$-x > -12$$
$$x < 12$$

3. For combined inequalities, solve each part separately
   - ➢ The solution is the intersection of the solution sets' results

$$3x - 2 < 4x < 5 - 3x$$
$$3x - 2 < 4x \text{ and } 4x < 5 - 3x$$
$$x > -2 \text{ and } x \leq \frac{5}{7}$$



$$\textit{Taking intersection of the 2 solution sets' results,} \quad -2 < x \leq \frac{5}{7}$$

2 - Polynomial

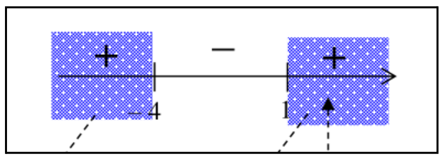
1. Factorize the polynomial expression
   - ➢ Use quadratic formula if that's not possible

2. Draw a number line and mark the critical values on it
   - ➢ There should be a few regions on the number line now

3. Sign Test
   - ➢ Pick any number from within each region and substitute it into the equation to determine the sign $(+\ /\ -)$ of that region
   - ➢ $+$: Graph is above $x - axis$
   - ➢ $-$: Graph is below $y - axis$
   - ★ If all factors are to the power of 1 , the signs will alternate

$$Solve\ x^2 - 3x + 4 > 0$$

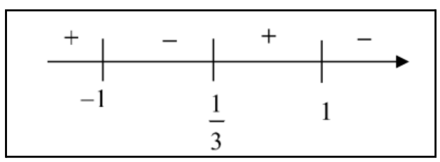
$$(x + 4)(x - 1) > 0$$



$$Since\ the\ inequality\ is\ greater\ than\ 0,\ focus\ on\ the\ positive\ regions$$

$$x < 4\ OR\ X > 1$$

$$Solve\ (1 - x)(1 + x)(3x - 1) < 0$$



$$Since\ the\ inequality\ is\ greater\ than\ 0,\ focus\ on\ the\ negative\ regions$$

$$-1 < x < \frac{1}{3}\ OR\ x > 1$$

3 - Rational

1. DO NOT cross multiply when solving rational inequalities
2. Make RHS 0 by moving all the terms from the RHS to the LHS
3. Simplify the expression into a single fraction
4. Follow the steps for polynomial inequalities
    - ★ The **denominator cannot be 0** so **exclude** those **critical values** from the solution set

$$\frac{12}{x-3} \le x+1 \qquad \text{(note that } x \neq 3\text{)}$$

$$\frac{12}{x-3} - (x+1) \le 0$$

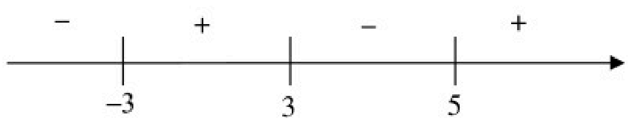
[step 1: make RHS zero and simplify LHS]

$$\frac{12-(x+1)(x-3)}{x-3} \le 0$$

$$\frac{-x^2+2x+15}{x-3} \le 0 \Rightarrow \frac{x^2-2x-15}{x-3} \ge 0$$

$$\frac{(x-5)(x+3)}{(x-3)} \ge 0$$

[step 2: __factorise__ the numerator completely, so that you can identify the critical values. Draw a number line with critical values labelled]



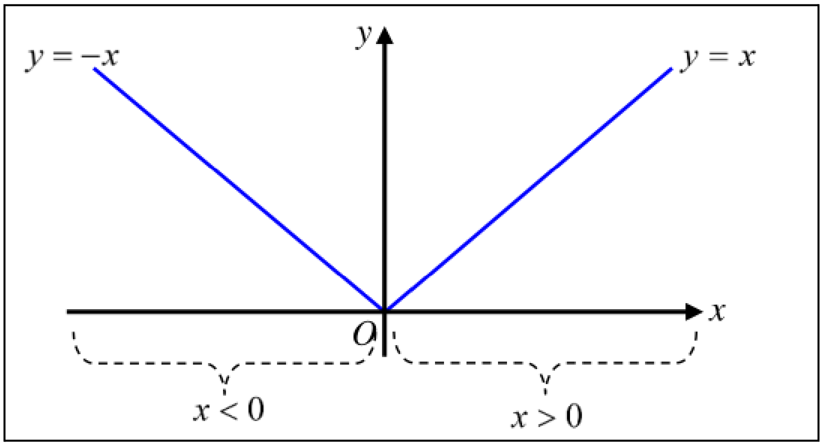
[step 3: carry out the sign test]

$\therefore$ the solution is $-3 \le x < 3$ **<u>or</u>** $x \ge 5$.

Similarly, since $x \neq 3$, we exclude 3 from the solution to the inequality.

4 - Modulus

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



- $|x| \geq 0$
- $|x|^2 = |x^2| = x^2$
- $\sqrt{x} = |x|$
- $\sqrt{x^2} \neq |x|$

★ $|x| < a \Leftrightarrow -a < x < a$

➢ $Solve\ |x - 5| < 3$

1. $-3 < x - 5 < 3$
2. $2 < x < 8$

★ $|x| > a \Leftrightarrow x < -a\ OR\ x > a$

➢ $Solve\ |x - 5| > 3$

1. $x - 5 < -3\ OR\ x - 5 > 3$
2. $x < 2\ OR\ x > 8$

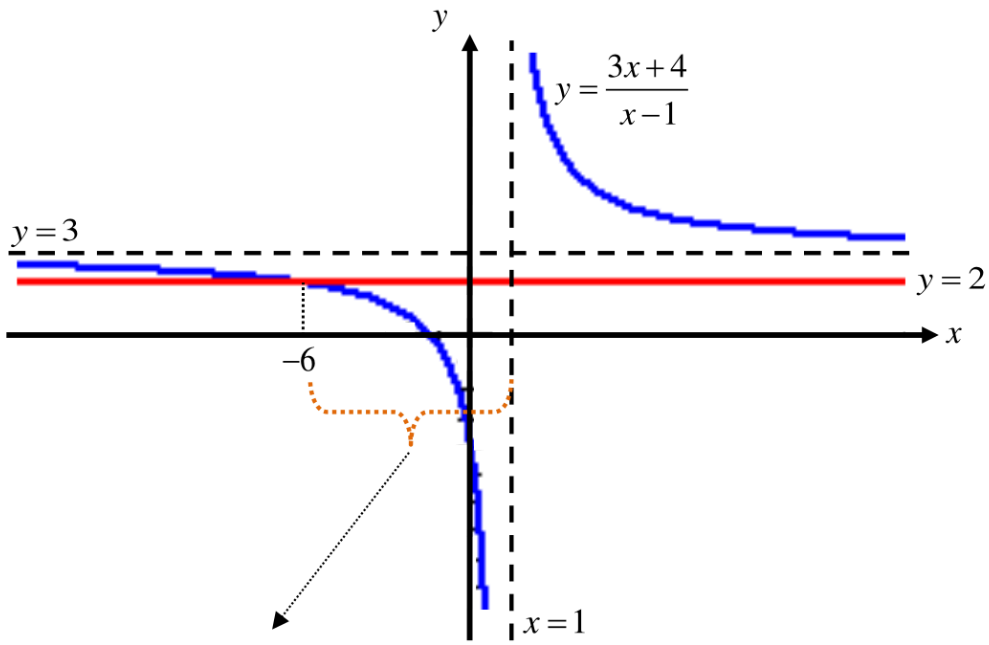
5 - Graphical

Method 1

1. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the GC
2. Find the $x - intercepts$
3. Find the range of values of x that satisfy the inequality
   - ★ $>$: Above $x - axis$
   - ★ $<$: Below $x - axis$

let's see how we can use a graphical method (Method 1) to solve the inequality $\frac{3x+4}{x-1} < 2$.

Sketch the graph of $y = \frac{3x+4}{x-1}$ and $y = 2$.



Since $\frac{3x+4}{x-1} < 2$, we will look for the range of values of x where the graph of $y = \frac{3x+4}{x-1}$ is lower than the graph of $y = 2$. $\therefore$ the solution is $-6 < x < 1$.

Method 2

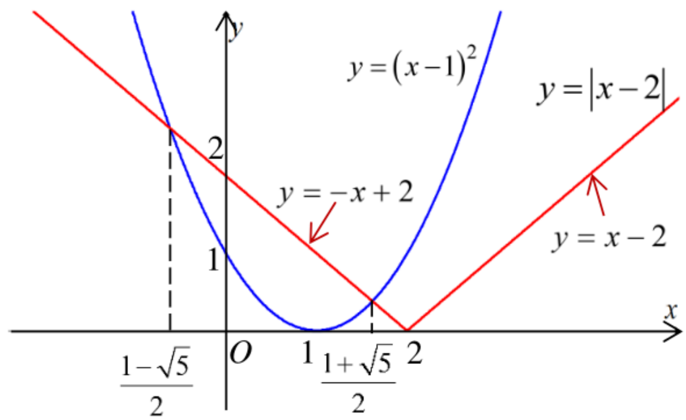
1. Sketch the graph of the $y = f(x) - g(x)$
2. Find the $x - intercepts$
3. Find the range of values of x that satisfy the inequality
   - ★ $>$: Above $x - axis$
   - ★ $<$: Below $x - axis$

Sketch the graphs of $y=(x-1)^2$ and $y=|x-2|$ on the same diagram. Hence, find the exact range of values of x for which $(x-1)^2 \le |x-2|$.

<u>Solution</u>

The graph of $y=|x-2|$ is made up of two lines, $y=x-2$ and. $y=-x+2$

The curve cuts the line $y=-x+2$ at two points.



To find the **<u>exact</u>** x-coordinates of these points of intersection, solve the equation

$$x^2 - 2x + 1 = -x + 2$$
$$x^2 - x - 1 = 0$$
$$x = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, from the graph, for $(x-1)^2 \le |x-2|$,

$$\frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2}.$$

We look for the range of values of x where the graph of $y=(x-1)^2$ is below or intersecting the graph of $y=|x-2|$.

2 - Sequences & Series

A - Sequence

- It's a list of numbers arranged in a specified order. It can either be **finite** (has a last term) or **infinite** (has no last term)
  - Finite : [1,2,3,4,5]
  - Infinite : [1,2,3,4,5,...]
- u_n is used to represent the n^{th} term of the sequence
- Sequences with patterns have a formula to determine the n^{th} term's value

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Recurrence / Recursive Relations

- It's an equation that connects a term in a sequence to one or more of its prior terms

- To define a sequence using a recurrence relation, an initial value / base case is required so as to end the recurrence / recursion

- $u_{n+1} = f(u_n)$

- Example
  - $u_0 = 5$
  - $u_n = 3u_{n-1} - 9,\ n \geq 1$

    1. $When\, n = 1,\ u_1 = 3u_0 - 9 \Rightarrow 3(5) - 9 = 6$
    2. $When\, n = 2,\ u_1 = 3u_1 - 9 \Rightarrow 3(6) - 9 = 9$
    3. $When\, n = 3,\ u_1 = 3u_2 - 9 \Rightarrow 3(0) - 9 = 18$

B - Series

- It's the addition of the terms in a sequence. It can either be **finite** (has an ending term k) or **infinite** (has no ending term k)

1. Infinite
    - ★ $S_{\infty} = u_1 + u_2 + u_3 + \ ...$
    - ★ $S_{\infty} = \lim_{n \to \infty} S_n$
    - ★ $S_{\infty} = k$ [Convergent]
    - ★ $S_{\infty} \neq k$ [Divergent]

---

2. Finite
    - ★ $S_n = u_1 + u_2 + u_3 + \ ... \ + u_n$

---

3. Convergence
    - ★ The series approaches a finite value
    - ➢ All finite series are convergent as they have an end term
        - $S_5 = 1 + 2 + 3 + 4 + 5 \Rightarrow 15$
    - ➢ Some infinite series can be convergent
        - $\sum_{n=0}^{\infty} ar^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... \Rightarrow 2, \ |r| < 1$

---

4. Divergence
    - ★ The series do not approach a finite value
    - ➢ All finite series cannot be divergent as they have an end term
        - $S_5 = 1 + 2 + 3 + 4 + 5 \Rightarrow 15$
    - ➢ Some infinite series can be divergent
        - $S = 1 + 2 + 3 + \ ... \ = \infty$

---

5. Properties
    - ★ $u_1 = S_1$
    - ★ $u_n = S_n - S_{n-1}, \ n \geq 2$
        - ■ For divergent infinite series, this property is limited to only finite n as the total sum $S_n \to \infty$ or does not exist
        - ■ Still works as per normal for convergent infinite series as its total sum S_n approaches a finite value which can be plugged into the formula
        - ■ The formula works for all finite series as they have an ending term

1 - Arithmetic

1 - Definition

- ➢ It is a sequence where each term, excluding the first term a, is obtained by adding a constant number, known as the command difference d, to the its prior term

- ★ To prove a sequence is an AP, it must be proven that the common difference between any 2 consecutive terms is the same constant k, and can either be positive or negative
    - ➢ $d = u_n - u_{n-1} = k \, for \, all \, n \in Z^+, n \geq 2$

- ★ The command difference between 3 consecutive terms a, b, and c are equal
    - ○ $a \rightarrow b \rightarrow c$
    - ○ $b - a = c - b$

---

2 - Formulas

1. To find the n^{th} term
    - ★ $u_n = a + (n - 1)d$

2. To find the command difference d
    - ★ $d = u_n - u_{n-1}$

3. To find the sum of the first n terms
    - ★ $S_n = \frac{n}{2}(a + l)$
        - ○ l is the last term
    - ★ $S_n = \frac{n}{2}(2a + (n - 1)d)$

2 - Geometric

1 - Definition

- ➢ It is a sequence where each term, except for the first term a, is obtained by multiplying its prior term by a constant, non-zero number, known as the common ratio r, which can either be positive or negative

- ★ To prove if a sequence is a GP, it must be proven that the common ratio between any 2 consecutive terms is the same constant k

$$\frac{u_n}{u_{n-1}} = k \; for \; all \; n \in Z^+, \; n \geq 2$$

- ★ The common ratio between 3 consecutive terms a, b, and c are equal
  - ○ $a \rightarrow b \rightarrow c$
  - ○ $\frac{b}{a} = \frac{c}{b}$

---

2 - Formulas

1. To find the n^{th} term
   - ★ $u_n = ar^{n-1}$

2. To find the common ratio r
   - ★ $r = \frac{u_n}{u_{n-1}}$

3. To find the sum of the first n terms
   - ★ $S_n = \frac{a(1-r^n)}{1-r}, \; |r| < 1$
   - ★ $S_n = \frac{a(r^n-1)}{r-1}$
   - ★ $S_\infty = \frac{a}{1-r}, \; |r| < 1$
     - ➢ Why ? $n \rightarrow \infty \Rightarrow r^n \rightarrow 0$

3 - Sigma Notation

A - Definition

- It's a way of expressing the sum of a series
- It's not unique for a particular series ; it can have multiple sigma notations that are expressed in different ways

$$\sum_{r=m}^{n} u_r$$

1. $\sum$
    - The summation sign
    - It tells the person to repeatedly add the general term of the series, starting from the lower bound (**inclusive**) and upper bound (**inclusive**)

---

2. u_r
    - The general term that's being summed
    - For every term in the addition, r get a value of m that increments by 1 for every term added
    - The last term's value of r get the values of n

---

3. r
    - The index of summation
    - It must be an integer
    - It's a placeholder that's used to keep track of which term is being added to the sequence, hence any letter can be used to express it
    - It signifies which term to use at each step
    - It doesn't appear in the addition expression

---

4. m
    - The lower limit of summating
    - The starting value of r

---

5. n
    - The upper limit of summation
    - The ending value of r

---

6. $(n - m) + 1)$
    - The number of terms in the sequence
    - Why add 1 ? It's to include the starting term in the count

B - Examples

$$\sum_{r=1}^{5} r = 1 + 2 + 3 + 4 + 5 \Rightarrow 15$$

$$\sum_{r=1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \Rightarrow 55$$

$$\sum_{r=1}^{n} r^3 = 1^2 + 2^2 + 3^2 + \ldots + n^3$$

$$\sum_{r=1}^{\infty} r^3 = 1^2 + 2^2 + 3^2 + \ldots$$

C - Creation

- To create a summation for a series, Identify the general term u_r by identifying a pattern in it

    1. $S_{100} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{100}$
    2. $u_r = \frac{1}{r}$
    3. $\sum_{r=1}^{100} \frac{1}{r}$

D - Arithmetic Operations

1. Constant Multiple
    - $\sum_{r=m}^{n} cu_r = c \sum_{r=m}^{n} u_r$

2. Addition & Subtraction
    - $\sum_{r=m}^{n} (u_r \pm v_r) = \sum_{r=m}^{n} (u_r) + \sum_{r=m}^{n} (v_r)$

3. Multiplication

    a. $\sum_{r=m}^{n} u_r v_r \neq (\sum_{r=m}^{n} u_r)(\sum_{r=m}^{n} v_r)$

E - Standard Results

1. $\sum_{r=m}^{n} a = a + a + a + \ldots + a \, for \, [n - (m + 1)] \, terms$
    - Repeatedly adding a constant a for the summation's total amount of terms $(n - m) + 1$
    - $\sum_{r=m}^{n} a = [(n - m) + 1]a$

---

2. $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n$
    - Sum of first n terms of an AP with first term 1 and last term n
    - ★ $\sum_{r=1}^{n} r = \frac{n}{2}(1 + n)$

---

3. $\sum_{r=1}^{n} a^{r} = a + a^{2} + a^{3} + \ldots + a^{n}$
    - Sum of first n terms of a GP with first term a and common ratio a
    - ★ $\sum_{r=1}^{n} a^{r} = \frac{a(1-a^{n})}{1-a}$

---

4. $\sum_{r=1}^{\infty} a^{r} = a + a^{2} + a^{3} + \ldots$
    - Sum to infinity of a GP
    - ★ $\sum_{r=1}^{\infty} = \frac{a}{1-a}, \; |a| < 1$

F - Formulas

- Will not be provided in MF27. However, the question will provide them whenever required
    1. $\sum_{r=1}^{n} r^{2} = \frac{n}{6}(n + 1)(2n + 1)$
    2. $\sum_{r=1}^{n} r^{3} = \frac{n^{2}}{4}(n + 1)^{2}$

G - Change Of Limits

- If the lower bounds needs to start from after the beginning ($r = 1$), understand the following

$$\sum_{r=1}^{n} u_r = \underbrace{u_1 + u_2 + \dots + u_{k-1}}_{\sum_{r=1}^{k-1} u_r} + \underbrace{u_k + u_{k+1} + \dots + u_{n-1} + u_n}_{\sum_{r=k}^{n} u_r}$$

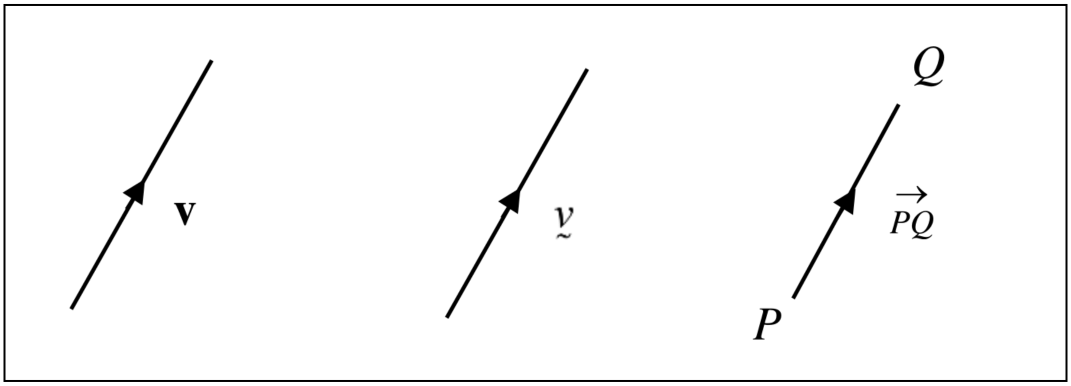
$$\therefore \sum_{r=k}^{n} u_r = \sum_{r=1}^{n} - \sum_{r=1}^{k},\ 1 < k < n$$

3 - Vectors

A - Introduction

1 - Definition

- It’s a quantity with both **magnitude** and **direction**
- Denoted as either
  1. Bold lowercase letter
  2. Handwritten lowercase letter with a wavy line (~) underneath
  3. Two capital letters representing its start and end terminal with an arrow



- Its magnitude / length is denoted by $|v|$

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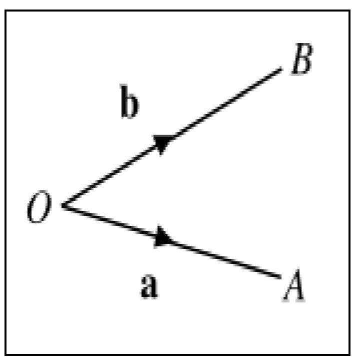
2 - Types

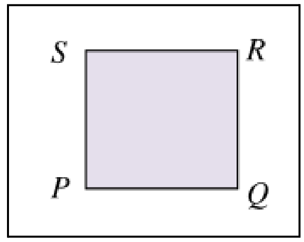
A - Position Vectors

- It’s a vector that’s defined relative to a given and fixed point
  - The origin O is a position vector
  - OA and OB are the position vectors of points A and B respectively

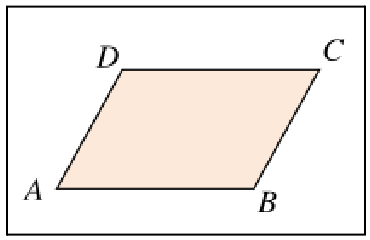


B - Equal Vectors

- 2 vectors are equal if and only if they have the same magnitude and direction
  - $\vec{PQ} = \vec{SR}$ & $\vec{PS} = \vec{QR}$ [Same magnitude & direction]
  - $\vec{PQ} \neq \vec{PS}$ [Same magnitude **BUT** different direction]



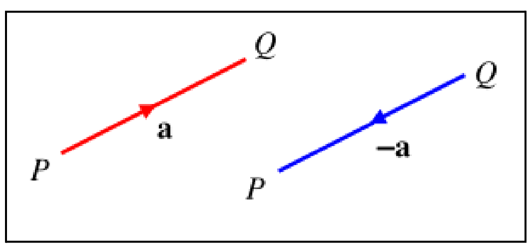
- A vector diagram is a parallelogram if and only if $\vec{AB} = \vec{DC}$ OR $\vec{AD} = \vec{BC}$



★ $\lambda a + \mu b = \gamma a + \delta b,\ a, b \neq 0,\ a \nparallel b$
  - $\lambda = \gamma,\ \mu = \delta$

C- Negative Vectors

- Same magnitude
  - ➢ $|-a| = |a|$
- Opposite Directions
  - ➢ Start and end points are reversed
  - ➢ $\vec{QP} = -\vec{PQ}$



D - Null / Zero Vectors

- No magnitude
- No direction
- It's a point and is denoted by **0** OR $\underset{\sim}{0}$

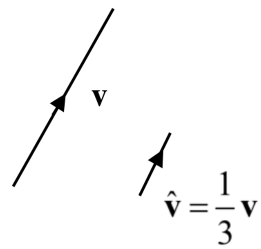
E - Unit Vectors

- It's a vector with a magnitude of 1, and is denoted as $\hat{v}$
  - $|\hat{v}| = 1$
- It, and its negative counterpart, are parallel to their parent vector **v**
- $\hat{v} = \frac{1}{|v|} v$
  - $v = (|v|)(\hat{v})$

Example: Consider a vector $\mathbf{v}$ such that $|\mathbf{v}| = 3$.

Then $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{3}\mathbf{v}$.

Note that $|\hat{\mathbf{v}}| = \left|\dfrac{1}{3}\mathbf{v}\right| = \dfrac{1}{\mathbf{3}}|\mathbf{v}| = \dfrac{1}{3}(3) = 1.$



$$\mathbf{v} = |\mathbf{v}|\,\hat{\mathbf{v}} = 3\hat{\mathbf{v}}.$$

F - Parallel Vectors

- $a \,//\, b \Leftrightarrow b = ka,\ k \in R,\ k \neq 0$
  - If $k > 0$, a and b are facing the same direction
  - If $k < 0$, a and b are in opposite directions

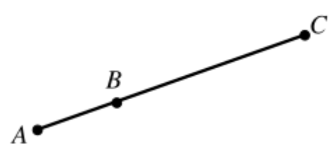
If $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$, then vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel since $\mathbf{b} = 3\mathbf{a}$

---

3 - Collinear Points

- Occurs when they lie on the **same straight line**
  - ★ They need to share a common point
  - ★ The bigger vector must be a scalar multiple of the smaller vector

$\overrightarrow{AB} = k\,\overrightarrow{AC}$ and $\overrightarrow{AB} = m\,\overrightarrow{BC}$ and $\overrightarrow{AC} = n\,\overrightarrow{BC}$, where k, m, n are non-zero scalars.



To prove that three points are collinear, we need to

- show that $\overrightarrow{AB} \,//\, \overrightarrow{BC}$, i.e. $\overrightarrow{AB} = \lambda\,\overrightarrow{BC}$ for some non-zero $\lambda \in \mathbb{R}$, <u>and</u>
- state that B is a common point.

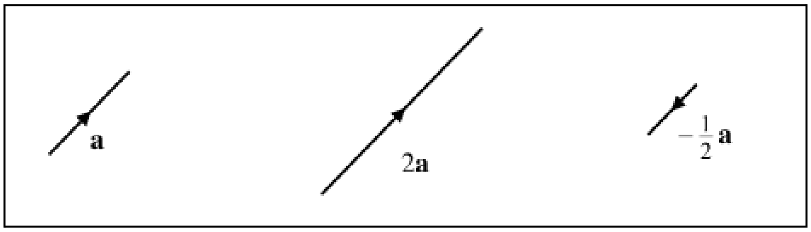
Alternatively, we may show that

- $\overrightarrow{AB} \,//\, \overrightarrow{AC}$ and state that A is a common point, or
- $\overrightarrow{AC} \,//\, \overrightarrow{BC}$ and state that C is a common point.

4 - Arithmetic Operations

A - Scalar λ Multiplication

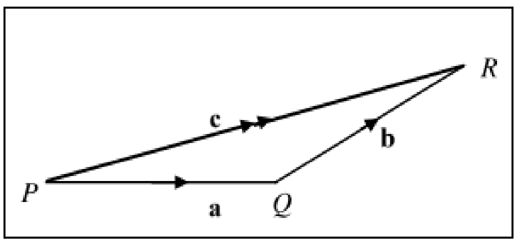
- ➢ λa is parallel to that of a
- ➢ λa's magnitude is $|\lambda||a|$
- ➢ If $\lambda > 0$, λa has the same direction as a
- ➢ If $\lambda < 0$. λa has the opposite direction of a
- ➢ If $\lambda = 0$, λa becomes the zero vector **0**



B - Vector Addition

1 - Triangle Law

- $\vec{PQ} + \vec{QR} = \vec{PR}$

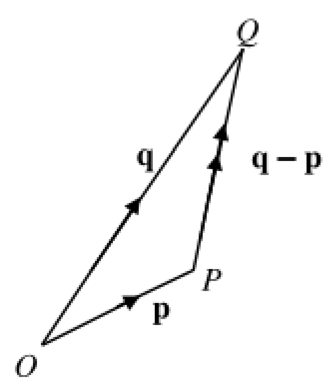


C - Vector Subtraction

- Subtraction of vectors is just adding a negative vector to another vector
  - ➢ $a - b = a + (-b)$

A vector $\vec{PQ}$ can be expressed in terms of the *position vectors*, $\vec{OP}$ and $\vec{OQ}$ as follows:

$$\begin{aligned}\vec{PQ} &= \vec{PO} + \vec{OQ}\\ &= -\vec{OP} + \vec{OQ}\\ &= \vec{OQ} - \vec{OP}\\ &= \mathbf{q} - \mathbf{p}\end{aligned}$$

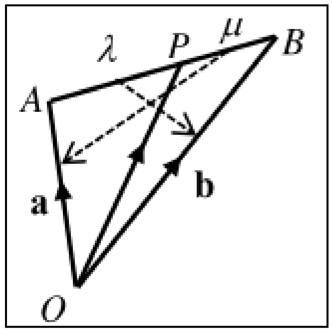


Thus $\vec{AB} = \vec{OB} - \vec{OA}$, $\vec{QR} = \vec{OR} - \vec{OQ}$ etc.

5 - Ratio Theorem

- If a point P divides the line segment AB internally in the ratio $\lambda : \mu$

$$\overrightarrow{OP} = \frac{\mu a + \lambda b}{\mu + \lambda}$$

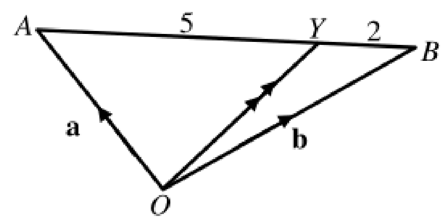


Relative to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point Y which divides AB in the ratio $5:2$.

Draw a diagram to identify the "criss-cross" pattern

Solution

$$\overrightarrow{OY} = \frac{2\overrightarrow{OA} + 5\overrightarrow{OB}}{7} = \frac{2}{7}\mathbf{a} + \frac{5}{7}\mathbf{b}$$



- If the point P divides the line segment AB internally in the ratio 1: 1

$$\overrightarrow{OP} = \frac{a+b}{2}$$

The points A and B have position vectors $2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ and $-2\mathbf{i} + 2\mathbf{k}$ respectively.

(a) Find the position vector of point M which is the midpoint of AB.

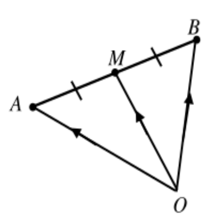
(b) Find the position vector of C on AB **produced** such that $AB = 3BC$.

Solution

(a) M is the mid-point of AB.

Using Ratio Theorem, we have

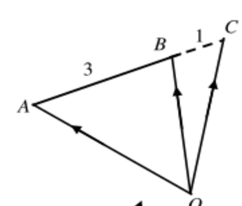
$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}.$$



$$\overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix}2\\-4\\-2\end{pmatrix} + \begin{pmatrix}-2\\0\\2\end{pmatrix}\right] = \begin{pmatrix}0\\-2\\0\end{pmatrix}$$

(b) Given $AB = 3BC$, that is, $AB:BC = 3:1$

Using Ratio Theorem,



C is on AB produced. So extend the line segment AB to get point C

$$\overrightarrow{OB} = \frac{\overrightarrow{OA} + 3\overrightarrow{OC}}{4}$$

$$\begin{pmatrix}-2\\0\\2\end{pmatrix} = \frac{1}{4}\left[\begin{pmatrix}2\\-4\\-2\end{pmatrix} + 3\overrightarrow{OC}\right]$$

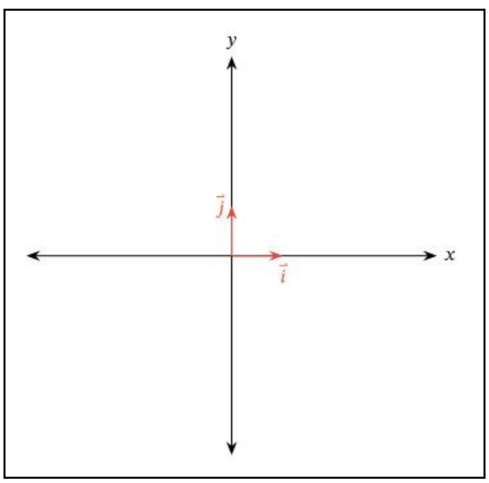
$$\begin{pmatrix}-8\\0\\8\end{pmatrix} = \begin{pmatrix}2\\-4\\-2\end{pmatrix} + 3\overrightarrow{OC}$$

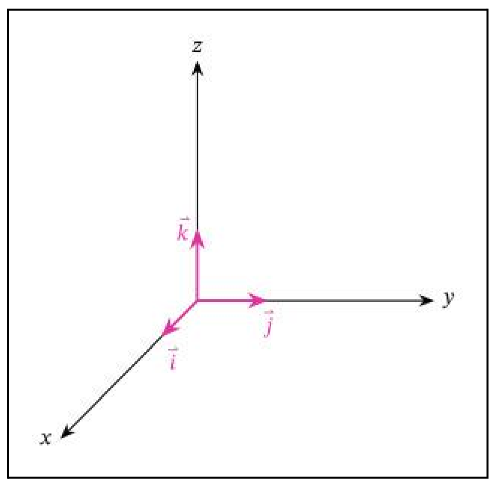
$$\begin{pmatrix}-10\\4\\10\end{pmatrix} = 3\overrightarrow{OC}$$

$$\overrightarrow{OC} = \frac{1}{3}\begin{pmatrix}-10\\4\\10\end{pmatrix} = \frac{2}{3}\begin{pmatrix}-5\\2\\5\end{pmatrix}$$

6 - 2D & 3D Vectors

- ★ i represents the positive $x - axis$
  - $i = [1, 0, 0]$
- ★ j represents the positive $y - axis$
  - $j = [0, 1, 0]$
- ★ k represents the positive $z - axis$
  - $k = [0, 0, 1]$
- ➢ They are unit vectors who are perpendicular to each other
- ➢ To find the magnitude, use Pythagoras' Theorem on their scalar multiples
  - If $v = 3i + 4j + 5k$
    - $|v| = \sqrt{3^2 + 4^2 + 5^2} \Rightarrow \sqrt{50}$
  - If $v = 3i + 4j$
    - $|v| = \sqrt{3^2 + 4^2} \Rightarrow 5$
- ➢ Their negative counterparts represent their corresponding axes in the $negative$ direction
  - $i = [-1, 0, 0]$
  - $j = [0, -1, 0]$
  - $k = [0, 0, -1]$





	vectors in 2-dimensions	vectors in 3-dimensions
1	$\overrightarrow{OP} = \mathbf{p} = a\mathbf{i} + b\mathbf{j} = \begin{pmatrix} a \\ b \end{pmatrix}$ Magnitude (or length) of vector $\overrightarrow{OP}$, $\left\lvert\overrightarrow{OP}\right\rvert = \lvert\mathbf{p}\rvert = \left\lvert\begin{pmatrix} a \\ b \end{pmatrix}\right\rvert = \sqrt{a^2+b^2}$ The unit vector in the direction of $\overrightarrow{OP}$ is $\hat{\mathbf{p}} = \frac{\mathbf{p}}{\lvert\mathbf{p}\rvert} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \end{pmatrix}$ *Example:* $\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $\lvert\mathbf{p}\rvert = \left\lvert\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right\rvert = \sqrt{3^2+(-4)^2} = 5$ Unit vector, $\hat{\mathbf{p}} = \frac{\mathbf{p}}{\lvert\mathbf{p}\rvert} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.	$\overrightarrow{OP} = \mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ Magnitude (or length) of vector $\overrightarrow{OP}$, $\left\lvert\overrightarrow{OP}\right\rvert = \lvert\mathbf{p}\rvert = \left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\rvert = \sqrt{a^2+b^2+c^2}$ The unit vector in the direction of $\overrightarrow{OP}$ is $\hat{\mathbf{p}} = \frac{\mathbf{p}}{\lvert\mathbf{p}\rvert} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *Example:* $\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ $\lvert\mathbf{p}\rvert = \left\lvert\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}\right\rvert = \sqrt{3^2+(-1)^2+(4)^2} = \sqrt{26}$ Unit vector, $\hat{\mathbf{p}} = \frac{\mathbf{p}}{\lvert\mathbf{p}\rvert} = \frac{1}{\sqrt{26}}\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.
2	Equal Vectors: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Leftrightarrow \quad a = p,\ b = q$	Equal Vectors: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Leftrightarrow \quad a = p,\ b = q,\ c = r$
3	Addition/Subtraction of Vectors: $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \end{pmatrix}$ *Example:* $\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 12 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$ $\begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 12 \\ -1 \end{pmatrix} = \begin{pmatrix} -14 \\ 4 \end{pmatrix}$	Addition/Subtraction of Vectors: $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \\ z_1 \pm z_2 \end{pmatrix}$ *Example:* $\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 12 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 9 \end{pmatrix}$

	vectors in 2-dimensions	vectors in 3-dimensions
4	Multiplication of a Vector by a Scalar: $\lambda\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$, where $\lambda \in \mathbb{R}$ *Example:* $2\begin{pmatrix} -2 \\ 3 \end{pmatrix}=\begin{pmatrix} -4 \\ 6 \end{pmatrix}$	Multiplication of a Vector by a Scalar: $\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$, where $\lambda \in \mathbb{R}$ *Example:* $-\frac{1}{2}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}=\begin{pmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{pmatrix}$

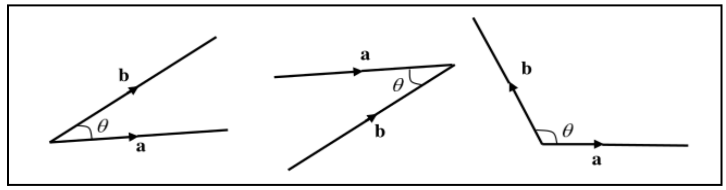
B - Scalar & Vector Products

1 - Scalar / Dot Product

A - Formula

★ $a \cdot b = |a||b|\cos\Theta$, $a, b \neq 0$, $0° \leq \Theta \leq 180°$
- ➢ $a \cdot b = |a||b|$, $\Theta = 0°$ [a & b are in the **same** direction]
- ➢ $a \cdot b = 0$, $\Theta = 90°$ [a & b are **perpendicular**]
- ➢ $a \cdot b = -|a||b|$, $\Theta = 180°$ [a & b are in **opposite** directions]

○ Vector Form
- ➢ $a = [x_1, y_1, z_1]$
- ➢ $b = [x_2, y_2, z_2]$
- ★ $a \cdot b = x_1x_2 + y_1y_2 + z_1z_2$



B - Properties

- $a \cdot b$ is a scalar
- $a \cdot b = b \cdot a$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- $\lambda(a \cdot b) = (\lambda a \cdot b) = a \cdot (\lambda b)$, $\lambda \notin R$
- $a \cdot a = |a|^2$

★ $a \cdot b = 0 \Rightarrow (a \perp b) \vee (a = 0) \vee (b = 0)$
- There is **no division** of vectors
  - ○ $a \cdot b = c \cdot d \rightarrow a \neq \frac{c \cdot d}{b}$

C - Use Cases

1 - Find the angle between 2 vectors

- ➢ $\cos\Theta = \frac{a \cdot b}{|a||b|}$, $a, b \neq 0$, $0° \leq \Theta \leq 180°$
- ➢ Acute Angle if $a \cdot b > 0 \Leftrightarrow \cos\Theta > 0$
- ➢ Obtuse Angle if $a \cdot b < 0 \Leftrightarrow \cos\Theta < 0$

(a) If $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, find the cosine of the angle between **a** and **b**.

(b) Find the angle between $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + \mathbf{k}$.

(c) The position vectors of points A and B relative to the origin O are $-3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{j} + 9\mathbf{k}$ respectively. Calculate the size of the angle ABO.

<u>Solution</u>

(a) Let the angle between **a** and **b** be θ.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\begin{pmatrix}1\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-5\end{pmatrix}}{\sqrt{1^2+4^2+1^2}\cdot\sqrt{2^2+1^2+5^2}} = \frac{2+4+5}{\sqrt{540}} = \frac{11}{6\sqrt{15}}$$

(b) Let θ be the angle between $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$.

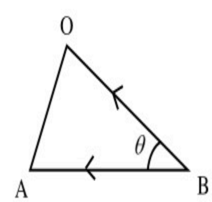
$$\cos\theta = \frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}}{\sqrt{2^2+1^2+(-1)^2}\sqrt{(-1)^2+0^2+1^2}} = \frac{-3}{\sqrt{6}\sqrt{2}}$$

$\Rightarrow \theta = 150°$

(c) To find $\angle ABO$, we need to take scalar product of $\overrightarrow{BO}$ and $\overrightarrow{BA}$.

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix}-3\\2\\1\end{pmatrix} - \begin{pmatrix}0\\1\\9\end{pmatrix} = \begin{pmatrix}-3\\1\\-8\end{pmatrix}$$

$$\overrightarrow{BO} = -\overrightarrow{OB} = \begin{pmatrix}0\\-1\\-9\end{pmatrix}$$



$$\cos\angle ABO = \frac{\begin{pmatrix}-3\\1\\-8\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\-9\end{pmatrix}}{\sqrt{(-3)^2+1^2+(-8)^2}\sqrt{(-1)^2+(-9)^2}} = \frac{71}{\sqrt{74}\sqrt{82}}$$

$\angle ABO = 24.3°$ (1 d.p.)

<u>**Note**</u>: To find $\angle ABO$, choose 2 vectors which are either both pointing outwards from B or inwards to B. Thus, we can use ($\overrightarrow{BO}$ and $\overrightarrow{BA}$) or ($\overrightarrow{OB}$ and $\overrightarrow{AB}$).

2 - Perpendicular Test

★ $a \cdot b = 0 \Leftrightarrow \theta = 90°$ ($cos90° = 0$)

➢ Can be used to find the values of the coefficients of i, j, and k

If $3\mathbf{i}+\lambda\mathbf{j}+6\mathbf{k}$ and $(2\lambda^2)\mathbf{i}-\mathbf{j}$ are perpendicular vectors, find the value(s) of λ.

<u>Solution</u>

$3\mathbf{i}+\lambda\mathbf{j}+6\mathbf{k}$ and $(2\lambda^2)\mathbf{i}-\mathbf{j}$ are perpendicular vectors $\Rightarrow \begin{pmatrix}3\\ \lambda\\ 6\end{pmatrix}\cdot\begin{pmatrix}2\lambda^2\\ -1\\ 0\end{pmatrix}=0$

$6\lambda^2-\lambda=0$

$\lambda(6\lambda-1)=0$

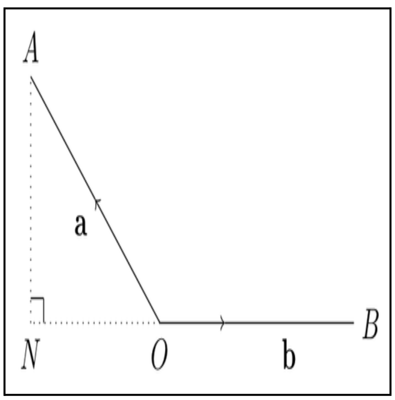
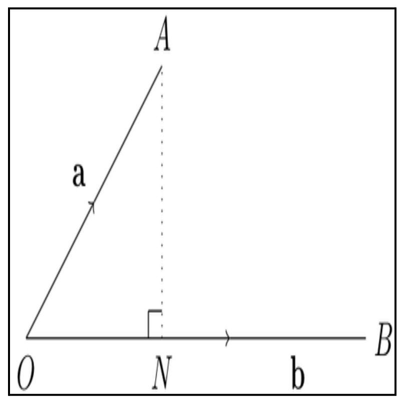
$\lambda=0$ or $\lambda=\dfrac{1}{6}$

---

3 - Find the length of projection

★ $\frac{|a \cdot b|}{|b|}$

➢ It's the **magnitude** representing the portion of a that lies lies on b

$\Theta > 90°$	$\Theta < 90°$
	

Let $\mathbf{a}=4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{j}$. Find the length of projection of **a** onto **b**.

<u>Solution</u>

Length of projection of **a** onto **b**

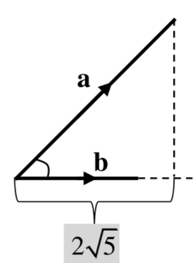
$=\dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}$

$=\dfrac{\left|\begin{pmatrix}4\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}\right|}{\sqrt{5}}$

$=\dfrac{4+6}{\sqrt{5}}$

$=\dfrac{10}{\sqrt{5}}$

$=2\sqrt{5}$ units



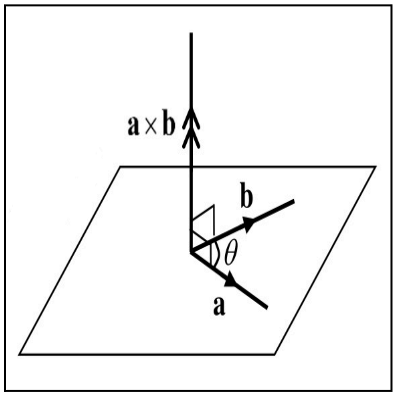
4 - To find the projection vector

- ★ $(a \cdot \hat{b})\ \hat{b}$
    - It's the **vector** representing the portion of a that is on b
    - $\Theta < 90°$: PV is the the same direction as b
    - $\Theta > 90°$: PV is the the opposite direction of b

2 - Vector / Cross Product

A - Formula

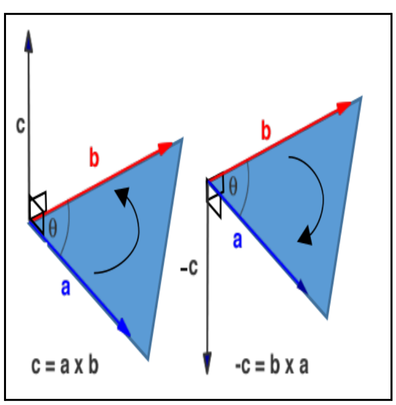
- ★ $a \times b = |a||b|sin\Theta\hat{n}$, $a, b \neq 0$, $0° < \Theta < 180°$, $\hat{n} \perp a, b$
    - ➢ $\hat{n}$ is a unit vector that's perpendicular to the plane in which a and b lie in
- ★ $|a \times b| = |a||b|sin\Theta$
- This vector will be perpendicular to both a and b



- Vector Form
    - $a = [x_1, y_1, z_1]$
    - $b = [x_2, y_2, z_2]$
    - ★ $a \times b = [y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2]$
        - Given in MF27

B - Properties

- $a \times b$ is a vector
- $a \times b \neq b \times a$
    - They will point in opposite directions
- $a \times (b + c) = (a \times b) + (a \times c)$
- $\lambda(a \times b) = [(\lambda a) \times b] = [a \times (\lambda b)]$, $\lambda \in R$
- $ma \times nb = (mn)(a \times b)$, $m, n \in R$
- ★ $a \times b = 0 \Rightarrow (a = 0) \vee (b = 0) \vee (a, b = 0) \vee (a \parallel b, sin\Theta = 0°)$
- $a \times a = 0$, $a \parallel a$
- $b \times b = 0$, $b \parallel b$
- ★ $a \times b = |a||b| \Rightarrow \Theta = 90°$, $a \perp b$



C - Use Cases (To Find…)

1 - A Triangle's Area

- ★ $\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$

The points A, B and C have coordinates $(1,2,1)$, $(1,0,3)$ and $(-1,2,-1)$ respectively.

Using a **vector product**, find the area of the triangle ABC.

Solution

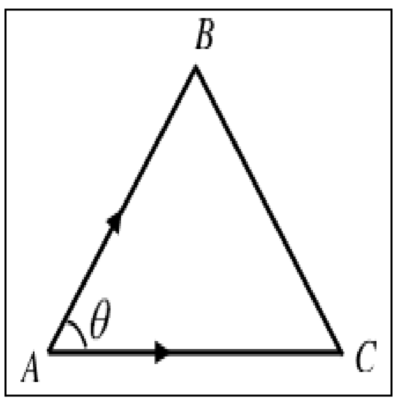
Tip: Extract the common factor to simplify your calculations.

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Area of triangle} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \times (-2)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right| = \frac{1}{2}|-4|\left|\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right|$$

$$= \frac{1}{2}|-4|\left|\begin{pmatrix} -1-0 \\ -(0-1) \\ 0-(-1) \end{pmatrix}\right| = 2\sqrt{(-1)^2 + 1^2 + 1^2}$$

$$= 2\sqrt{3} \text{ units}^2$$



2 - A Parallelogram's Area

★ $\left|\vec{OA} \times \vec{OC}\right|$

The points A, B and C have coordinates (4, 1, 0), (−1, −1, 1) and (3, 0, 1) respectively, and O is the origin.

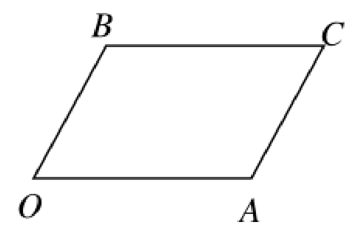
(i) Show that $OACB$ is a parallelogram.

(ii) Using a vector product, find the area of parallelogram $OACB$.

<u>Solution</u>

Given: $\vec{OA} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, $\quad \vec{OB} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $\quad \vec{OC} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

(i) Show: $OACB$ is a parallelogram



$$\vec{BC} = \vec{OC} - \vec{OB}$$

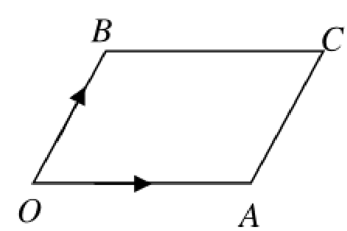
$$= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$= \vec{OA}$$

$\vec{BC} = \vec{OA} \Rightarrow$ opposite sides parallel and equal length

$\Rightarrow$ $OACB$ is a parallelogram

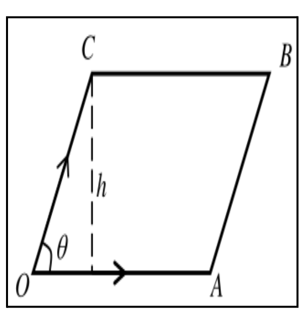
(ii) Area of parallelogram $OACB$



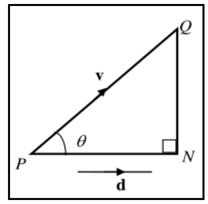
$$= \left|\vec{OA} \times \vec{OB}\right|$$

$$= \left|\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}\right|$$

$$= \sqrt{26} \ \text{ units}^2$$



3 - A Right-Angled Triangle's Sides' Lengths



- ★ $PN = \left|v \cdot \hat{d}\right|$
- ★ $NQ = \left|v \times \hat{d}\right|$
- ★ $NQ = \sqrt{(PQ)^2 - (PN)^2}$

The points A, B and C have coordinates (3, –1, 4), (5, 2, –4) and (–2, 0, 1) respectively, and O is the origin.

(i) Find the length of projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$.

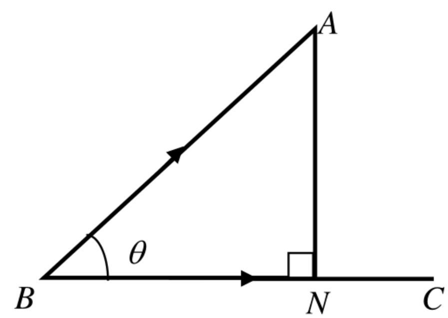
(ii) Find the projection vector of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$.

(iii) Using vector product, calculate the length of perpendicular from A to $\overrightarrow{BC}$.

<u>Solution</u>

(i) $\overrightarrow{BA} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 8 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ 5 \end{pmatrix}$



Length of projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$ is

$$= \frac{\left|\overrightarrow{BA} \cdot \overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$$

$$= \frac{\left|\begin{pmatrix} -2 \\ -3 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ -2 \\ 5 \end{pmatrix}\right|}{\sqrt{(-7)^2 + (-2)^2 + 5^2}}$$

$$= \frac{|14 + 6 + 40|}{\sqrt{78}}$$

$$= \frac{10\sqrt{78}}{13} \text{ units}$$

(ii) Projection vector of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$ is $\overrightarrow{BN}$.

$$\overrightarrow{BN} = \frac{10\sqrt{78}}{13} \cdot \frac{\begin{pmatrix} -7 \\ -2 \\ 5 \end{pmatrix}}{\sqrt{(-7)^2 + (-2)^2 + 5^2}}$$

$$= \frac{10}{13}\begin{pmatrix} -7 \\ -2 \\ 5 \end{pmatrix}$$

(iii) Let the foot of perpendicular from A to $\overrightarrow{BC}$ be N.

Length of perpendicular from A to $\overrightarrow{BC}$ is

$$NA = \frac{\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|} = \frac{\left|\begin{pmatrix} -2 \\ -3 \\ 8 \end{pmatrix} \times \begin{pmatrix} -7 \\ -2 \\ 5 \end{pmatrix}\right|}{\sqrt{(-7)^2 + (-2)^2 + 5^2}}$$

$$= \frac{\left|\begin{pmatrix} -15-(-16) \\ -(-10+56) \\ 4-21 \end{pmatrix}\right|}{\sqrt{78}}$$

$$= \frac{\left|\begin{pmatrix} 1 \\ -46 \\ -17 \end{pmatrix}\right|}{\sqrt{78}}$$

$$= \frac{\sqrt{1+(-46)^2+(-17)^2}}{\sqrt{78}}$$

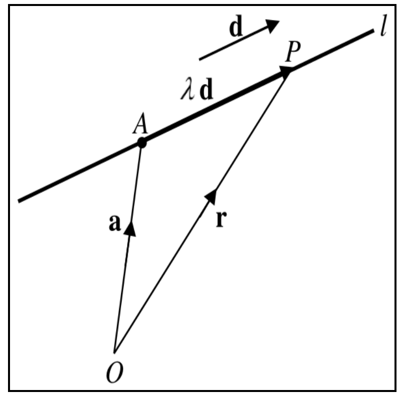
$$= \sqrt{\frac{401}{13}} \text{ units}$$

C - Lines

1 - Forms

A - Vector Form

- $r = a + \lambda d$
  - r is the **position vector of any point** on the line l
  - a is **positive vector of a fixed point** on the line l
  - d is a **vector that is parallel** to the line l
  - λ is the scalar multiple that scales d, forming r



B - Parametric Form

- $x = a_1 + \lambda d_1$
- $y = a_2 + \lambda d_2$
- $z = a_3 + \lambda d_3$

C - Cartesian Form

$$\lambda = \frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$$

- A point lies on a line if its position vector satisfies the line's equation
  - There needs to be a unique / one solution for λ only

A line l has vector equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ \frac{1}{2} \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

Determine whether the point $(-1, 3, -2)$ lies on the line l.

$$\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ \frac{1}{2} \end{pmatrix}$$

$$5 + 3\lambda = -1 \Rightarrow \lambda = \frac{-6}{3} = -2$$

$$1 - \lambda = 3 \quad \Rightarrow \lambda = 1 - 3 = -2$$

$$-2 + \frac{1}{2}\lambda = -2 \ \Rightarrow \lambda = 0$$

Since there is no unique solution for λ, the point $(-1, 3, -2)$ does not lie on the line l.

2 - Relationship

A - Point & Line

- Foot Of ⊥
- ⊥'s distance
- Reflection Of A Point P In A Line
  - ★ $\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP}$

The equation of a straight line l is $\mathbf{r} = (1+4\lambda)\mathbf{i} + 3\lambda\,\mathbf{j} + 2\mathbf{k}$ and the point P has coordinates $(2, 7, -1)$.

(i) Find the position vector of the foot of the perpendicular from point P to line l.

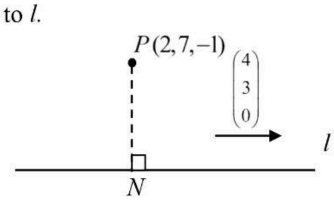
(ii) Hence find the shortest distance from point P to line l.

(iii) Find the position vector of the image of point P when reflected in the line l.

<u>Solution</u>

(i) Given the equation of line l: $\mathbf{r} = \begin{pmatrix} 1+4\lambda \\ 3\lambda \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Let N be the foot of the perpendicular from P to l.
N is a point on l



$$\Rightarrow \overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

$$\therefore \overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

$$\overrightarrow{PN} \perp l \Rightarrow \overrightarrow{PN} \cdot \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = 0 \Rightarrow \left[\begin{pmatrix} -1 \\ -7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow (-4-21) + \lambda(16+9) = 0 \Rightarrow \lambda = \frac{25}{25} \Rightarrow \lambda = 1$$

Hence, $\overrightarrow{ON} = \begin{pmatrix} 1+4(1) \\ 3(1) \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$.

(ii) $\overrightarrow{PN} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$

$$\therefore \left| \overrightarrow{PN} \right| = \sqrt{9+16+9} = \sqrt{34}$$

(iii) Using Mid-point Theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$

$$\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP} = \begin{pmatrix} 8 \\ -1 \\ 5 \end{pmatrix}$$

B - 2 Lines

1 - Parallel Lines

- ★ $l_1 = kl_2$, $\theta = 0^\circ / 180^\circ$
  - ➢ l_1 and l_2 do not intersect
  - ➢ l_1 and l_2 lie on the same plane
- ★ Distance = $\frac{\vec{AB} \times d}{|d|}$
  - ○ $\vec{AB}$ is the **vector connecting** the 2 lines
  - ○ d is the **direction vector** of the 2 lines

Example 9:

Find the distance between the lines $l_1:\ \mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix},\ \lambda\in\mathbb{R}$ and $l_2:\ \mathbf{r}=\begin{pmatrix}3\\-2\\4\end{pmatrix}+\mu\begin{pmatrix}2\\1\\2\end{pmatrix},\ \mu\in\mathbb{R}$.

<u>Solution</u>

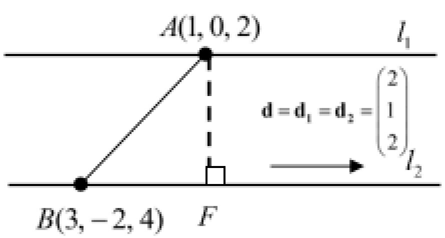
Let F be the foot of perpendicular from point A (on l_1) with position vector $\begin{pmatrix}1\\0\\2\end{pmatrix}$ to l_2.

$$\overrightarrow{AB}=\begin{pmatrix}3\\-2\\4\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}2\\-2\\2\end{pmatrix}=2\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Note that both l_1 and l_2 have the same direction vector $\mathbf{d}=\begin{pmatrix}2\\1\\2\end{pmatrix}$.

$\therefore l_1$ is parallel to l_2

Distance between l_1 and l_2 = length of $AF=\dfrac{\left|\overrightarrow{AB}\times\mathbf{d}\right|}{|\mathbf{d}|}$



$$=\frac{\left|\begin{pmatrix}2\\-2\\2\end{pmatrix}\times\begin{pmatrix}2\\1\\2\end{pmatrix}\right|}{\sqrt{4+1+4}}$$

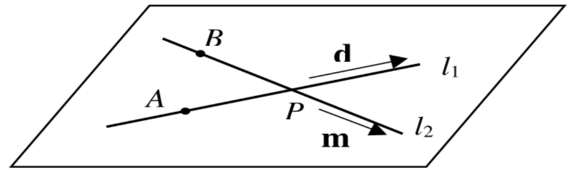
$$=\frac{\left|\begin{pmatrix}-6\\0\\6\end{pmatrix}\right|}{3}=\frac{\left|6\begin{pmatrix}-1\\0\\1\end{pmatrix}\right|}{3}$$

$$=2\left|\begin{pmatrix}-1\\0\\1\end{pmatrix}\right|=2\sqrt{1^2+0^2+1^2}$$

$$=2\sqrt{2}$$

2 - Intersecting Lines

- ➢ l_1 and l_2 are not parallel
- ➢ l_1 and l_2 lie on the same plane
- ★ There is **only 1 solution** set for both λ and μ



Determine whether the following two lines are intersecting:

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad ; \quad \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},\ s,t \in \mathbb{R}.$$

If both lines do intersect each other, find the position vector of the point of intersection.

$$\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+s\begin{pmatrix}1\\-1\\1\end{pmatrix} \quad \text{------- (*)}; \qquad \mathbf{r}=\begin{pmatrix}2\\4\\6\end{pmatrix}+t\begin{pmatrix}2\\1\\3\end{pmatrix}$$

Since $\begin{pmatrix}1\\-1\\1\end{pmatrix}\neq k\begin{pmatrix}2\\1\\3\end{pmatrix}, k\in\mathbb{R}$, the two lines are non-parallel.

Consider $\begin{pmatrix}1\\-1\\3\end{pmatrix}+s\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\4\\6\end{pmatrix}+t\begin{pmatrix}2\\1\\3\end{pmatrix} \Rightarrow \begin{cases}1+s=2+2t\\-1-s=4+t\\3+s=6+3t\end{cases}$

Thus, we have $\begin{cases}s-2t=1 & \text{-------- (1)}\\-s-t=5 & \text{------- (2)}\\s-3t=3 & \text{------- (3)}\end{cases}$

Solving (1) and (2) simultaneously, we get $t=-2$ and $s=-3$.
(or Using GC to solve eqn (1), (2), (3), we get $t=-2$ and $s=-3$.)

Sub. $t=-2$ and $s=-3$ into (3):

LHS = $-3-3(-2)=3=$ RHS
Hence the two lines intersect.

Sub. $s=-3$ into (*) : position vector of the point of intersection is

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}-3\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}-2\\2\\0\end{pmatrix}.$$

3 - Skew Lines

- ○ l_1 and l_2 are not parallel
- ○ l_1 and l_2 do not intersect
- ○ l_1 and l_2 do not lie on the same plane
- ★ There are **no solutions sets** for both λ and μ

Show that l_1 and l_2 are skew lines, given that

$$l_1 : \frac{x+1}{-2} = \frac{y-1}{2} = \frac{z-3}{1} \quad ; \quad l_2 : \frac{x-1}{3} = \frac{y-3}{1} = 2z.$$

$l_1 : \frac{x+1}{-2} = \frac{y-1}{2} = \frac{z-3}{1} \ (= \lambda)$

$$\Rightarrow \begin{cases} x = -2\lambda - 1 \\ y = 2\lambda + 1 \\ z = \lambda + 3 \end{cases}$$

$$\Rightarrow l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$l_2 : \frac{x-1}{3} = \frac{y-3}{1} = 2z \ (= \mu)$

$$\Rightarrow \begin{cases} x = 1 + 3\mu \\ y = 3 + \mu \\ z = \frac{1}{2}\mu \end{cases}$$

$$\Rightarrow l_2\text{:} \ \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

Since $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 3 \\ 1 \\ \frac{1}{2} \end{pmatrix}, k \in \mathbb{R}$, l_1 is not parallel to l_2.

Consider $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ \frac{1}{2} \end{pmatrix} \Rightarrow \begin{cases} -1 - 2\lambda = 1 + 3\mu \\ 1 + 2\lambda = 3 + \mu \\ 3 + \lambda = \frac{1}{2}\mu \end{cases}$

Thus, we have $2\lambda + 3\mu = -2$ --- (1)

$-2\lambda + \mu = -2$ --- (2)

$-\lambda + \frac{1}{2}\mu = 3$ --- (3)

Solving equation (1) and (2), we have $\lambda = \frac{1}{2}$ and $\mu = -1$.

Substituting both values into (3),

LHS = $-\lambda + \frac{1}{2}\mu = -\frac{1}{2} + \frac{1}{2}(-1) = -1$

RHS = 3

Since LHS $\neq$ RHS, the two lines do not intersect.
(Alternatively using GC to solve eqn (1), (2) and (3), there is no real solution for λ and μ. Hence, the two lines do no intersect.)

Since the two lines are not parallel to each other and do not intersect, they are skew lines.

3 - Angle

★ $cos\,\theta = \frac{|d\cdot m|}{|d||m|}$

- ➢ $d \perp m \Leftrightarrow cos\,\theta = 0$
- ➢ $d\, //\, m \Leftrightarrow cos\,\theta = 1^{\circ}$

(a) Find the acute angle between the line with equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},\ t \in \mathbb{R}\text{, and the } z\text{-axis.}$$

(b) The vector equations of two lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix},\ \lambda \in \mathbb{R} \qquad l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix},\ \mu \in \mathbb{R}$$

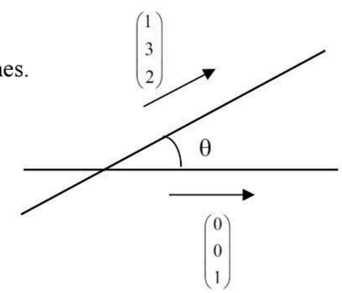
respectively.

Find the acute angle between lines l_1 and l_2.

<u>Solution</u>

(a) Vector equation of z-axis: $\mathbf{r} = s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $s \in \mathbb{R}$

Let θ be the acute angle between the two lines.



$$\cos\,\theta = \frac{\left|\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}{\sqrt{1+9+4}\sqrt{1}} = \frac{2}{\sqrt{14}}$$

$\therefore \theta = 57.7^{\circ}$ (to 1 decimal place)

(b) Let θ be the acute angle between l_1 and l_2 .

$$\cos\,\theta = \frac{\left|\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}\right|}{\sqrt{9+1+4}\sqrt{4+25}} = \frac{|-4|}{\sqrt{406}} = \frac{4}{\sqrt{406}}$$

$\therefore \theta = 78.5^{\circ}$ (to 1 decimal place)

4 - Reflection

3.4 Reflection of a Line in another Coplanar Line

Two lines are coplanar if they are intersecting or parallel.

Video 2C.10

Example 10:

The point A has position vector $3\mathbf{i}-6\mathbf{j}+\mathbf{k}$ and the line l has equation $\mathbf{r}=7\mathbf{i}+\mathbf{j}-5\mathbf{k}+\lambda(-\mathbf{i}+3\mathbf{j}+\mathbf{k})$, $\lambda\in\mathbb{R}$.

(i) Find the position vector of the foot of perpendicular, F, from A to l, and hence the position vector of the reflection of A in l.

(ii) Let B be a point on l with position vector $6\mathbf{i}+4\mathbf{j}-4\mathbf{k}$. Find the equation of the reflection of line AB in l.

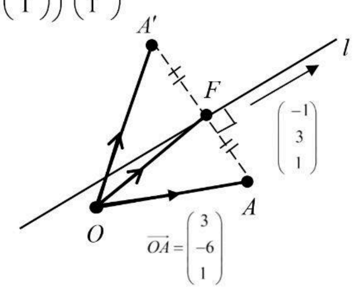
<u>Solution</u>

(i) Since F lies on l, $\overrightarrow{OF}=\begin{pmatrix}7\\1\\-5\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

$$\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}=\begin{pmatrix}7\\1\\-5\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\1\end{pmatrix}-\begin{pmatrix}3\\-6\\1\end{pmatrix}=\begin{pmatrix}4\\7\\-6\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\1\end{pmatrix}$$

Since $\overrightarrow{AF}\perp l$, $\overrightarrow{AF}\cdot\begin{pmatrix}-1\\3\\1\end{pmatrix}=0$, i.e. $\left(\begin{pmatrix}4\\7\\-6\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\1\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\3\\1\end{pmatrix}=0$

$$(-4+21-6)+\lambda(1+9+1)=0$$
$$11+11\lambda=0$$
$$\lambda=-1$$

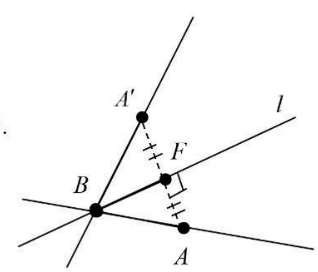


$$\overrightarrow{OF}=\begin{pmatrix}7\\1\\-5\end{pmatrix}+(-1)\begin{pmatrix}-1\\3\\1\end{pmatrix}=\begin{pmatrix}8\\-2\\-6\end{pmatrix}$$

$$\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)$$

$$\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}8\\-2\\-6\end{pmatrix}-\begin{pmatrix}3\\-6\\1\end{pmatrix}=\begin{pmatrix}13\\2\\-13\end{pmatrix}$$

(ii) The reflection of line AB in the line l is the line $A'B$.



$$\overrightarrow{A'B}=\overrightarrow{OB}-\overrightarrow{OA'}=\begin{pmatrix}6\\4\\-4\end{pmatrix}-\begin{pmatrix}13\\2\\-13\end{pmatrix}=\begin{pmatrix}-7\\2\\9\end{pmatrix}\text{ t}$$

Vector equation of line $A'B$: $\mathbf{r}=\begin{pmatrix}6\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-7\\2\\9\end{pmatrix}$, $\mu\in\mathbb{R}$.

Tutorial 2C Q9 – Q10

Note: The parameter for line $A'B$ is μ . λ cannot be used because it has been used as the parameter for line l.

D - Planes

1 - Forms

A - Parametric Form

- $r = a + \lambda u + \mu v$
    - r is the **position vector of any point** on the plane
    - a is the **position vector of a fixed point** on the plane
    - u , v are non-parallel **vectors that are parallel** to the plane

> Write down a vector equation in **parametric form** of the plane that passes through the point $(2,-1,-2)$ and is parallel to $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{i}-3\mathbf{k}$.
>
> Solution
>
> A vector equation, in parametric form, of the plane is
>
> $$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \text{ where } \lambda, \mu \in \mathbb{R}.$$
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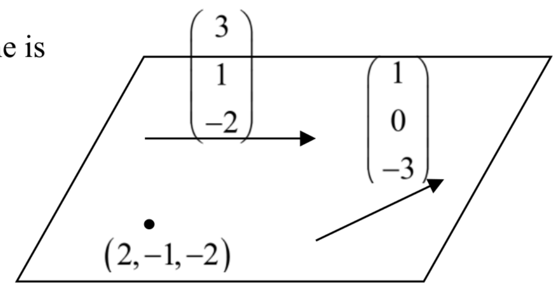
> 
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B - Scalar Product Form

- $r \cdot n = D$, $D = a \cdot n$
    - r is the position vector of any point on the plane
    - n is a vector perpendicular to the plane
    - a is a position vector of a fixed point on the plane
    - D is a scalar

★ Any point on the plane, r, when dot multiplied by the plane's normal,n, all give the same scalar output

- $r \cdot n = 0 \Rightarrow$ **origin** lies on the plane

Example 3:

Find a vector equation in **scalar product form** of the plane which

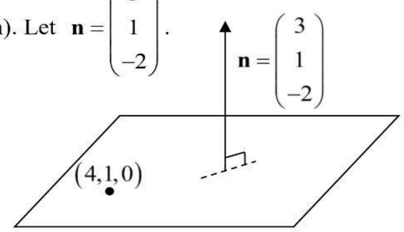
(a) passes through the point $(4, 1, 0)$ and is perpendicular to $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$,

(b) passes through the point $(1,0,1)$ and is parallel to $\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and $8\mathbf{i}-\mathbf{j}-\mathbf{k}$.

(c) passes through the points $A(0,0,1)$, $B(2, 3, 4)$ and $C(3,-2,-1)$.

Solution

(a) A normal vector to the plane $=\begin{pmatrix}3\\1\\-2\end{pmatrix}$ (given). Let $\mathbf{n}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$.

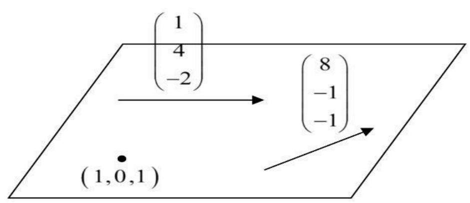


$$\mathbf{r}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}=\begin{pmatrix}4\\1\\0\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

$\therefore\ \mathbf{r}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}=13$ is a vector equation of the plane in scalar product form.

(b) A normal vector to the plane $=\begin{pmatrix}1\\4\\-2\end{pmatrix}\times\begin{pmatrix}8\\-1\\-1\end{pmatrix}=\begin{pmatrix}-6\\-15\\-33\end{pmatrix}=-3\begin{pmatrix}2\\5\\11\end{pmatrix}$

Let $\mathbf{n}=\begin{pmatrix}2\\5\\11\end{pmatrix}$.



$$\mathbf{r}\cdot\begin{pmatrix}2\\5\\11\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\5\\11\end{pmatrix}$$

$\therefore\ \mathbf{r}\cdot\begin{pmatrix}2\\5\\11\end{pmatrix}=13$ is a vector equation of the plane in scalar product form.

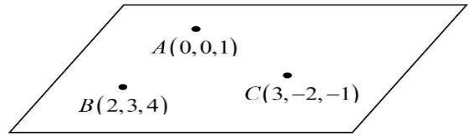
<u>Note:</u> Any scalar multiple of $\begin{pmatrix}-6\\-15\\-33\end{pmatrix}$ can be used as the normal vector to the plane.

Thus, we use $\begin{pmatrix}2\\5\\11\end{pmatrix}$ for simpler calculations in subsequent parts.

(c) Two vectors parallel to the plane are

$$\overrightarrow{BA} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -3 \end{pmatrix} \quad \text{and}$$

$$\overrightarrow{BC} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix}$$



A normal vector to the plane $= \begin{pmatrix} -2 \\ -3 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ -13 \\ 13 \end{pmatrix} = 13 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Let $\mathbf{n} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$$\mathbf{r} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

This is the position vector of point A. You may also use $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

$\therefore\ \mathbf{r} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 1$ is a vector equation of the plane in scalar product form.

C - Cartesian Form

- $n_1 x + n_2 y + n_3 z = D$, $D = a \cdot n$
  - n_{1-3} are the x, y, z coordinates of the vector perpendicular to the plane

The plane p contains the line $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \gamma(2\mathbf{i} + \mathbf{j})$, where $\gamma \in \mathbb{R}$, and is perpendicular to the plane q with equation $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3$.

(a) Find a cartesian equation of p.

(b) Hence determine if the points $A(2,1,-1)$ and $B(-1,1,1)$ lie on p.

<u>Solution</u>

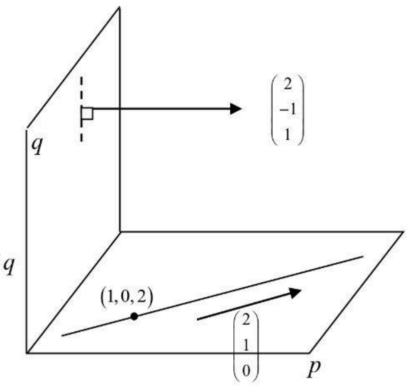
(a) Line: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\gamma \in \mathbb{R}$

Plane q: $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 3$

Vectors parallel to p are the normal vector of q and the direction vector of the line.

Thus, a normal vector to the plane p:

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$$



$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} = -7 \text{ (vector equation in scalar product form)}$$

i.e. $x - 2y - 4z = -7$ (cartesian equation)

(b) Substitute the coordinates of point A into the LHS of the cartesian equation of p:

LHS $= 2 - 2(1) - 4(-1) = 4$, **RHS** $= -7$

LHS $\neq$ RHS

$\therefore$ Point A does not lie on p.

Substitute the coordinates of point B into the LHS of the cartesian equation of p:

LHS $= -1 - 2(1) - 4(1) = -7 =$ **RHS**

$\therefore$ Point B lies on p.

2 - Special Planes

xy - plane : z - coordinate $=0$

- The z - plane is perpendicular to it

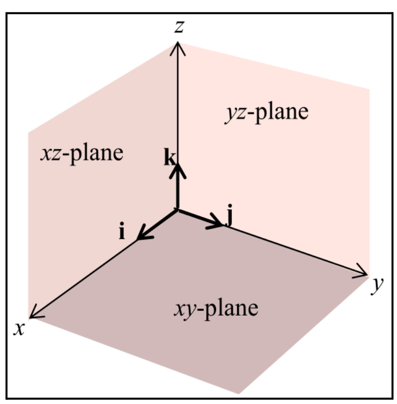
yz - plane : x - coordinate $=0$

- The x - plane is perpendicular to it

xz - plane : y - coordinate $=0$

- The y - plane is perpendicular to it

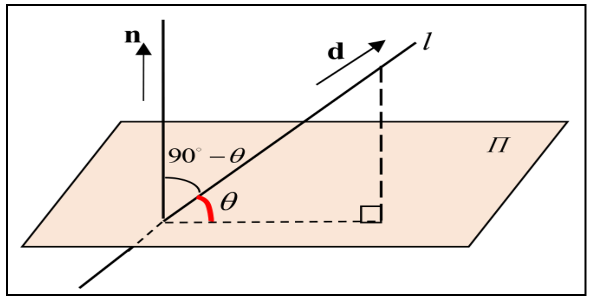
Plane	**Vector Equation in Parametric form**	**Vector Equation in Scalar Product form**	**Cartesian equation**
xy-plane	$\mathbf{r}=\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\0\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$	$\mathbf{r}\cdot\mathbf{k}=\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$	$z=0$
yz-plane	$\mathbf{r}=\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\0\\1\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$	$\mathbf{r}\cdot\mathbf{i}=\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=0$	$x=0$
xz-plane	$\mathbf{r}=\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\0\\1\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$	$\mathbf{r}\cdot\mathbf{j}=\mathbf{r}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0$	$y=0$



3 - Relationship

A - Line & Plane

1 - Angle



- $sin\,\theta = \frac{|d \cdot n|}{|d||n|}$
    - d is the **direction** vector
    - n is the **normal** vector

The plane π and the line l have equations $\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j}+\mathbf{k})=3$ and $\mathbf{r}=\mathbf{i}+\mathbf{j}+\mu(\mathbf{i}+\mathbf{j}-2\mathbf{k})$, $\mu\in\mathbb{R}$ respectively. Find the **acute** angle between π and l.

<u>Solution</u>

Let θ be the acute angle between π and l.

$$\sin\theta = \frac{\left|\begin{pmatrix}5\\2\\1\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-2\end{pmatrix}\right|}{\sqrt{25+4+1}\,\sqrt{1+1+4}} = \frac{5}{\sqrt{180}}$$

$$\theta = \sin^{-1}\frac{5}{\sqrt{180}}$$

$$= 21.9^\circ \quad (1\text{ d.p.})$$

The plane p has equation $x = 3y$. Find the sine of the acute angle between the plane p and the y-axis.

Solution

$$p:\ x = 3y \ \Rightarrow\ x - 3y = 0 \ \Rightarrow\ \mathbf{r} \cdot \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = 0$$

Equation of the y-axis: $\mathbf{r} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$

Let θ be the acute angle between plane p and the y-axis.

$$\sin\theta = \frac{\left| \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|}{\sqrt{1+9}\sqrt{1}} = \frac{|0-3+0|}{\sqrt{10}} = \frac{3}{\sqrt{10}}$$

$$\therefore\ \sin\theta = \frac{3}{\sqrt{10}}$$

2 - l Lies On Π

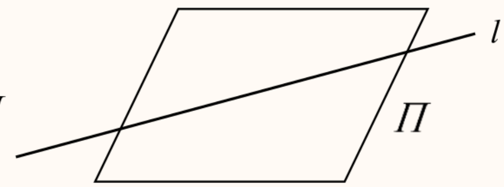
Case I: $\boldsymbol{l}$ lies in Π

l lies in Π

$\Leftrightarrow$ **every** point on line l is a point on plane Π

$\Leftrightarrow$ $(\mathbf{a}+\lambda\mathbf{d})\cdot\mathbf{n}=D$ for **all** values of λ

$\Leftrightarrow$ $\mathbf{d}\cdot\mathbf{n}=0$ and $\mathbf{a}\cdot\mathbf{n}=D$



The plane Π has equation $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}-2\mathbf{k})=4$. The lines l_1 and l_2 have equations $\mathbf{r}=2\mathbf{i}-2\mathbf{j}-4\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}-\mathbf{k})$ and $\mathbf{r}=-2\mathbf{i}+6\mathbf{j}+\mu(5\mathbf{i}-4\mathbf{j}+\mathbf{k})$ respectively.

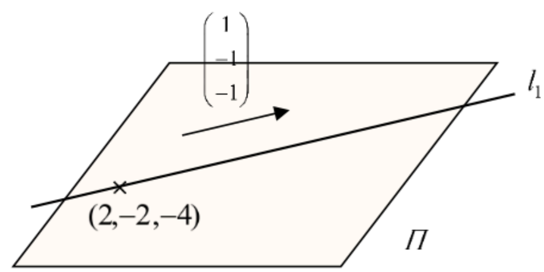
(i) Show that l_1 lies in Π.

(ii) Find the position vector of the point where l_2 meets Π.

Solution

(i) Plane Π: $\mathbf{r}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}=4$ ---- (1)

Line l_1: $\mathbf{r}=\begin{pmatrix}2\\-2\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ ---- (2)



Substitute equation (2) into the LHS of (1),

$$\left[\begin{pmatrix}2\\-2\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$=\begin{pmatrix}2\\-2\\-4\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$=(2-6+8)+\lambda(1-3+2)$$

$=4$, for all values of λ.

$\therefore l_1$ lies in Π.

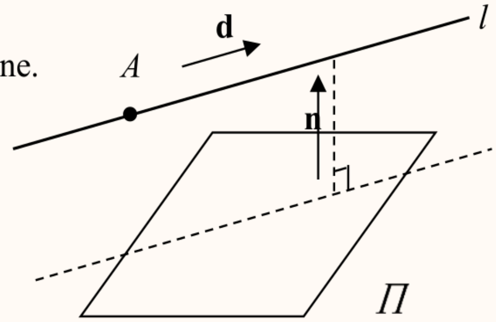
Note:
If l_1 lies in Π, the equation of l_1 would satisfy that of Π.
Thus we substitute equation (2) into the LHS of (1) and show that it is equal to the RHS of (1).

3 - l // Π But Does Not Lie On Π

Case II: l is parallel to Π <u>but</u> does not lie in Π

Check that the line is parallel to the plane.

Verify that the point A on the line is not on the plane.



l is parallel to Π <u>but</u> does not lie in Π

$\Leftrightarrow$ $\mathbf{d}\cdot\mathbf{n}=0$ <u>and</u> $\mathbf{a}\cdot\mathbf{n}\neq D$

The plane Π has equation $\mathbf{r}\cdot(3\mathbf{i}-3\mathbf{j}+4\mathbf{k})=3$ and the line l has equation $\mathbf{r}=\mathbf{i}-3\mathbf{j}+5\mathbf{k}+s(\mathbf{i}-3\mathbf{j}-3\mathbf{k})$.

(a) Show that l is parallel to Π but does not lie in Π.

(b) Find a vector equation of the plane that is parallel to Π and contains line l.

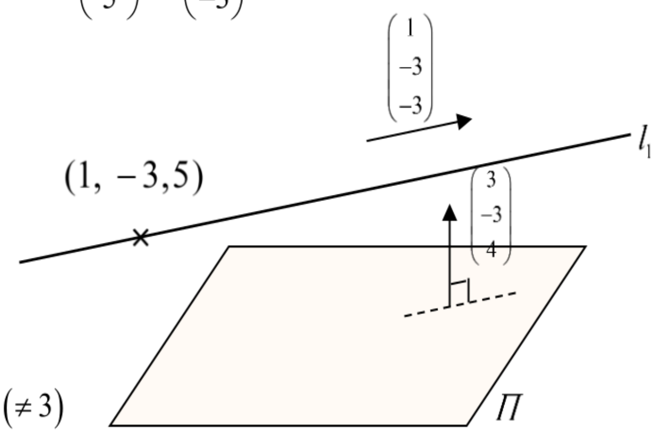
<u>Solution</u>

(a) Plane Π : $\mathbf{r}\cdot\begin{pmatrix}3\\-3\\4\end{pmatrix}=3$; Line l : $\mathbf{r}=\begin{pmatrix}1\\-3\\5\end{pmatrix}+s\begin{pmatrix}1\\-3\\-3\end{pmatrix}$, $s\in\mathbb{R}$

$$\mathbf{d}\cdot\mathbf{n}=\begin{pmatrix}1\\-3\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\-3\\4\end{pmatrix}=3+9-12=0$$

$\Rightarrow$ $\mathbf{d}\perp\mathbf{n}$.

$\therefore$ l is parallel to Π



$$\text{However, }\begin{pmatrix}1\\-3\\5\end{pmatrix}\cdot\begin{pmatrix}3\\-3\\4\end{pmatrix}=3+9+20=32\ (\neq 3)$$

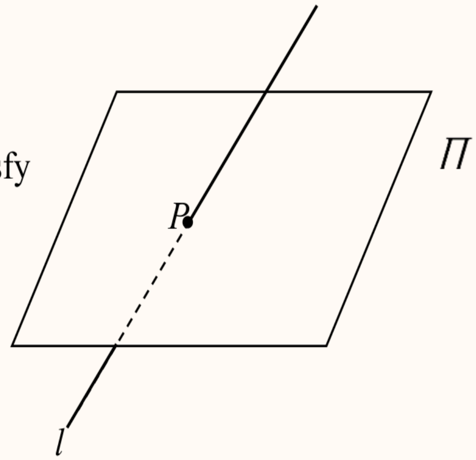
$\therefore$ the fixed point $(1, -3, 5)$ on l does not lie on Π.

Thus l is parallel to Π but does not lie in Π.

4 - l Intersects Π

Case III: l intersects Π at a point

If line l **intersects** plane Π at a point, then the position vector of the point of intersection must satisfy both the equation of l and equation of Π .



l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ ------ (1)

Π : $\mathbf{r}.\mathbf{n} = D$ ------ (2)

Substitute (1) into (2) :

Therefore $(\mathbf{a} + \lambda\mathbf{d})\cdot\mathbf{n} = D$ for **a particular** value of λ

The line l passes through the points A and B with coordinates $(1, 2, -3)$ and $(5, -4, -13)$ respectively. The plane p has equation $\mathbf{r} = \mathbf{j} + \mathbf{k} + s\,(2\mathbf{i} - 4\mathbf{k}) + t\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. Find the coordinates of the point of intersection of l and p.

Plane p: $\mathbf{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} + s\begin{pmatrix}2\\0\\-4\end{pmatrix} + t\begin{pmatrix}1\\2\\1\end{pmatrix}$ --- (1)

Line: $\mathbf{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-5\end{pmatrix}$ --- (2)

At the point of intersection,

$$\begin{pmatrix}1\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-5\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} + s\begin{pmatrix}2\\0\\-4\end{pmatrix} + t\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$1 + 2\lambda = 2s + t \quad \Rightarrow 2\lambda - 2s - t = -1$ ---(1)

$2 - 3\lambda = 1 + 2t \quad \Rightarrow 3\lambda + 2t = 1$ ---(2)

$-3 - 5\lambda = 1 - 4s + t \quad \Rightarrow 5\lambda - 4s + t = -4$ ---(3)

Using GC to solve the above system of linear equations:

$\lambda = 1,\ s = 2,\ t = -1$

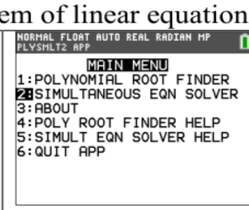
Position vector of the point of intersection

$$= \begin{pmatrix}1\\2\\-3\end{pmatrix} + \begin{pmatrix}2\\-3\\-5\end{pmatrix} = \begin{pmatrix}3\\-1\\-8\end{pmatrix}$$

$\therefore$ the **coordinates** of the point of intersection are $(3, -1, -8)$.

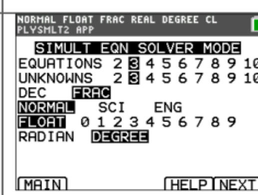
GC steps to solve system of linear equations:

Step 1:
- Press APPS and select **PlySmlt2**.
- Press any key to go to the next screen.
- Select **2: SIMULTANEOUS EQN SOLVER**

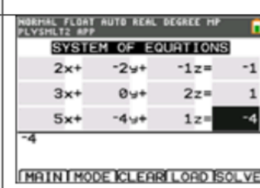


Step 2:
In this example, we have **3 equations with 3 unknowns**. Select the values as shown in the screenshot.

Go to **NEXT** page by pressing the button GRAPH which is below "**NEXT**".



Step 3:
Key in the coefficients and the operation signs of the equations.



Step 4:
Obtain the solutions by pressing the button GRAPH which is below "SOLVE".

B - Point & Plane

1 - Foot Of ⊥

To find $\overrightarrow{OF}$:

1. Find the equation of the line passing through P that is perpendicular to Π (i.e. parallel to $\mathbf{n}$).

$$\text{Line } PF: \quad \mathbf{r} = \mathbf{p} + \lambda\mathbf{n} \quad \text{----- (1)}$$

2. Find the point of intersection of this line and the plane Π with equation

$$\mathbf{r} \cdot \mathbf{n} = D \quad \text{-------- (2)}$$

(Substitute (1) into (2) and solve for the value of λ, then substitute λ back into (1) to find $\overrightarrow{OF}$.)

Example 11:

The plane Π has equation $-2x + y + z = 4$ and the point P has position vector $2\mathbf{j} + 5\mathbf{k}$.

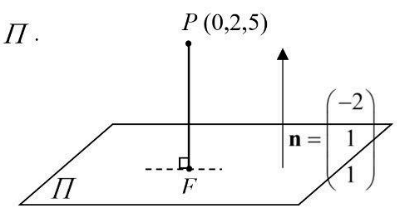
(i) Find the position vector of the foot of the perpendicular from P to Π.

(ii) Hence find the exact distance from P to Π.

(iii) Given that point Q is the point of reflection of P in Π, find the coordinates of Q.

Solution

(i) Let F be the foot of perpendicular from P to Π.



Line PF: $\mathbf{r} = \begin{pmatrix}0\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}$, $\lambda \in \mathbb{R}$ --- (1)

Plane Π : $\mathbf{r} \cdot \begin{pmatrix}-2\\1\\1\end{pmatrix} = 4$ --- (2)

Since F is the point of intersection of line PF and plane Π, substitute (1) into (2):

$$\left[\begin{pmatrix}0\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}\right] \cdot \begin{pmatrix}-2\\1\\1\end{pmatrix} = 4$$

$$(0+2+5) + \lambda(4+1+1) = 4$$

$$\lambda = -\frac{1}{2}$$

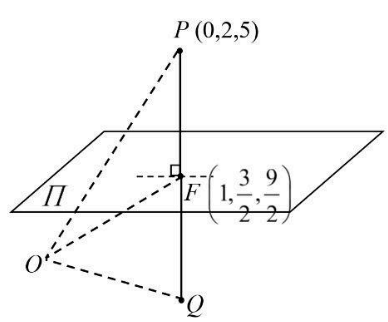
$$\therefore \; \overrightarrow{OF} = \begin{pmatrix}0\\2\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-2\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\3\\9\end{pmatrix} \quad \textbf{------- (*)}$$

Note: From equation (*), $\frac{1}{2}\begin{pmatrix}2\\3\\9\end{pmatrix} - \begin{pmatrix}0\\2\\5\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}-2\\1\\1\end{pmatrix}$

(ii) Shortest distance from P to Π $= \left|\overrightarrow{PF}\right|$

$$= \left|\overrightarrow{OF} - \overrightarrow{OP}\right|$$

$$= \left|\frac{1}{2}\begin{pmatrix}-2\\1\\1\end{pmatrix}\right| = \frac{1}{2}\sqrt{6}$$



(iii) Since F is the midpoint of PQ,

using ratio theorem, $\overrightarrow{OF} = \dfrac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$

$$\Rightarrow \; \overrightarrow{OQ} = 2\overrightarrow{OF} - \overrightarrow{OP}$$

$$= \begin{pmatrix}2\\3\\ \end{pmatrix} - \begin{pmatrix}0\\2\\ \end{pmatrix} = \begin{pmatrix}2\\1\\ \end{pmatrix}$$

$\therefore$ the coordinates of Q are $(2,1,4)$.

2 - ⊥ Distance From Point To Plane

- $\frac{\left|(\vec{OP}\cdot n)-D\right|}{|n|}$

The equation of a plane Π is $3x-4y+z+3=0$ and the point A has coordinates (2, 1, 0). Find the exact length of the perpendicular from A to Π.

$$\text{Length of perpendicular from } A \text{ to } \Pi = \frac{\left|\vec{OA}\cdot\mathbf{n}-D\right|}{|\mathbf{n}|}$$

$$= \frac{\left|\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\1\end{pmatrix}-(-3)\right|}{\sqrt{3^2+4^2+1^2}}$$

$$= \frac{|6-4+0+3|}{\sqrt{26}}$$

$$= \frac{5}{\sqrt{26}}$$

3 - ⊥ Distance From Origin To Plane

- $\frac{|D|}{|n|}$

Find the distance from the origin to the plane Π with equation $3x-4y+z=-3$.

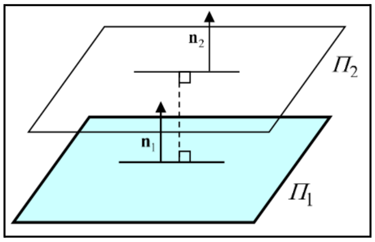
Solution

$$\text{Distance from the origin to the plane } \Pi = \frac{|D|}{|\mathbf{n}|} = \frac{|-3|}{\left|\begin{pmatrix}3\\-4\\1\end{pmatrix}\right|} = \frac{3}{\sqrt{26}}$$

C - Plane & Plane

1 - Parallel Planes

- ★ $n_1 = kn_2$



(a) $\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 1$ and $\Pi_2 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = 3$ are parallel planes.

(b) $\Pi_3 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 2$ and $\Pi_4 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = 4$ are the same plane.

- ★ **Either**
  - ➢ They **do not intersect**
  - ➢ They **coincide** as a single plane

- ★ Distance = $\frac{|D_2 - D_1|}{|n|}$

The plane π_1 has equation $\mathbf{r}\cdot\begin{pmatrix}7\\2\\-3\end{pmatrix}=-4$. The plane π_2 passes through (2, 4, 5) and is parallel to π_1.

(i) Find the equation of the plane π_2.

(ii) Find the perpendicular distances of the origin O from π_1 and from π_2. Hence deduce the distance between π_1 and π_2.

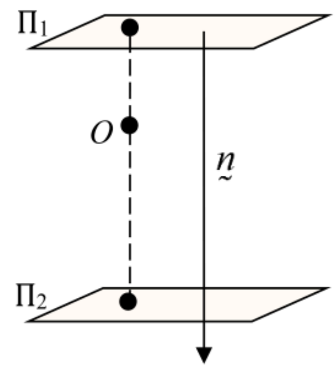
<u>Solution</u>

(i) Since the plane π_2 passes through (2, 4, 5) and is parallel to π_1,

$$\text{Equation of } \pi_2: \underset{\sim}{r}\cdot\begin{pmatrix}7\\2\\-3\end{pmatrix}=\begin{pmatrix}7\\2\\-3\end{pmatrix}\cdot\begin{pmatrix}2\\4\\5\end{pmatrix}$$
$$=7$$

(ii)

$$\text{Perpendicular distance of } O \text{ from } \pi_1=\frac{|-4|}{\left|\begin{pmatrix}7\\2\\-3\end{pmatrix}\right|}=\frac{4}{\sqrt{62}}$$

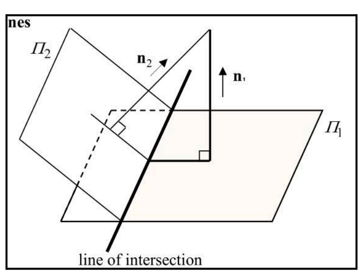


$$\text{Perpendicular distance of } O \text{ from } \pi_2=\frac{|7|}{\left|\begin{pmatrix}7\\2\\-3\end{pmatrix}\right|}=\frac{7}{\sqrt{62}}$$

$$\text{Distance between } \pi_1 \text{ and } \pi_2=\frac{7}{\sqrt{62}}+\frac{4}{\sqrt{62}}=\frac{11}{\sqrt{62}}$$

2 - Non-Parallel Planes

★ $n_1 \neq kn_2$



★ $cos\,\theta = \frac{|n_1\cdot n_2|}{|n_1||n_2|}$

Find the acute angle between the planes $\mathbf{r}=3\mathbf{i}+2\mathbf{j}-\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j})+\mu(3\mathbf{i}-7\mathbf{j}-\mathbf{k})$ and $x+2y+z=6$.

<u>Solution</u>

$$\Pi_1 : \mathbf{r}=\begin{pmatrix}3\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}3\\-7\\-1\end{pmatrix} \quad \text{and} \quad \Pi_2 : \mathbf{r}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=6$$

A normal vector to $\Pi_1 = \begin{pmatrix}1\\-2\\0\end{pmatrix}\times\begin{pmatrix}3\\-7\\-1\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$

Let θ be the acute angle between the two planes.

$$\cos\theta=\frac{\left|\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}\right|}{\sqrt{6}\,\sqrt{6}}=\frac{1}{2}$$

$\therefore\ \theta=60^\circ$

★ There is a **line of intersection** between them

- GC

Find a vector equation of the line of intersection l of the planes Π_1 and Π_2 with equations $\mathbf{r}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=6$ and $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=4$ respectively.

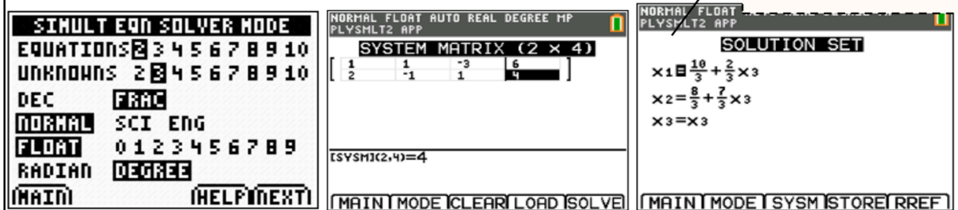
<u>Solution</u>

Method 1: Using GC (<u>APPS (PolySmlt2)</u>

$\Pi_1:\quad x+y-3z=6$ --- (1)

$\Pi_2:\quad 2x-y+z=4$ --- (2)

Thus, $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{10}{3}+\frac{2}{3}z\\ \frac{8}{3}+\frac{7}{3}z\\ z\end{pmatrix}=\begin{pmatrix}\frac{10}{3}\\ \frac{8}{3}\\ 0\end{pmatrix}+\frac{z}{3}\begin{pmatrix}2\\7\\3\end{pmatrix}$



A vector equation of the line of intersection l is $\underset{\sim}{r}=\frac{1}{3}\begin{pmatrix}10\\8\\0\end{pmatrix}+t\begin{pmatrix}2\\7\\3\end{pmatrix},\ t\in\mathbb{R}$.

- $r = \vec{OA} + \lambda(n_1 \times n_2)$
  - $\vec{OA}$ is the position vector of the 2 planes' **common point**

Find the vector equation of the line of intersection of the two planes

$$\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})=6 \quad\text{and}\quad \mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})=2.$$

<u>Solution</u>

A direction vector of the line, $\mathbf{d}=\begin{pmatrix}1\\1\\-1\end{pmatrix}\times\begin{pmatrix}2\\-3\\2\end{pmatrix}=\begin{pmatrix}-1\\-4\\-5\end{pmatrix}=-\begin{pmatrix}1\\4\\5\end{pmatrix}$

To find a common point on both planes:

Cartesian equation of π_1**:** $\quad x+y-z=6$ ------- (1)

Cartesian equation of π_2: $\quad 2x-3y+2z=2$ ------- (2)

Put $x=0$, $\quad$ (1) $\Rightarrow$ $\quad y-z=6$

$\qquad\qquad\quad$ (2) $\Rightarrow$ $\quad -3y+2z=2$

From GC, $\ y=-14$, $\ z=-20$

$\therefore$ The point A (0, –14, –20) lies on both planes.

Hence a vector equation of the line of intersection is $\mathbf{r}=\begin{pmatrix}0\\-14\\-20\end{pmatrix}+t\begin{pmatrix}1\\4\\5\end{pmatrix},\ t\in\mathbb{R}$.

A - Introduction

- $z = x + iy$
  - z is the complex number
  - x is the real portion $Re(z) = x$
  - y is the imaginary portion $Lm(z) = y$
  - i is the imaginary number $i = \sqrt{-1}$

- If $Re(Z) = 0 \Rightarrow z = iy \Rightarrow z$ is purely imaginary
- If $IM(Z) = 0 \Rightarrow z = x \Rightarrow z$ is purely real
  - Hence, the real numbers are a subset of the complex numbers

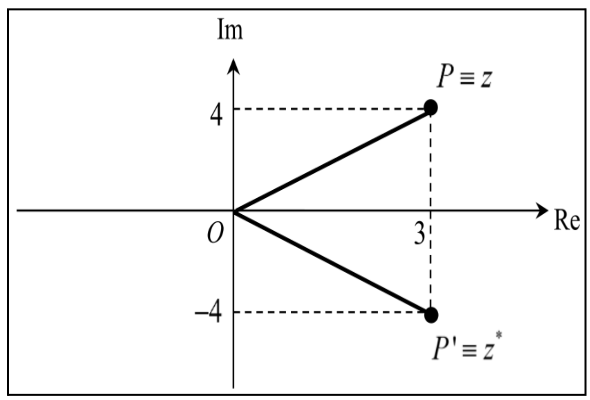
- $z = w \Leftrightarrow x_1 = x_2 ,\ y_1 = y_2$

B - Conjugate

1 - Definition

- The sign of the imaginary part is reversed
- Denoted using an asterisk *
  - $z = x + iy$
  - $z^* = x - iy$

2 - Properties



★ $zz^* = x^2 + y^2$
- $(z^*)^* = z$
- $z^* \pm w^* = (z \pm w)^*$
- $(zw)^* = z^* w^*$
- $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$, $w \neq 0$
- $z + z^* = 2\,Re(z)$
- $z - z^* = 2i\,Lm(z)$

★ $z^*\ (x, -y)$ is a **reflection** of $z\ (x, y)$ in the $x - axis$ in the argand diagram

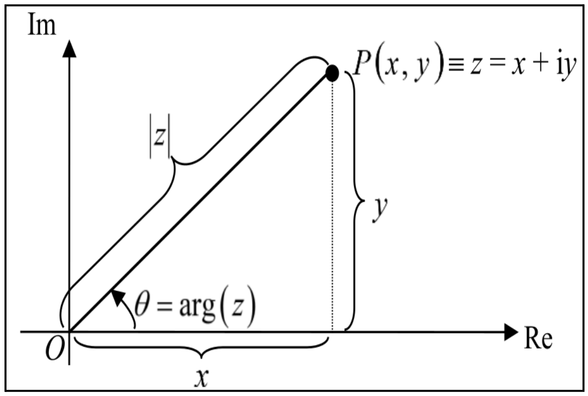
3 - Theorem

- If a polynomial with **only real coefficients** and degree ≥ 2 has non-real roots, it will appear in conjugate pairs
  - If $x + iy$ is a root of $P(z) = 0$, then $x - iy$ is also a root

C - Argand Diagram

1 - Definition

- It the the graph of the z , with the $x - axis$ being real and the $y - axis$ being imaginary



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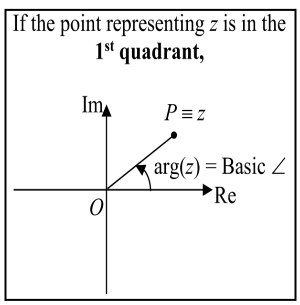
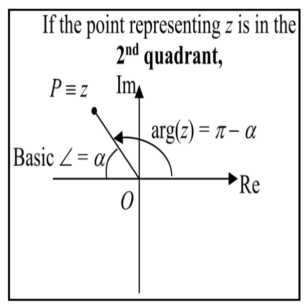
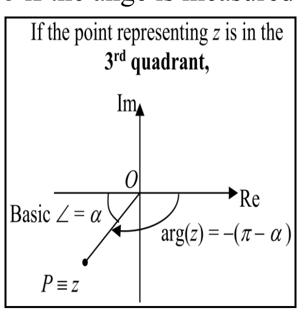
2 - Properties

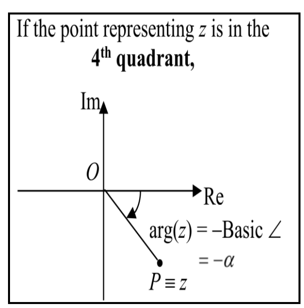
- (x, y) are the coordinates
- $OP = |z| = \sqrt{x^2 + y^2}$
- $arg(z)$ is the principal angle θ, $-\pi < \theta \leq \pi$ that OP makes with the $+ve$ real axis
  - $arg(z) \geq 0$ if the angle is measured **anti-clockwise** from the $+ve$ real axis





  - $arg(z) < 0$ if the ange is measured **clockwise** from the $+ve$ real axis





★ $\frac{\pi}{2}$ / $quadrant$

★ Basic angle $\alpha = tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

D - Arithmetic Operations

$z_1 = x_1 + iy_1$

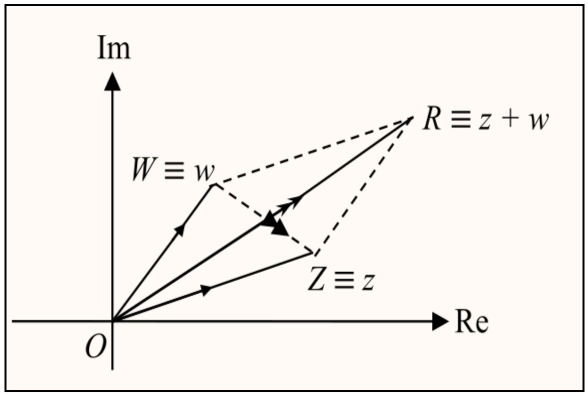
$z_2 = x_2 + iy_2$

---

1 - Addition / Subtraction

$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

- Use the **parallelogram law** to visualise on the argand diagram



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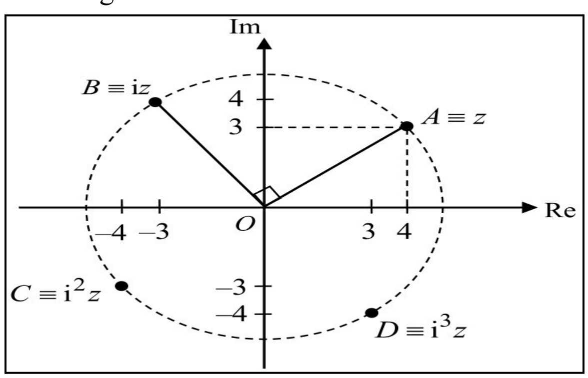
2 - Multiplication

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

- Multiplication of a complex number by i rotates the vector on the argand diagram 90° **anti-clockwise** about the origin



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3 - Division

$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$

$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$

$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$

- Multiply the denominator by its conjugate so as to remove the imaginary part

A - Differentiation

1 - Fundamental Rules

Scalar Multiple	$\frac{d}{dx}(ku) = k\frac{d}{dx}(u)$
Addition & Subtraction Rule	$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(v) \pm \frac{d}{dx}(u)$
Chain Rule	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
Product Rule	$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
Quotient Rule	$\frac{d}{dx}(\frac{u}{v}) = \frac{v(\frac{du}{dx})-u(\frac{dv}{dx})}{v^2}$

2 - Standard Results

A - Algebraic Functions

$\frac{d}{dx}(a) = 0$, *where a is a constant*

$\frac{d}{dx}[f(x)]^n = n \times \{f'(x)[f(x)^{n-1}]\}$, $n \neq 0$

B - Trigonometric Functions

$\frac{d}{dx} sin[f(x)] \quad = f'(x) \times cos[f(x)]$

$\frac{d}{dx} cos[f(x)] \quad = f'(x) \times \{-\ sin[f(x)]\}$

$\frac{d}{dx} tan[f(x) \quad = f'(x) \times sec^2[f(x)]$

$\frac{d}{dx} cot[f(x)] \quad = f'(x) \times \{-\ cosec^2[f(x)]\}$

$\frac{d}{dx} sec[f(x)] \quad = f'(x) \times sec[f(x)]tan[f(x)]$ **[MF27]**

$\frac{d}{dx} cosec[f(x)] = f'(x) \times \{-\ cosec[f(x)]cot[f(x)]$ **[MF27]**

C - Inverse Trigonometric Functions [MF27]

$\frac{d}{dx}(sin^{-1}[f(x)]) = f'(x) \times \frac{1}{\sqrt{1-[f(x)]^2}}$, $-1 < f(x) < 1$

$\frac{d}{dx}(cos^{-1}[f(x)] = f'(x) \times \{-\frac{1}{\sqrt{1-[f(x)]^2}}\}$, $-1 < f(x) < 1$

$\frac{d}{dx}(tan^{-1}[f(x)] = f'(x) \times \frac{1}{1+[f(x)]^2}$

D - Exponential Functions

$\frac{d}{dx} e^{f(x)} = f'(x) \times e^{f(x)}$

E - Logarithmic Functions

$\frac{d}{dx}[ln(f(x)] = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(log_a f(x)) = \frac{d}{dx}(\frac{ln(f(x))}{ln\ a}) \Rightarrow \frac{1}{ln\ a}(\frac{f'(x)}{f(x)})$, $a > 0$

3 - Techniques
A - Explicit Differentiation
1 - Definition

- It is a way of differentiating when y is isolated on the LHS, and hence, it is the subject

---

2 - Method

1. Make y the subject, if it is not isolated yet
2. Differentiate both sides
3. Simplify

---

3 - Example

0. Find the derivative of the following

$$2y = x^3 + 6x^2 + 11x + 6 + y$$

1. Make y the subject, if it is not isolated yet

$$y = x^3 + 6x^2 + 11x + 6$$

2. Differentiate both sides

$$\frac{dy}{dx} = 3x^2 + 12x + 11$$

3. Simplify, if needed

$$\frac{dy}{dx} = 3x^2 + 12x + 11$$

B - Implicit Differentiation

1 - Definition

- It is a way of differentiating an equation when y is mixed with other variables, such as x, and hence, is not isolated on the LHS

- Implicit differentiation is done if making y the subject makes explicit differentiation difficult or impossible to do

---

2 - Method

1. To do this, simply differentiate y normally, then multiply by $\frac{dy}{dx}$
    - $\frac{d}{dx}g(y) = \left[\frac{d}{dy}g(y)\right] \times \frac{dy}{dx}$
2. Make $\frac{dy}{dx}$ the subject
3. Simplify

---

3 - Example

0. Solve the following equation implicitly

$$x^2 + y^2 = 25$$

1. Differentiate both sides

$$2x + 2y\frac{dy}{dx} = 0$$

2. Make $\frac{dy}{dx}$ the subject

$$\frac{dy}{dx} = -\frac{x}{y}$$

3. Simplify

$$\frac{dy}{dx} = -\frac{x}{y}$$

C - Logarithmic Differentiation

1 - Definition

- It is a way of differentiating if the base is not e

---

2 - Method

1. Equate to y
2. ln both sides
3. Differentiate
4. Make $\frac{dy}{dx}$ the subject
5. Simplify

---

3 - Example

0. Find the derivative of the following

$$2^x$$

1. Equate to y

$$y = 2^x$$

2. ln both sides

$$ln\, y = ln(2^x)$$
$$ln\, y = x\, ln(2)$$

3. Differentiate

$$\frac{1}{y} \times \frac{dy}{dx} = ln2$$

4. Make $\frac{dy}{dx}$ the subject

$$\frac{dy}{dx} = y\, ln2$$

5. Simplify

$$\frac{dy}{dx} = (2^x)(ln2)$$

D - Parametric Differentiation

1 - Definition

- It is a way of differentiating parametric equations by using the **chain rule**

---

2 - Method

1. Differentiate both equations
2. Use the chain rule
3. Simplify

---

3 - Example

0. Given that $x = acos\theta$, $y = bsin\theta$, $a, b \in Z^{+}$, find $\frac{dy}{dx}$

1. Differentiate both equations

$$x = acos\Theta \qquad\qquad y = bsin\theta$$

$$\frac{dx}{d\Theta} =- asin\Theta \qquad\qquad \frac{dy}{d\theta} = bcos\theta$$

2. Use the chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = bcos\Theta \times \frac{1}{-asin\Theta}$$

3. Simplify

$$\frac{dy}{dx} = bcos\Theta \times \frac{1}{-asin\Theta}$$

$$\frac{dy}{dx} = \frac{bcos\Theta}{-asin\Theta}$$

$$\frac{dy}{dx} = \frac{b}{-a} \times \frac{cos\Theta}{sin\Theta}$$

$$\frac{dy}{dx} =- \frac{b}{a}cot\Theta$$

E - MacLaurin Series

A - Definition

- It is a way of expressing a function as the infinite polynomial sum whereby the derivatives, when evaluated at $x = 0$, are the coefficients

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \quad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \quad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \quad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

- The graph's **tangent** is made up of the expansion's first 2 terms

---

B - Method

1. Find the derivatives up till the question's requirements
2. Evaluate them at $x = 0$
3. Plug the values into the appropriate formula

C - Example

0. Show that the MacLaurin Series for $secx$, up to and including the term in x^2, is $1 + \frac{1}{2}x^2$

1. Find the derivatives up till the question's requirements **[The 2nd term in this case]**

$$Let\ f(x) = secx$$
$$f'(x) = secxtanx$$
$$f''(x) = sec^3x + (tan^2x)(secx)$$

2. Evaluate them at $x = 0$

$$f(0) = sec(0) \Rightarrow 1$$
$$f'(0) = sec(0)tan(0) \Rightarrow 0$$
$$f''(0) = sec^3(0) + (tan^2(0))(sec((0)) \Rightarrow 1$$

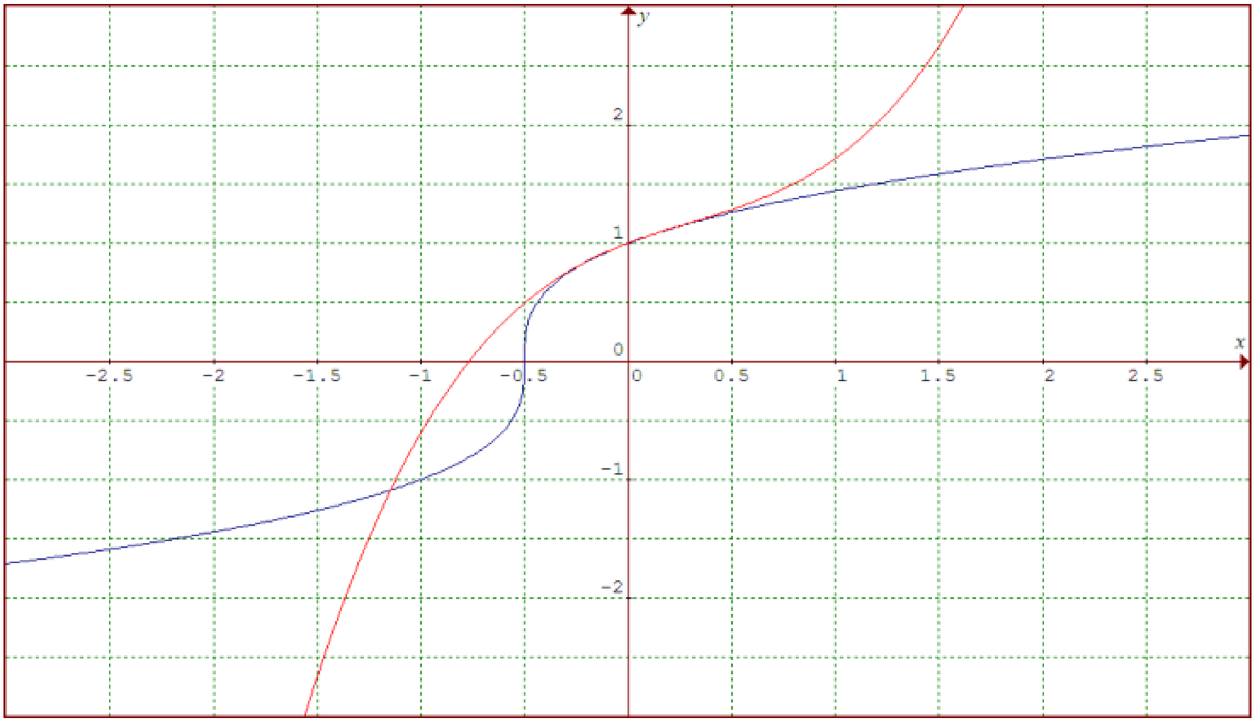
3. Insert the values into the appropriate formula

$$secx = 1 + (0)x + \frac{x^2}{2!}(1) + \ldots$$
$$secx = 1 + \frac{1}{2}x^2 + \ldots \text{ (shown)}$$

D - Use Cases

1 - General Approximation

- It's used to **approximate** a function's value
- When a function is approximated using its Maclaurin series, it gets more accurate for values of x that are closer to 0, as the series is centered at $x = 0$



Red : $y = 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 + ...$

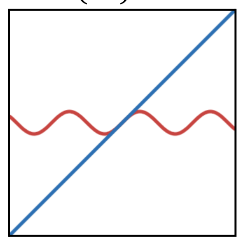
Blue : $y = (1 + 2x)^{\frac{1}{3}}$

The MacLaurin Series' graph is the **closest to the function's graph** for values of x that are the **closest to** 0. Hence, **making it more and more accurate**

2 - Small Angle Approximation

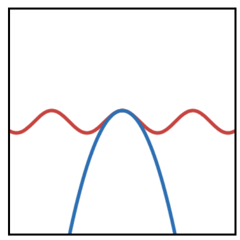
- If x $(radians)$ is small enough such that powers from x^3 onwards can be ignored, then

$$sin(kx) \approx kx$$



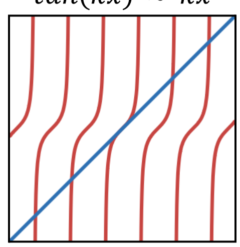
$y = kx$ gives a good approximation of $y = sin(kx)$ for small values of x

$$cos(kx) \approx 1 - \frac{(kx)^2}{2}$$



$y = 1 - (kx)^2$ gives a good approximation of $y = cos(kx)$ for small values of x

$$tan(kx) \approx kx$$



$y = kx$ gives a good approximation of $y = tan(kx)$ for small values of x

Given that x is sufficiently small for x^3 and higher powers of x to be neglected, and that $\cos x - 4\sin x = 6x$, show that an approximate quadratic equation for x is $x^2 + 20x - 2 = 0$. Hence find an approximate value for x, giving your answer correct to 3 significant figures.

Solution:

Since x is sufficiently small for x^3 and higher powers of x to be neglected,
$\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$.

Hence,

$$\cos x - 4\sin x = 6x$$

$$\left(1 - \frac{1}{2}x^2\right) - 4x \approx 6x$$

$$1 - \frac{1}{2}x^2 - 10x \approx 0$$

$x^2 + 20x - 2 = 0$ is the approximate quadratic equation (Shown)

By using GC, $x \approx 0.0995$ or $x \approx -20.1$ (NA since – 20.1 is not small.)

4 - Applications

A - Finding the tangent & normal

1 - Explicitly

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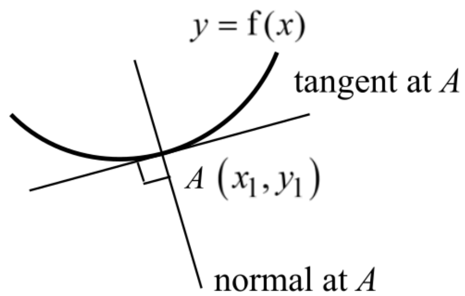
<tr>
<td>
</td>
<td>The gradient of tangent to the curve $y = \mathrm{f}(x)$ at point A = $m = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_1}$

The equation of <u>tangent</u> to the curve at point A is
$$y - y_1 = m(x - x_1)$$
The equation of <u>normal</u> to the curve at point A is
$$y - y_1 = -\frac{1}{m}(x - x_1).$$</td>
</tr>
</table>

Video 8.1

Example 2: (Cartesian Equation, A-level) [J92/P1/Q15a]

The line l is the tangent to the curve $y=1-\frac{1}{x}$, where $x \neq 0$, at the point where $x=a$. Show that the equation of l may be expressed in the form $a^2y-x=a^2-2a$. Find the value of a for which the line l passes through the origin, and deduce the equation of l in this case.

<u>Solution</u>

$$y=1-\frac{1}{x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x^2}$$

At the point where $x=a$,

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{a^2}$ and $y=1-\frac{1}{a}$.

Recall that equation of tangent at point $(x_1\ ,\ y_1)$ is $y-y_1=m(x-x_1)$.
In this question, $x_1=a$, $y_1=1-\frac{1}{a}$ and $m=\frac{1}{a^2}$.

Equation of l : the tangent to the curve at $x=a$ is

$$y-\left(1-\frac{1}{a}\right)=\frac{1}{a^2}(x-a)$$

$$a^2y-a^2\left(1-\frac{1}{a}\right)=x-a$$

$$a^2y-a^2+a=x-a$$

$$\therefore\ a^2y-x=a^2-2a \quad \text{(shown)}$$

Since l passes through (0, 0), we substitute $x=0$ and $y=0$ into the equation. Thus,

$$0=a^2-2a$$

$$a(a-2)=0$$

$$a=0 \text{ or } a=2$$

$a=0 \Rightarrow x=0$. But it was given that $x \neq 0$. Thus reject $a=0$.

$\therefore\ a=2$

Thus equation of l is : $2^2y-x=2^2-2(2)$

i.e., $4y-x=0$

2 - Implicitly

1. Use implicit differentiation to find $\frac{dy}{dx}$
2. Substitute $x = x_1$ and $y = y_1$ into $\frac{dy}{dx}$ to find the gradient

Example 3: (The curve is defined implicitly.)
The equation of a curve is $x^2y + xy^2 = 12$. Find
(i) the equation of the normal at the point (1, 3), and
(ii) the equation of the tangent which is parallel to the y-axis.

Solution

$$x^2y + xy^2 = 12 \quad \cdots\cdots(1)$$

(i) Differentiate both sides of equation (1) w.r.t x,

$$x^2\frac{dy}{dx} + (2x)y + x\left(2y\frac{dy}{dx}\right) + (1)y^2 = 0$$

Implicit differentiation using product rule.

$$\left(x^2 + 2xy\right)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}$$

At the point (1, 3), $\frac{dy}{dx} = \frac{-3(2\times1+3)}{(1)(1+2\times3)} = -\frac{15}{7}$

Gradient of normal $= -\frac{1}{\left(-\frac{15}{7}\right)} = \frac{7}{15}$

Equation of normal at (1, 3) is $y - 3 = \frac{7}{15}(x-1)$

$$y = \frac{7}{15}x + \frac{38}{15}$$

(ii) If tangent is **<u>parallel to the y-axis</u>**, then gradient is undefined,
i.e., the denominator of $\frac{dy}{dx}$ is 0: $x(x+2y) = 0$

$$x = 0 \quad \text{or} \quad x = -2y.$$

Think: Why is the denominator 0 when the tangent is // to the y-axis?

What will happen if the tangent is parallel to the x-axis?

If $x = 0$, using equation (1) where $x^2y + xy^2 = 12$,
L.H.S. $= x^2y + xy^2 = 0$ but R.H.S = 12.
This is impossible and so we reject $x = 0$. Thus,
$x = -2y$.
Substitute $x = -2y$ into equation (1) to get

$$(-2y)^2y + (-2y)y^2 = 12$$

$$2y^3 = 12 \Rightarrow y = \sqrt[3]{6}$$

At this particular point, $x = -2y$. Thus $x = -2\left(\sqrt[3]{6}\right)$.

hus, the point on the curve with coordinates $\left(-2\left(\sqrt[3]{6}\right), \sqrt[3]{6}\right)$ has a vertical tangent.

herefore, the equation of tangent which is parallel to the y-axis is $x = -2\left(\sqrt[3]{6}\right)$.

Note that equation of a vertical line is in the form $x = k$. Since the vertical tangent passes hrough the point $\left(-2\left(\sqrt[3]{6}\right), \sqrt[3]{6}\right)$, thus $k = -2\left(\sqrt[3]{6}\right)$.

3 - Parametrically

1. Find $\frac{dx}{dt}$ and then inverse it to find $\frac{dt}{dx}$
2. Find $\frac{dy}{dt}$
3. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dx}{dt}$
4. Substitute $t = t_1$ into $\frac{dy}{dx}$ to find the gradient

Example 4: (Parametric Equations) [N2003/P1/Q12]

8.3

Sketch the curve with parametric equations

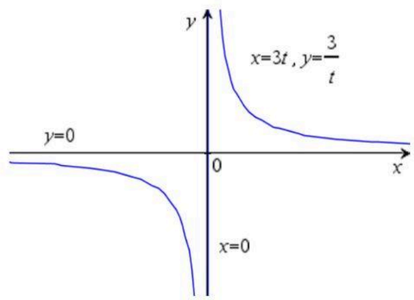
$$x = 3t, \quad y = \frac{3}{t}, \text{ where } t \neq 0$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q.

(i) Show that the normal at P has equation $2y = 8x - 45$.

(ii) Find the value of t at Q.

Solution



(i) $x = 3t \ , \quad y = \frac{3}{t}$

$\frac{dx}{dt} = 3 \ , \quad \frac{dy}{dt} = -\frac{3}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{3}{t^2} \times \frac{1}{3} = -\frac{1}{t^2}$

At point P, $t = 2$, $\frac{dy}{dx} = -\frac{1}{4} \quad \Rightarrow$ gradient of normal = 4.

Also, when $t = 2$, $x = 6$ and $y = \frac{3}{2}$.

The equation of normal at P is $\quad y - \frac{3}{2} = 4(x - 6)$

$$2y - 3 = 8x - 48$$

$$2y = 8x - 45 \quad \text{(shown)}$$

(ii) At point Q, the normal meets the curve again.

Substitute $x = 3t$ and $y = \frac{3}{t}$ into the equation of normal:

$$2\left(\frac{3}{t}\right) = 8(3t) - 45$$

$$8t^2 - 15t - 2 = 0$$

$$(8t + 1)(t - 2) = 0$$

$$t = -\frac{1}{8} \quad \text{or} \quad t = 2$$

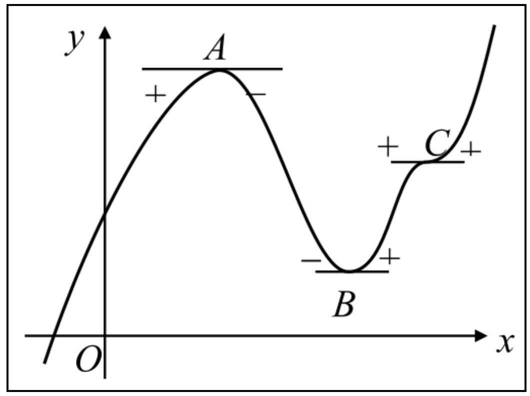
At Q, $\quad t = -\frac{1}{8} \qquad$ (ans)

At P, $t = 2$. Since Q is a different point from P, $t = -\frac{1}{8}$ at Q.

Recall that when finding the points of intersection of a line $y = mx + c$ and a curve $y = f(x)$, we simply equate the eqns to form ONE equation $mx + c = f(x)$ and solve for x.

Now, for parametric eqns, we use exactly the same idea. Substitute $x = f(t)$ and $y = g(t)$ into $y = mx + c$ to **form ONE equation** and solve for t.

B - Finding the stationary points



- $\frac{dy}{dx} = 0$ when at these points
- Types
    A. Maximum Point
        - $+\ ve \rightarrow 0 \rightarrow -\ ve$

    B. Minimum Point
        - $-\ ve \rightarrow 0 \rightarrow +\ ve$

    C. Stationary Point Of Inflexion
        - $+\ ve \rightarrow 0 \rightarrow +\ ve$
        - $-\ ve \rightarrow 0 \rightarrow -\ ve$
- To determine the nature of the stationary points, do the **first** derivative test

x	a^- a a^+	a^- a a^+	a^- a a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+$ 0 $-$	$-$ 0 $+$	$+$ 0 $+$ **or** $-$ 0 $-$
Slope			**or**
Nature of stationary point	maximum point	minimum point	stationary point of inflexion

- a^- : Values slightly **lesser** than a
- a^+ : Values slightly **greater** than a

By using differentiation, show that the curve with equation $y = x^4(x-2)$ has two stationary points and determine the nature of these points. Hence, sketch the graph of $y = x^4(x-2)$.

Solution

$$y = x^4(x-2) = x^5 - 2x^4$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4 - 8x^3 = x^3(5x-8)$$

At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$x^3(5x-8) = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{8}{5}$$

$\therefore$ There are two stationary points.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 20x^3 - 24x^2 = 4x^2(5x-6)$$

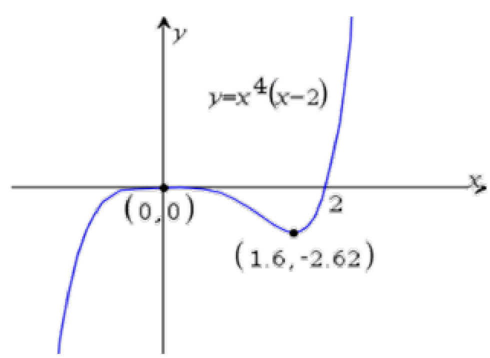
When $x = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. (No conclusion, so use first derivative test.)

x	-0.01	0	0.01
$\frac{\mathrm{d}y}{\mathrm{d}x}$	8.13×10^{-6}	0	-8.03×10^{-6}
Slope	/	—	\

$\therefore$ The point at $x = 0$ is a maximum point.

When $x = \frac{8}{5}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\left(\frac{8}{5}\right)^2(8-6) > 0$.

$\therefore$ The point at $x = \frac{8}{5}$ is a minimum point.



C - Finding The Maxima & Minima

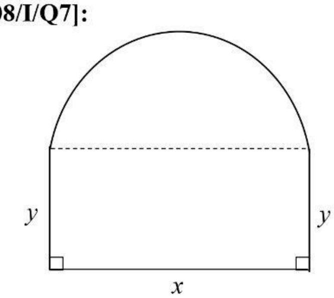
1. Use the restriction to obtain a relationship between 2 variables

2. Express the variable to be maximised/minimised in terms of 1 variable only, by using the relation from step 1

3. Differentiate $w.r.t$ to the expressed variable and equate the derivative to 0

4. Simplify and solve

Example 7 [N2008/I/Q7]:



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flower-bed.

The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area.

<u>Solution</u>

Let the area of the flower-bed be A m^2.

Total time = 180 h

$$\Rightarrow \quad 3(x+2y)+9\pi\left(\frac{x}{2}\right)=180$$

$$\Rightarrow \quad y=\frac{1}{6}\left(180-\frac{9\pi x}{2}-3x\right)=30-\frac{3\pi x}{4}-\frac{x}{2}$$

$$\begin{aligned} A &= xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \\ &= x\left(30-\frac{3\pi x}{4}-\frac{x}{2}\right)+\frac{1}{8}\pi x^2 \\ &= 30x-\left(\frac{1}{2}+\frac{5\pi}{8}\right)x^2 \end{aligned}$$

$$\frac{dA}{dx}=30-2\left(\frac{1}{2}+\frac{5\pi}{8}\right)x=30-\left(\frac{4+5\pi}{4}\right)x$$

For A to be maximum, $\frac{dA}{dx}=0.$

$$30-\left(\frac{4+5\pi}{4}\right)x=0$$

$$x=\frac{120}{4+5\pi}=6.0889$$

$$y=12.609$$

$$\frac{d^2A}{dx^2}=-\left(\frac{4+5\pi}{4}\right)<0$$

Thus the flower-bed has a maximum area when x = 6.09 and y = 12.6 (3 s.f.).

D - Finding The Connected Rates Of Change

- ★ It is **usually** taken with **respect to time** t unless otherwise stated
- ★ **Increasing** rates of change have a **positive** sign
- ★ **Decreasing** rates of change have a **negative** sign

---

1. **Identity** the variable in the question
   - ➢ Let r be the radius of the spherical balloon
   - ➢ Let v be the volume of the spherical balloon

2. **Create** a $D.E$ to **link** the identified variables
   - ➢ $V = \frac{4}{3}\pi r^3$

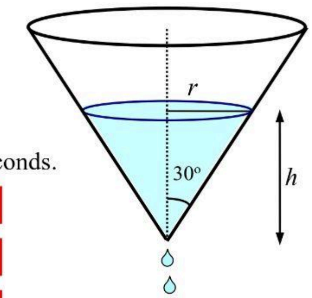
3. **Differentiate** both sides $w.r.t$ to the **appropriate** variable
   - ➢ $\frac{dV}{dr} = 4\pi r^2$

4. Apply the **chain rule**
   a. $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

- ★ Alternatively, you may **implicitly differentiate** the D.E from step 2 with respect to t

Video 8.8

Example 9:
An inverted right circular conical tank has a $30°$ semi-vertex angle. Water is leaking out of the tank from the vertex at a rate of 2 $cm^3\ s^{-1}$. Find the rate of decrease of the depth of water when the depth is 4 cm. Find also the rate at which the area of the water surface is decreasing at that instant.



Solution

Let V cm^3 be the volume of the water in the cone,
A cm^2 be the area of the water surface,
r cm be the radius of the water surface,
h cm be the depth of the water in the cone, at time t seconds.

$$V = \frac{1}{3}\pi r^2 h \quad \text{-------- (1)}$$

Note: V is expressed in terms of **two variables,** r & h. Need to eliminate one of them via substitution.
***Qn*:** Should we eliminate r or h?
Ans: Since we need to find $\frac{dh}{dt}$, we should eliminate r.

$$\tan 30^\circ = \frac{r}{h}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{h}$$

$$\therefore\ r = \frac{1}{\sqrt{3}}h \quad \text{------ (2)}$$

Sub. (2) into (1) : $V = \frac{1}{3}\pi\left(\frac{1}{\sqrt{3}}h\right)^2 h$

$$= \frac{1}{9}\pi h^3$$

Differentiating V w.r.t. h,

$$\frac{dV}{dh} = \frac{1}{3}\pi h^2$$

Using chain rule, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

Since $\frac{dV}{dt} = -2$, a constant rate,

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \times (-2) = -\frac{6}{\pi h^2}$$

When $h = 4$, $\frac{dh}{dt} = -\frac{6}{\pi(4)^2} = -\frac{3}{8\pi}$

Thus, when the depth is 4 cm, the rate of **decrease** of the depth of water is $\frac{3}{8\pi}$ cm s^{-1}.

Please take note that the "**rate of decrease**" is given as a **positive** value. You may also present this answer as 0.119 cm s^{-1}, correct to 3 s.f.

$$A = \pi r^2$$

$$= \pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{1}{3}\pi h^2$$

$$\Rightarrow \frac{dA}{dh} = \frac{2}{3}\pi h$$

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$$

$$= \left(\frac{2}{3}\pi h\right)\left(-\frac{6}{\pi h^2}\right)$$

$$= -\frac{4}{h}$$

When $h = 4$,

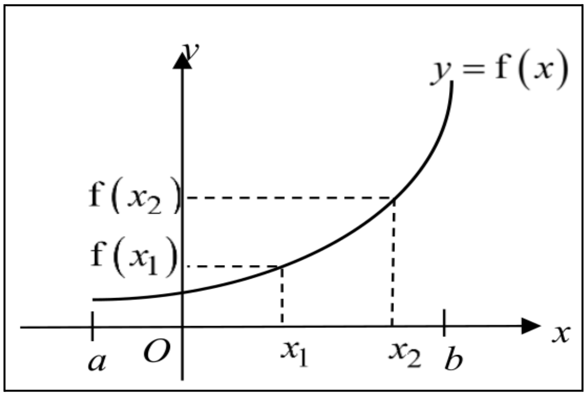
$$\frac{dA}{dt} = -\frac{4}{4} = -1$$

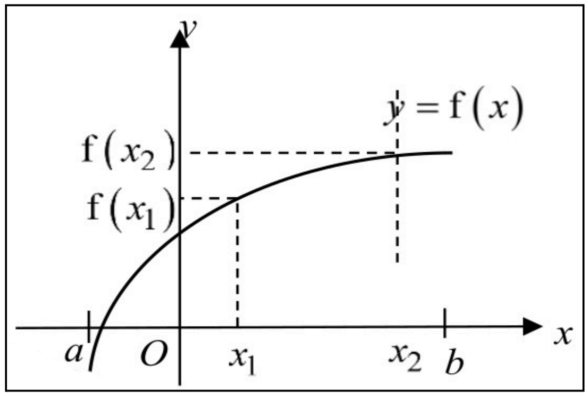
$\therefore$ Required rate of decrease of the area of water surface is 1 $cm^2\ s^{-1}$.

5 - Properties

A - Increasing Functions

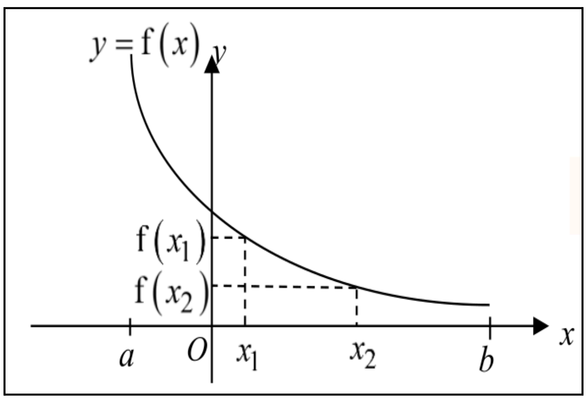
- ★ A function is said to be **increasing** on an *interval* if its **output values increase as the input values increase** within that interval
- ★ The **gradient** is **positive**

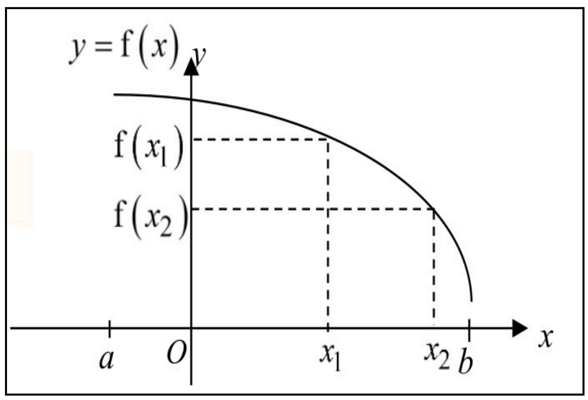




B - Decreasing Functions

- ★ A function is said to be **decreasing** on an *interval* if its **output values decrease as the input values increase** within that interval
- ★ The **gradient** is **negative**





The function f is defined by $f(x) = e^{-2x} + \frac{3}{x} + 3$ for $x > 0$.

(i) Find $f'(x)$.

(ii) Hence prove that f is a decreasing function.

Solution

(i) $$f(x) = e^{-2x} + \frac{3}{x} + 3$$

$$f'(x) = -2e^{-2x} - \frac{3}{x^2}$$

(ii) $$f'(x) = -\left(2e^{-2x} + \frac{3}{x^2}\right)$$

Since $2e^{-2x} > 0$ and $\frac{3}{x^2} > 0$ for all $x > 0$,

$\therefore \quad f'(x) < 0$

Hence f is a decreasing function for $x > 0$.

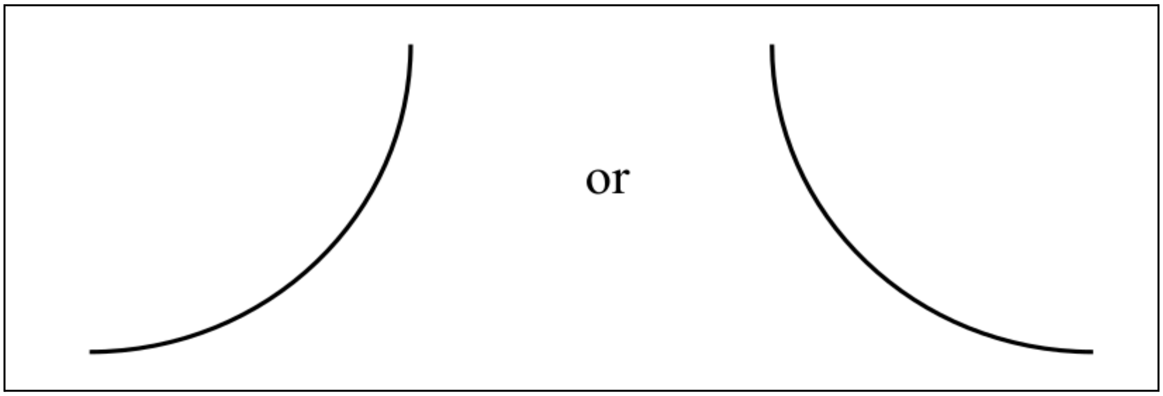
To prove that f is decreasing, we need to show that $f'(x) < 0$ in the domain given.

Likewise, to show that f is increasing, we show that $f'(x) > 0$.

C - Upward Concavity

★ $f''(x) > 0$

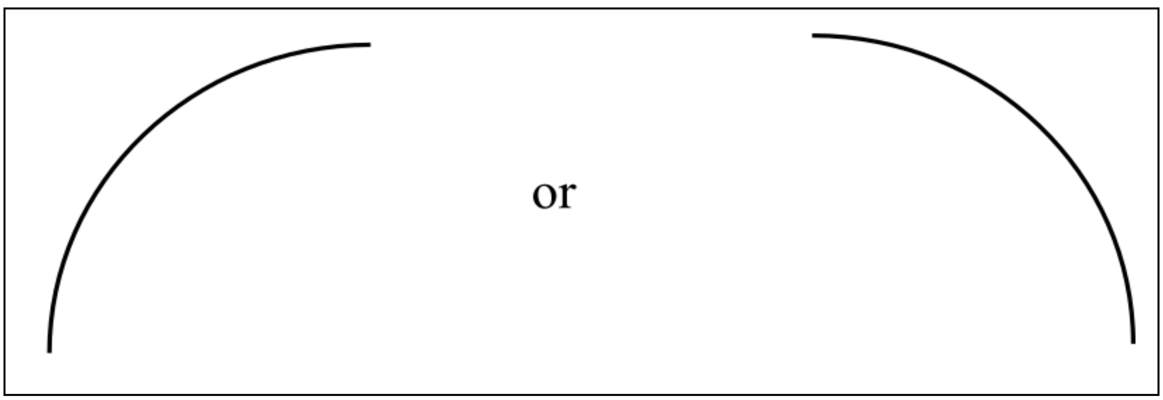
- The function's gradient is **increasing** as x **increases**
- Hence, the graph is getting **steeper**



D - Downward Concavity

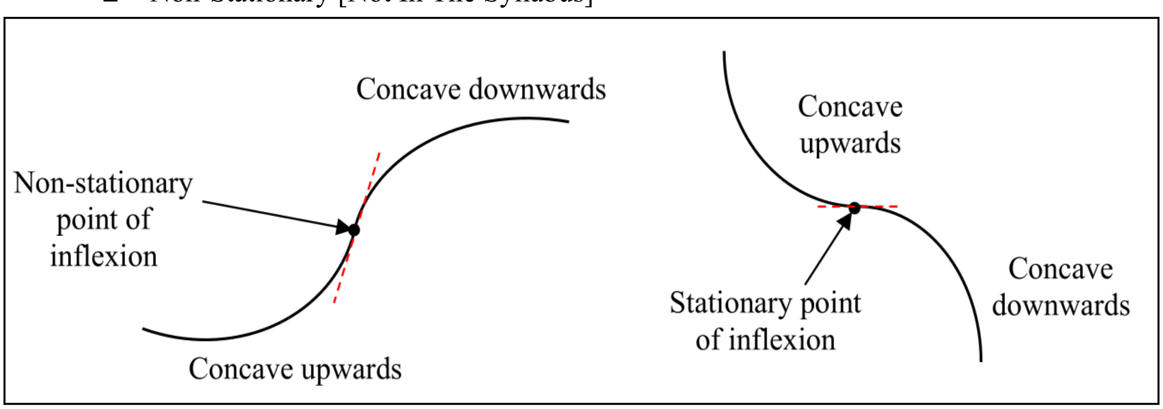
★ $f''(x) < 0$

- The function's gradient is **decreasing** as x **increases**
- Hence, the graph is getting **gentler**



E - Points Of Inflexion

- The point on the curve where the concavity changes
- It could either be
  - Stationary
  - Non-Stationary [Not In The Syllabus]

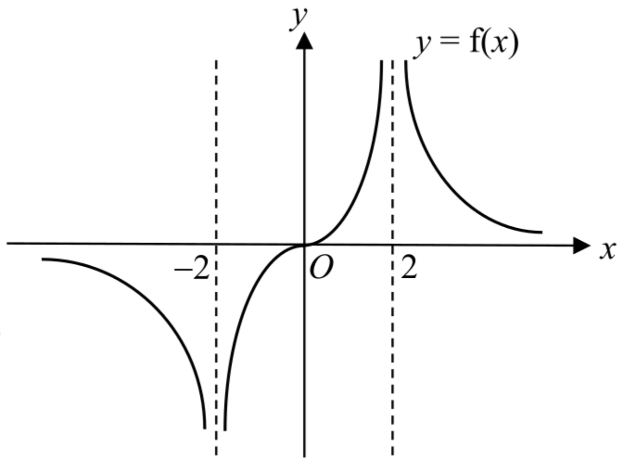


The diagram below shows the graph of $y = \mathrm{f}(x)$, and the origin O is a stationary point on the graph. Determine the set of values of x for which the graph

(i) is stationary,

(ii) is concave upwards,

(iii) is concave downwards.



Solution

(i) Graph is stationary $\Rightarrow \mathrm{f}'(x) = 0$,

Set of values of x $= \{0\}$

(ii) Graph is concave upwards

Set of values of x $= \{x \in \mathbb{R} : 0 < x < 2 \text{ or } x > 2\}$

(iii) Graph is concave downwards

Set of values of x $= \{x \in \mathbb{R} : x < -2 \text{ or } -2 < x < 0\}$

B - Integration

1 - Identity

$$\int f'(x)\,dx = f(x) + C$$

- It is the process of undoing a derivative to get back the original function
- Multiple functions with different constants have the same derivate, because differentiating constants returns 0
- Hence, a variable C is needed to account for this

2 - Fundamental Rules

Scalar Multiple	$\int kf(x)\,dx = k\int f(x)\,dx$
Addition & Subtraction Rule	$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3 - Standard Results

- $f(x) = ax + b$

A - Trigonometric Functions

$\int sin[f(x)]\, dx = \frac{1}{f\,'(x)} \times \{-\ cos[fx(x)]\} + C$
$\int cos[f(x)]\, dx = \frac{1}{f\,'(x)} \times sin[fx(x)] + C$
$\int tan[f(x)]\, dx = \frac{1}{f\,'(x)} \times ln\|sec[f(x)]\| + C$
$\int cot[f(x)]\, dx = \frac{1}{f\,'(x)} \times ln\|sin[f(x)]\| + C$
$\int sec[f(x)]\, dx = \frac{1}{f\,'(x)} \times ln\|sec[f(x)] + tan[f(x)]\| + C$
$\int sec^2[f(x)]\, dx = \frac{1}{f\,'(x)} \times tan[f(x)] + C$
$\int cosec[f(x)]\, dx = \frac{1}{f\,'(x)} \times \{-\ ln\|cosec[f(x)] + cot[f(x)]\|\} + C$
$\int cosec^2[f(x)]\, dx = \frac{1}{f\,'(x)} \times \{-\ cot[f(x)]\} + C$
$\int sec[f(x)]tan[f(x)]\, dx = \frac{1}{f\,'(x)} \times sec[f(x)] + C$
$\int cosec[f(x)]\, cot[f(x)]\, dx = \frac{1}{f\,'(x)} \times \{-\ cosec[f(x)]\} + C$

B - Exponential Functions

$\int e^{f(x)} = \frac{1}{f'(x)} \times e^{f(x)} + C$
$\int [f'(x)][e^{f(x)}] = e^{f(x)} + C$
$\int a^{f(x)} = \frac{1}{f'(x)} \times \frac{a^{f(x)}}{\ln a} + C$

C - Logarithmic Functions

$\int \frac{f'(x)}{f(x)} = \ln\lvert f(x)\rvert + C$
$\int \frac{1}{f(x)} = \frac{1}{f'(x)} \times \ln\lvert f(x)\rvert + C$

D - Algebraic Functions

$\int a\,dx = ax + C$, *where a is a constant*
$\int [f(x)]^n\,dx = \{\frac{1}{f'(x)} \times \frac{f(x)^{n+1}}{n+1}\} + C,\ n \neq 1$
$\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$
$\int \frac{1}{[f(x)]^2 - a^2}\,dx = \frac{1}{f'(x)} \times \{\frac{1}{2a}\ln\left\lvert\frac{f(x)-a}{f(x)+a}\right\rvert\} + C$
$\int \frac{1}{a^2 - [f(x)]^2}\,dx = \frac{1}{f'(x)} \times \{\frac{1}{2a}\ln\left\lvert\frac{a+f(x)}{a-f(x)}\right\rvert\} + C$
$\int \frac{1}{\sqrt{a^2 - [f(x)]^2}}\,dx = \frac{1}{f'(x)} \times \{\sin^{-1}(\frac{f(x)}{a})\} + C,\ f(x) < a$
$\int \frac{1}{a^2 + [f(x)]^2}\,dx = \frac{1}{f'(x)} \times \{\frac{1}{a}\tan^{-1}(\frac{f(x)}{a})\} + C$

4 - Techniques

A - Integration By Substitution

1 - Definition

- It is a way of integrating an expression by substituting it to a variable, such as u

---

2 - Method

1. From the substitution
   - ➢ Make x the subject
   - ➢ Differentiate x with respect to u
   - ➢ Make dx the subject
2. Eliminate the x terms from the integrand by using the 2 stuff from step 1
3. Simplify & Solve
4. Revert the u terms back into x terms

---

3 - Example

0. Use the substitution $u = 1 + 2x$ to find $\int x(1 + 2x)^{11}\, dx$

1. From the substitution
   - ➢ Make x the subject

$$u = 1 + 2x$$
$$x = \frac{u-1}{2}$$

   - ➢ Differentiate x with respect to u

$$\frac{dx}{du} = \frac{1}{2}$$

   - ➢ Make dx the subject

$$dx = (\frac{1}{2})(du)$$

2. Eliminate the x terms from the integrand by using the 2 stuff from step 1

$$\int x(1 + 2x)^{11}\, dx = \int (\frac{u-1}{2})(u^{11})(\frac{1}{2})(du)$$

3. Simplify & Solve

$$= \frac{1}{4}\int (u^{12} - u^{11})\, du$$

$$= \frac{1}{4}(\frac{u^{13}}{13} - \frac{u^{12}}{12}) + C$$

$$= \frac{u^{13}}{52} - \frac{u^{12}}{48} + C$$

4. Revert the u terms back into x terms

$$= \frac{(1+2x)^{13}}{52} - \frac{(1+2x)^{12}}{48} + C$$

B - Integration By Parts

1 - Definition

- It is based on the **product rule**
- The **goal** is to make the **integral** only have **one function**, making it **simpler** to **simplify** and **solve**

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

L	-	Logarithmic Functions	[**Most** Preferred]
I	-	Inverse Trigonometric Functions	
A	-	Algebraic Functions	
T	-	Trigonometric Functions	
E	-	Exponential Functions	[**Least** Preferred]

2 - Method

1. Determine $f(x)$ by using **LIATE** and find $f'(x)$
2. Let $g'(x)$ be the other term and find $g(x)$
3. Plug these 2 values into the formula
4. Solve & Simplify

3 - Example

0. Find $\int x^2 e^{4x}\,dx$

1. Determine $f(x)$ by using **LIATE** and find $f'(x)$

$$f(x) = x^2$$
$$f'(x) = 2x$$

2. Let $g'(x)$ be the **other term** and find $g(x)$

$$g'(x) = e^{4x}$$
$$g(x) = \frac{1}{4}e^{4x}$$

3. Plug these 2 values into the formula

$$\int x^2 e^{4x}\,dx = [(x^2)(\frac{1}{4}e^{4x})] - \int (2x)(\frac{1}{4}e^{4x})\,dx$$

4. Simplify & Solve

$$\int x^2 e^{4x}\,dx = \frac{1}{4}(x^2)(e^{4x}) - \frac{1}{2}\int (x)(e^{4x})\,dx$$

★ Some equations need **integration by parts** to be done **more than once** as the **resulting integrand** still **contains more than one function**

1. Determine $f(x)$ by using **LIATE** and find $f'(x)$

$$f(x) = x$$
$$f'(x) = 1$$

2. Let $g'(x)$ be the other term and find $g(x)$

$$g'(x) = e^{4x}$$
$$g(x) = \frac{1}{4}e^{4x}$$

3. Plug these 2 values into the equation from the first step 4

$$\int x^2 e^{4x}\,dx = \frac{1}{4}(x^2)(e^{4x}) - \frac{1}{2}[(x)(\frac{1}{4}e^{4x}) - \int(1)(\frac{1}{4}e^{4x})\,dx]$$

4. Simplify & Solve

$$\int x^2 e^{4x}\,dx = \frac{1}{4}(x^2)(e^{4x}) - \frac{1}{2}[\frac{1}{4}(x)(e^{4x}) - \frac{1}{16}e^{4x}] + C$$
$$= \frac{e^{4x}}{32}(8x^2 - 4x + 1) + C$$

C - Definite Integrals

1 - Definition

- Contains a lower boundary a
- Contains an upper boundary b
- Used to find the **area** and **volume** under the curve

$\int_a^b f(x)\,dx = F(b) - F(a)\,,\ f(x) = \frac{d}{dx}(F(x))$

2 - Fundamental Properties

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

D - Differential Equations

1 - Definition

- It's an equation which contains one or more terms involving derivatives of one variable, such as y, with respect to another variable, such as x
  - $\frac{dy}{dx} = f(x)g(y)$
- The **order** is the highest derivative present in the differential equation
- The **general solution** contains the arbitrary constant C
- The **particular solution** is the **general solution** but when the value of C has been found
- To solve them, they must be in the following form

$$\boxed{\int \frac{1}{g(y)}\, dy = \int f(x)\, dx}$$

---

2 - Methods

A - Method 1

1. **Manipulate** the D.E into the **required form**
2. **Integrate** both sides to **find** the **general** solution
3. If the question requires it, **find** the **particular** solution

B - Method 2

- If the given D.E is not in the form $\frac{dy}{dx} = f(x)g(y)$,
  1. **Differentiate** the given **substitution**
  2. Use them to **determine** the new D.E
  3. **Solve** and simplify the new D.E using **method 1**
  4. **Express** the final answer in terms of the **original variables**

3 - Examples

A - Example 1

$Find\ the\ D.E\ \frac{dy}{dx} = x + 6sec^2(2x)\ ,\ find\ its\ general\ solution\ and\ then\ its\ particular\ solution\ for\ which\ y = 3\ when\ x = 0$

1. **Manipulate** the D.E into the **required form**

$$\frac{dy}{dx} = [x + 6sec^2(2x)](1)$$

$$f(x) = x + 6sec^2(2x)$$

$$g(y) = 1 \Rightarrow \frac{1}{g(y)} = 1$$

$$\therefore \int (1)\, dy = \int [x + 6sec^2(2x)]\, dx$$

2. **Integrate** both sides to **find** the **general** solution

$$General\ Solution:\ y = \frac{x^2}{2} + 6 \times \frac{tan(2x)}{2} + C$$

$$= \frac{x^2}{2} + 3tan(2x) + C\ ---\ ①$$

3. If the question requires it, **find** the **particular** solution

$$Substituting\ x = 0\ ,\ y = 3\ into\ ①$$

$$3 = \frac{(0)^2}{2} + 3tan[2(0)] + C$$

$$C = 3$$

$$\therefore Particular\ Solution: y = \frac{x^2}{2} + 3tan(2x) + 3$$

B - Example 2

Show that the substitution $x + y = z$ *reduces the D.E* $\frac{dy}{dx} = (x + y)^2$ *to* $\frac{dz}{dx} = 1 + z^2$ *Hence, solve the D.E* $\frac{dy}{dx} = (x + y)^2$, *expressing the final answer in the form* $y = f(x)$

1. **Differentiate** the given **substitution**

$$x + y = z --- ①$$
$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$
$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \quad --- \quad ②$$

2. **Use them to solve** and simplify the new D.E

$$\frac{dy}{dx} = (x + y)^2 \quad --- \quad ③$$
$$Sub\ ①^2\ \&\ ②\ into\ ③ : \ \frac{dz}{dx} - 1 = z^2$$
$$: \ \frac{dz}{dx} = z^2 + 1\ (shown)$$

3. **Solve** and simplify the new D.E using **method 1**

$$\frac{dz}{dx} = (z^2 + 1)(1)$$
$$f(x) = 1$$
$$g(x) = z^2 + 1 \Rightarrow \frac{1}{g(x)} = \frac{1}{z^2+1}$$
$$\therefore \int \frac{1}{z^2+1}\, dz = \int (1)\, dx$$
$$tan^{-1} z = x + c$$
$$z = tan(x + C)$$

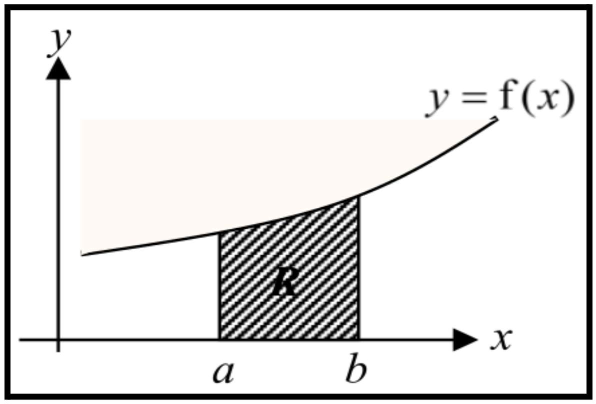
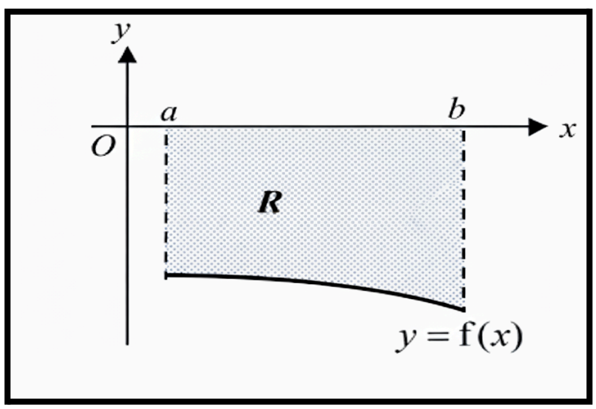
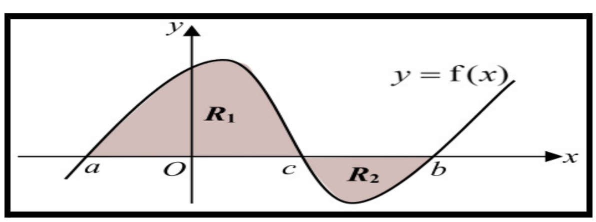
4. **Express** the final answer in terms of the **original variables**

$$z = tan(x + C)$$
$$x + y = tan(x + C)$$
$$y = tan(x + C) - x$$

5 - Applications

A - Finding Area

1 - Bounded by the $x - axis$

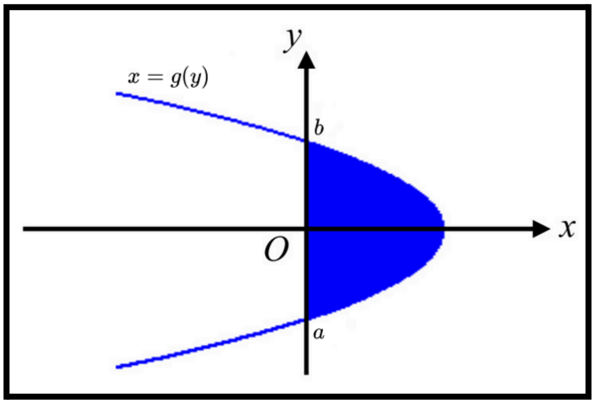
$$Area = \int_a^b y\, dx$$
 Region **above** $x - axis$
$$Area = -\int_a^b y\, dx$$
 Region **below** $x - axis$
$$Area = \int_a^c y\, dx + \left(-\int_c^b y\right) dx$$
 Regions that are in both **above** and **below** the $x - axis$

2 - Bounded by the $y - axis$

<table>
<tr><td>

$$Area = \int_{a}^{b} x\,dy$$

</td></tr>
<tr><td>

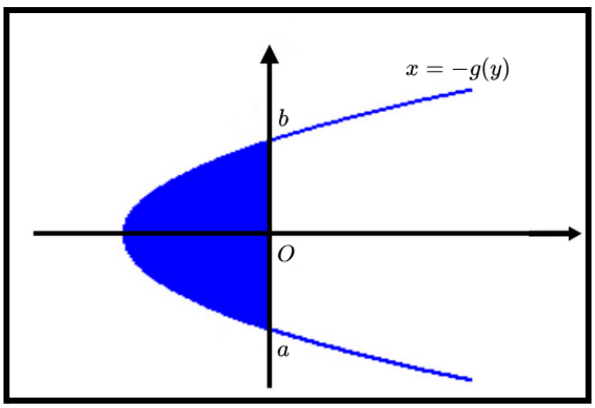


Region where $x \geq 0$

</td></tr>
<tr><td>

$$Area = -\int_{a}^{b} x\,dy$$

</td></tr>
<tr><td>

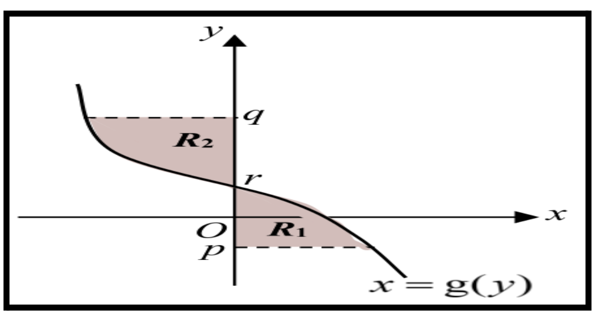


Region where $x \leq 0$

</td></tr>
<tr><td>

$$Area = \int_{p}^{r} x\,dy + \left(-\int_{r}^{q} x\,dy\right)$$

</td></tr>
<tr><td>

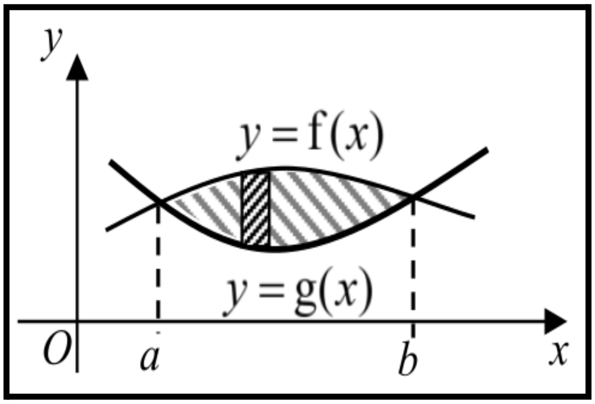


Regions that are in **both** sides

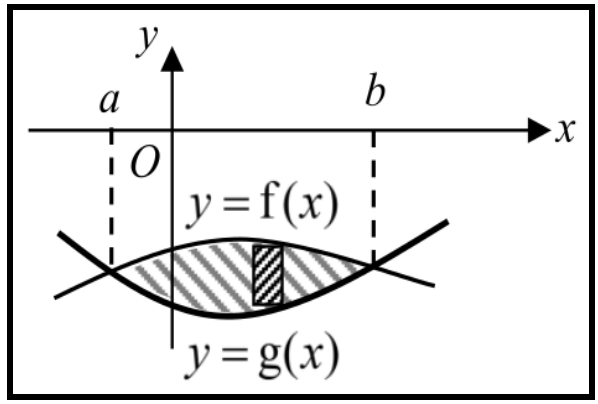
</td></tr>
</table>

3 - Bounded By 2 Curves $y = f(x)$ & $y = g(x)$

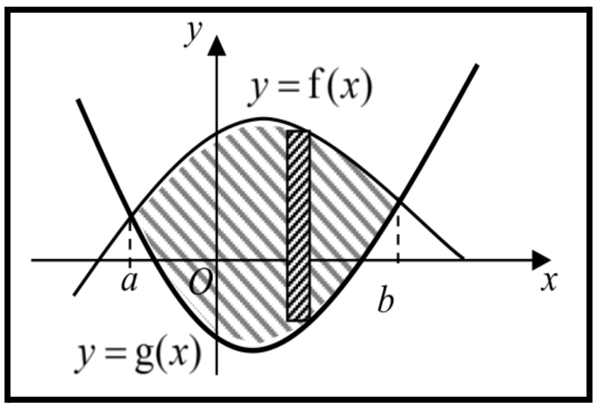
$$Area = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$



Region **above** $x - axis$



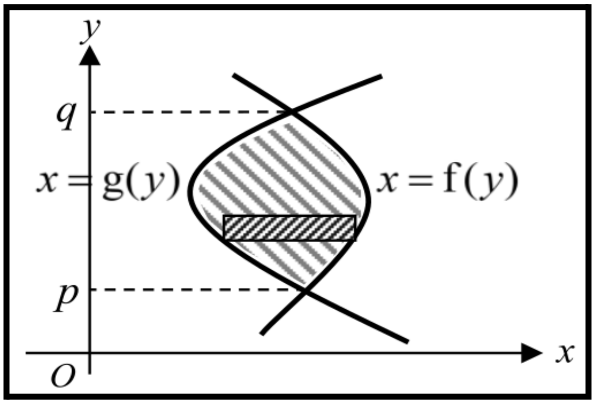
Region **below** $x - axis$



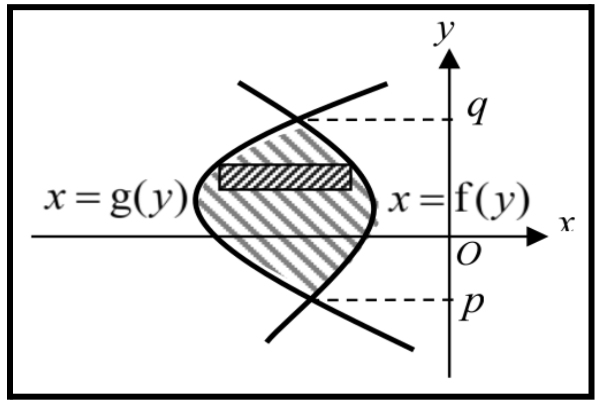
Region **crosses** the $x - axis$

4 - Bounded By 2 Curves $x = f(y)$ & $x = g(y)$

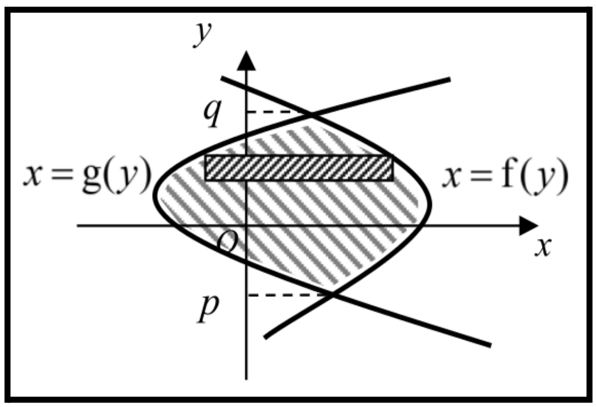
$$Area = \int_p^q f(y)\, dy - \int_p^q f(y)\, dy$$



Region on the **RHS** of the $y - axis$



Region on the **LHS** of the $y - axis$



Region **crosses** the $y - axis$

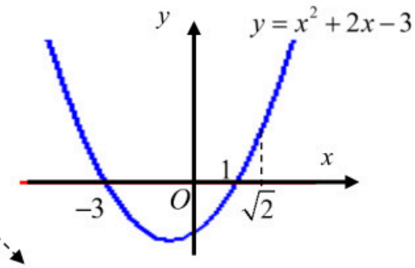
5 - Modulus Sign

$$Area = \int_a^b f(x)\,dx + \int_b^c - f(x)\,dx$$
$$a\,,\ b \Leftrightarrow f(x) > 0$$
$$b\,,\ c \Leftrightarrow f(x) < 0$$

Find the exact value of $\int_0^{\sqrt{2}} \left|x^2+2x-3\right| \mathrm{d}x$.

<u>Solution</u>

$$\int_0^{\sqrt{2}} \left|x^2+2x-3\right| \mathrm{d}x$$
$$= \int_0^1 \left(-(x^2+2x-3)\right)\mathrm{d}x + \int_1^{\sqrt{2}} \left(x^2+2x-3\right)\mathrm{d}x$$



From the graph, it can be observed that the curve lies <u>below the x-axis</u> (i.e. $f(x) < 0$) when $0 \le x < 1$. $\left|x^2+2x-3\right|$ takes the expression $-(x^2+2x-3)$ when $0 \le x < 1$.

It can also be observed that the curve lies <u>above the x-axis</u> (i.e. $f(x) > 0$) when $1 < x \le \sqrt{2}$. $\left|x^2+2x-3\right|$ takes the expression x^2+2x-3 when $1 < x \le \sqrt{2}$.

$$= -\left[\frac{x^3}{3}+x^2-3x\right]_0^1 + \left[\frac{x^3}{3}+x^2-3x\right]_1^{\sqrt{2}}$$
$$= -\left[\frac{1}{3}+1-3-0\right]+\left[\frac{\sqrt{8}}{3}+2-3\sqrt{2}-\frac{1}{3}-1+3\right]$$
$$= \frac{5}{3}+\frac{2\sqrt{2}}{3}+2-3\sqrt{2}+\frac{5}{3}$$
$$= \frac{16}{3}-\frac{7}{3}\sqrt{2}$$

B - Finding Volume

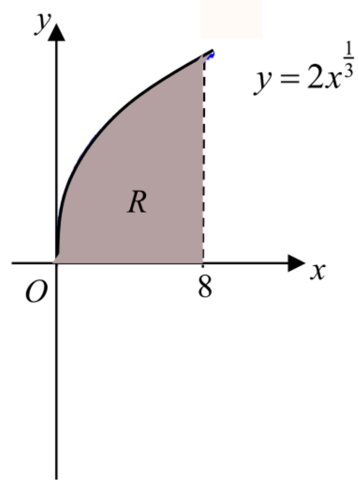
1 - About the $x - axis$

$$Volume = k\pi \int_a^b [f(x)]^2 \, dx$$

k represents k rounds (360° / 2π) of rotation

The region R, in the first quadrant, is bounded by the curve $y = 2x^{\frac{1}{3}}$, the x-axis and the line $x = 8$. Find the exact value of the volume generated when R is rotated through four right angles about the x-axis.

<u>Solution</u>



Required volume
$$= \pi \int_0^8 y^2 \, dx$$
$$= \pi \int_0^8 \left(2x^{\frac{1}{3}}\right)^2 dx$$
$$= 4\pi \left[\frac{3}{5}x^{\frac{5}{3}}\right]_0^8$$
$$= \frac{12}{5}\pi[32 - 0]$$
$$= \frac{384}{5}\pi \text{ units}^3$$

2 - About the $y - axis$

$$Volume = k\pi \int_p^q [g(y)]^2 \, dy$$

k represents k rounds (360° / 2π) of rotation

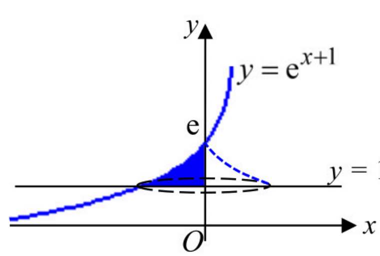
The region bounded by the curve $y = e^{x+1}$, the line $y = 1$ and the y-axis is rotated through 360° about the y-axis. Find the **<u>numerical value</u>** of the volume of the solid formed.

<u>Solution</u>

Making x the subject: $y = e^{x+1} \Rightarrow x = \ln y - 1$

Volume of solid $= \pi \int_1^e (\ln y - 1)^2 \, dy$

≈ 1.37 units^3



3 - Parallel to the $x\ /\ y\ -\ axis$

- The axis of rotation may not be about the $x/y\ -\ axis$ and instead be parallel to them

- Hence, **translate** the graph until it is about either the $x/y\ -\ axis$

- Translation **does not affect** the graph's **shape** and **size**, hence the translated graph's **volume** will be the **same** as that of the original graph

Example 17:

The region R is bounded by the curve C with equation $y = 2e^{-x}$, the lines $x = 1$ and $y = 2$.

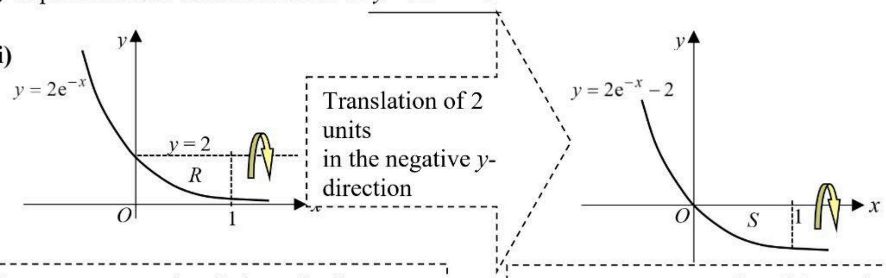
(i) Write down the equation of the curve obtained when C is translated by 2 units in the negative y-direction.

(ii) Hence, or otherwise, find the exact value of the volume of the solid formed when R is rotated completely about the line $y = 2$.

Solution

(i) Equation of the translated curve is $y = 2e^{-x} - 2$

(ii)



For $y = 2e^{-x}$: rotating R about the line $y =$

For $y = 2e^{-x} - 2$: rotating S about the x-axis.

Observe that the volume of revolution formed when rotating R about the line $y = 2$ is the same as the volume of revolution formed by rotating S about the x-axis.

We can now apply the formula $\pi\int_a^b y^2\ dx$ to the rotation of region S since the axis of rotation is the x-axis, where y is the equation of the curve after translation.

$$
\begin{aligned}
\text{Volume of the solid formed} &= \pi\int_0^1 \left(2e^{-x} - 2\right)^2 dx \\
&= \pi\int_0^1 \left(4e^{-2x} - 8e^{-x} + 4\right) dx \\
&= \pi\Big[-2e^{-2x} + 8e^{-x} + 4x\Big]_0^1 \\
&= \pi\Big[(-2e^{-2} + 8e^{-1} + 4) - (-2 + 8)\Big] \\
&= 2\pi\Big[-e^{-2} + 4e^{-1} - 1\Big] \text{ units}^3
\end{aligned}
$$

4 - Bounded by 2 curves

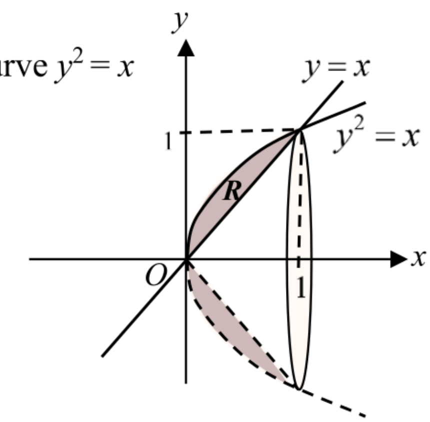
$$Total\, Volume = Volume_{Bigger\ Curve} - Volume_{Smaller\ Curve}$$

The region R is bounded by the curve $y^2 = x$ and the line $y = x$. Find the exact volume of the solid formed when R is rotated through one revolution about the x-axis.

Solution

Required Volume

= Volume generated by the region bounded by the curve $y^2 = x$ and the x-axis for $0 \le x \le 1$ about x-axis – Volume of cone, height 1, radius 1



$= \pi \int_0^1 x \, \mathrm{d}x - \frac{1}{3}\pi(1)(1)$

$= \pi \left[\frac{x^2}{2} \right]_0^1 - \frac{\pi}{3}$

$= \frac{\pi}{2} - \frac{\pi}{3}$

$= \frac{\pi}{6}$ units^3

5 - Involving Composite Solids

$$Total\ Volume\ =\ \Sigma\, Volume_{Composite\ Solids}$$

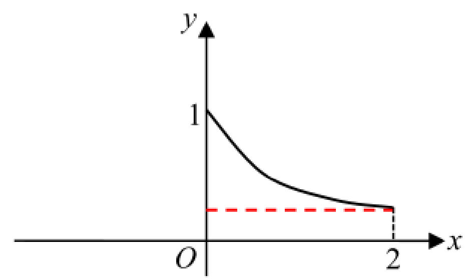
The region bounded by the curve $y=\dfrac{1}{1+2x}$, the line $x=2$ and the axes is rotated completely about the y-axis. Prove that the volume of the solid formed is $\dfrac{1}{2}\pi\left(4-\ln 5\right)$.

Solution

Required volume about $y-$axis

= Volume generated by region bounded by curve and y-axis for $\dfrac{1}{5}\le y\le 1$ about $y-$axis

+ Volume of cylinder, of radius 2, height $\dfrac{1}{5}$



$$=\pi\int_{\frac{1}{5}}^{1} x^2\ \mathrm{d}y\ +\ \pi\left(2^2\right)\left(\frac{1}{5}\right)$$

$$=\pi\int_{\frac{1}{5}}^{1}\left(\frac{1}{2y}-\frac{1}{2}\right)^2 \mathrm{d}y\ +\ \frac{4\pi}{5}$$

$$=\pi\int_{\frac{1}{5}}^{1}\left(\frac{1}{4y^2}-\frac{1}{2y}+\frac{1}{4}\right)\mathrm{d}y+\frac{4\pi}{5}$$

$$=\pi\left[\frac{1}{4}\left(-\frac{1}{y}\right)-\frac{1}{2}\ln|y|+\frac{1}{4}y\right]_{\frac{1}{5}}^{1}+\frac{4\pi}{5}$$

$$=\pi\left[-\frac{1}{4}+\frac{1}{4}-\left(-\frac{5}{4}-\frac{1}{2}\ln\left(\frac{1}{5}\right)+\frac{1}{20}\right)\right]+\frac{4\pi}{5}$$

$$=\pi\left(\frac{5}{4}-\frac{1}{2}\ln 5-\frac{1}{20}\right)+\frac{4\pi}{5}$$

$$=\pi\left(\frac{25-1+16}{20}-\frac{1}{2}\ln 5\right)$$

$$=\frac{\pi}{2}\left(4-\ln 5\right)\ \text{units}^3\ \text{(shown)}$$

$$y=\frac{1}{1+2x}$$

$$\frac{1}{y}=1+2x$$

$$2x=\frac{1}{y}-1$$

$$x=\frac{1}{2y}-\frac{1}{2}$$

C - Modelling

A - Definition

- It is a process in which a real world problem is simplified into a mathematical problem, whose solutions may be used to help solve the corresponding real world problem

---

B - Method

1. **Identify** the required variables
2. **Construct** the D.E
3. **Solve** the D.E
4. **Check** if the solution makes sense with respect to the real world problem's context

---

C - Example

The rate at which a certain radioactive substance decays is proportional to the amount present at that time. A block of such substance having an initial quantity of m_0 units are observed. After 100 days, 30% of its initial amount has decayed. Find the half-life of this substance. Half-life refers to the time taken for a radioactive element to decay to $\frac{1}{2}$ of its initial quantity

1 - **Identify** the required variables

Let x be the amount of radioactive substance

Let t be the amount of radioactive substance at a specific time in days

---

2 - **Construct** the D.E

$\frac{dx}{dt} =- kx\, ,\ k > 0$

3 - **Solve** the D.E

$\frac{dx}{dt} = (-\ k)(x)$

$f(t) = \ \ -\ k$

$g(x)\ = x \Rightarrow \frac{1}{g(x)} = \frac{1}{x}$

$\therefore \int \frac{1}{x}\ dx = \int -\ k\ dt$

$ln(x)\ = -\ kt\ +\ C$

$x\ = e^{-kt+C}$

$x\ = e^{-kt+C} \Rightarrow e^{-kt} \times e^{C}$

$x\ = e^{-kt+C} \Rightarrow Ae^{-kt}\ ,\ A\ = e^{c}$

When $t\ = \ 0\ ,\ x\ = m_0 \Rightarrow A\ = m_0$

When $t\ = \ 100,\ x\ = \ 0.7m_0$

$0.7m_0 = m_0e^{-100k}$

$ln(0.7)\ = -\ 100k$

$-\ k\ = \frac{1}{100}ln(0.7)$

Hence, $x\ = m_0e^{(\frac{t}{100}ln(0.7))}$

At half-life

$x\ = \frac{1}{2}m_0$

$e^{\left(\frac{t}{100}ln(0.7)\right)} = \frac{1}{2}$

$\frac{t}{100}ln(0.7)\ = \ ln(\frac{1}{2})$

$t\ = \ 100(\frac{ln(0.5)}{ln(0.7)})$

$\ \ = \ 194\ (3sf)$

---

4 - **Check** if the solution makes sense with respect to the real world problem’s context

Hence, the half-life of the radioactive substance is approximately 194 days

B - Probability & Statistics
1 - Combinatorics
A - Permutations
1 - Definition

- It's an **order-sensitive** arrangement of objects
  - Different arrangements of the **same entities** will be seen as **different objects**
  - (**A,B**) and (**B,A**) are seen as 2 **distinct objects**, and hence, 2 **distinct permutations**
- It's used when the **sequence** of the chosen elements **changes the outcome**

---

2 - Formulae

1. Permutation of n **distinct** objects, taken r at a time
   - ★ **Without** repetition [Each entity can only be used **ONCE**]
     - $P(n, r) = \frac{n!}{(n-r)!}$
   - ★ **With** repetition [Each entity can be used **MORE THAN ONCE**]
     - $P(n, r) = n^r$

2. Permutation of n objects with **some identical entities**
   - ★ $\frac{n!}{n_1! \times n_2! \times n_3! \times \ldots \times n_k!}$ [n_1 , n_2 , n_3 , … , n_k are counts of identical entities]

3. Circular Permutations
   - ★ Not Rotation-Sensitive
     - $(n - 1)!$
     - Each position **IS NOT** fixed and labelled, making them not unique
     - Hence, each rotation is considered the same
   - ★ Rotation-Sensitive
     - $n!$
     - Each position **IS** fixed and labelled, making them unique
     - Hence, each rotation **IS NOT** considered the same

B - Combinations
1 - Definition

- It's an **order-insensitive** arrangement of objects
  - Different arrangements of the **same entities** will be seen as **identical objects**
  - (**A,B**) and (**B,A**) are seen as 2 **identical objects**, and hence, 2 **identical combinations**
- It's used when the **sequence** of the chosen elements **does not change the outcome**

---

2 - Formulae

1. Combination of n distinct objects, taken r at time
   - ★ $\binom{n}{r} = \frac{n!}{r! \times (n-r)!}$

Combination of n identical objects, taken r at a time $\Rightarrow 1$

2 - Probability Theory

A - Event Types

1 - Mutually Exclusive Events

- They cannot occur at the same time
- ★ $n_1 + n_2 + n_3 + \ \dots \ + n_k$
    - Used to count all possible outcomes when only one of the sub-events occur one one any given time, by **adding** them all

---

2 - Independent Events

- The probability of any event does not affect the probability of another event
- ★ $n_1 \times n_2 \times n_3 \times \ \dots \ \times n_k$
    - Used to count all the possible outcomes when all of sub-events occur, and each is independent of the previous ones, by multiplying them all

- Independence **cannot be directly inferred** from a Venn Diagram

B - Probability Types

1 - Unconditional

- It's the probability of an event E occurring, without assuming anything about other events
- $P(E) = \frac{No.\, of\, elements\, in\, E}{No.\, of\, elements\, in\, S} = \frac{N(E)}{N(S)}$
- Sample space S contains a finite number of equally likely outcomes

★ $0 \le P(E) \le 1$
  - ➢ The probability of an event E occurring is always between 0 and 1 inclusive
  - ➢ 0 means it will never happen $[P(E) = 0]$ $[P(\emptyset) = 0]$
  - ➢ 1 means it will always happen $[P(E) = 1]$

---

2 - Conditional

- The probability of an event A occurring, given that another event B occurred
  - The conditional probability of A given B

★ $P(A|B) = \frac{P(A\cap B)}{P(B)}$
  - ➢ $P(A \cap B) = P(A|B) \times P(B)$
  - ➢ $P(A \cap B) = P(B|A) \times P(A)$

➢ To check if A OR B are **independent events**, their conditional probability of them has to be equal to their unconditional probability of happening, as they are not dependent on the probability of each other happening
  - ★ $P(A|B) = P(A)$
  - ★ $P(B|A) = P(B)$

C - Terminology

- Experiment / Trial
  - Any course of action having an observable outcome, and can be repeated under the same set of conditions
  - Tossing two coins

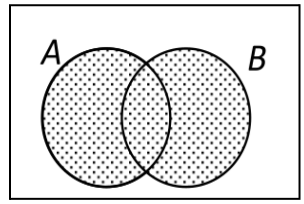
- Outcome
  - The result(s) obtained from the experiment
  - One Outcome : Two Heads (HH)

- Sample / Possibility Space S
  - The set of all possible outcomes of the experiment
    - $S = \{HH, TT, HT, TH\}$

- Event E
  - A subset of the S
  - $E = \{HT, TH\}$ is a subvert of $S = \{HH, TT, HT, TH\}$
  - It represents getting one head and one tail

D - Venn Diagrams

1 - Union

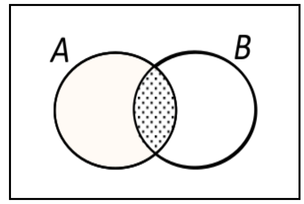


- $A \cup B$
- Either A or B or both occurs

★ $P(A \cup B) = [P(A) + P(B)] - P(A \cap B)$

---

2 - Intersection



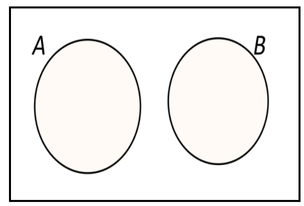
- $A \cap B$
- The set of outcomes common to both A and B

➢ To check if A and B are **independent events,** their intersection must be the **multiplication** of both of their values

★ $P(A \cap B) = P(A) \times P(B)$

---

3 - Mutually Exclusive Events

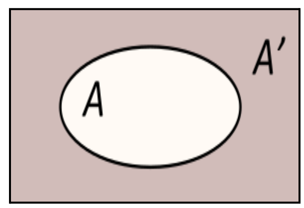


- A and B cannot occur at the same time
- $(A \cap B) = \emptyset$

★ $P(A\, OR\, B) = P(A) + P(B)$

---

4 - Complement



- $A^{'}$
- The event that A does not occur

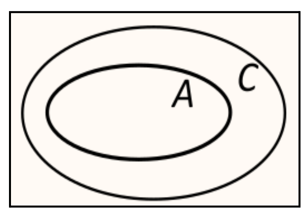
★ $P(A^{'}) = 1 - P(A)$

★ $(A \cup B)^{'} = A^{'} \cap B^{'}$

★ $(A \cap B)^{'} = A^{'} \cup B^{'}$

---

5 - Subset



- $A \subseteq C$
- Every element in A is also in C

---

6 - Empty / Null Set

- {} / $\emptyset$
- A set with no elements

E - Diagrams

1 - Sample Space Diagram

- It's the table of all possible outcomes
- This method can only be used if the outcomes have an equal chance of occurring

An unbiased tetrahedral dice has its faces numbered 1, 2, 3, 4. When the dice is thrown on a horizontal table, the number on the face in contact with the table is taken to be the score. The dice is thrown twice. Find the probability that

(i) the score on the first throw is greater than the score on the second throw,

(ii) the sum of the scores is divisible by 4.

Solution

(i) Required probability = $\frac{6}{16} = \frac{3}{8}$

1st \ 2nd	**1**	**2**	**3**	**4**
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,1)	(4,2)	(4,3)	(4,4)

The possible outcomes of having the score on the 1st throw is greater than the score on the 2nd throw are (2, 1), (3,1), (4,1), (3,2), (4,2) and (4,3). Hence, the number of required outcomes = 6

Total no. of possible outcomes = 16

(ii) P(sum of scores is divisible by 4) $= \frac{4}{16} = \frac{1}{4}$

1st \ 2nd	**1**	**2**	**3**	**4**
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

The possible sums of scores that are divisible by 4 are 4 and 8 (Possible outcome is highlighted in grey). Hence, the number of required outcomes = 4.

2 - Tree Diagram

- It's the visualisation of all the possible outcomes of a sequence of experiment, especially when the probability of an event is dependent on the previous event
  - Along the same branch, multiply the probabilities

Video 15.9

Example 10:

Bag A contains 4 red balls and 4 black balls. Bag B contains 3 red balls and 9 black balls. All the balls are identical apart from colour. One ball is chosen at random from Bag A and put into Bag B. A second ball is chosen from Bag B afterwards.

(i) Draw a tree diagram to represent the possible outcomes. [2]

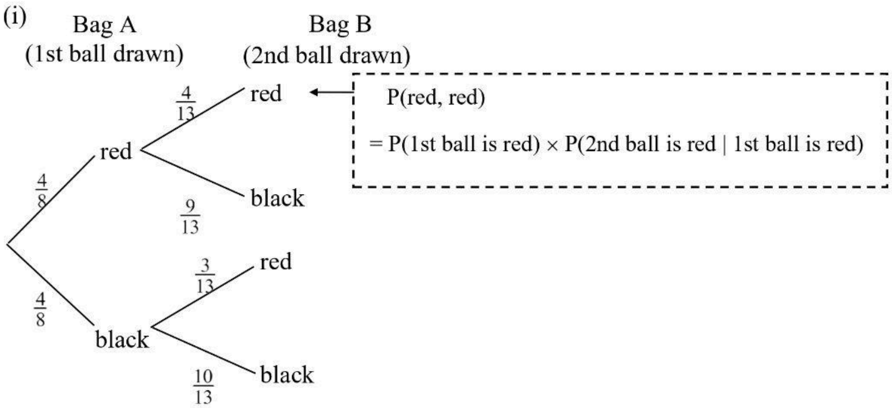
(ii) Write down the conditional probability that the second ball drawn is black given that the first ball drawn is red. [1]

(iii) Find the probability that the second ball drawn is red. [2]

(iv) Find the conditional probability that the first ball drawn is red, given that the second ball drawn is black. [3]

<u>Solution</u>

(i)



(ii) $\text{P(2nd ball is black | 1st ball is red)} = \dfrac{9}{13}$

(iii) $\text{P(2nd ball drawn is red)} = \text{P(red, red)} + \text{P(black, red)}$

$$= \left(\frac{4}{8}\times\frac{4}{13}\right)+\left(\frac{4}{8}\times\frac{3}{13}\right)=\frac{7}{26}$$

(iv) $$\text{P(1st ball is red | 2nd ball is black)} = \frac{\text{P(1st ball is red and 2nd ball is black)}}{\text{P(2nd ball is black)}}$$

$$= \frac{\text{P(red, black)}}{\text{P(red, black)}+\text{P(black, black)}}$$

$$= \frac{\left(\frac{4}{8}\times\frac{9}{13}\right)}{\left(\frac{4}{8}\times\frac{9}{13}\right)+\left(\frac{4}{8}\times\frac{10}{13}\right)}$$

$$= \frac{9}{19}$$

3 - Discrete Random Variables

A - Definition

1. A random variable X is said to be **discrete** if its **possible values** are **countable**
2. $X = \{x_1, x_2, x_3, \dots, x_n\}$
3. $P(X = x)$: If the D.R.V X takes on the value x

B - Measurements

1 - Expectation

A - Definition

- The average value of X
- Denoted as
  - $E(X)$
  - μ

B - Formulae

★ $E(X) = \sum_{all\ x} [x \times P(X = x)]$

★ $E(g(x)) = \sum_{all\ x} [g(x) \times P(X = x)]$

- ➢ $E(a) = a$
- ➢ $E(aX) = aE(X)$
- ➢ $E(aX + b) = aE(X) + b$
- ➢ $E(X^2) \neq [E(x)]^2$

C - Properties

- If the p.d.f of X is symmetrical about the central value
  - $E(X) = Central\ Value\ Of\ X\ (If\ n\ is\ odd)$
  - $E(X) = Average\ Of\ The\ Two\ Central\ Values\ Of\ X\ (If\ n\ is\ even)$

For each of the following parts, find $\mathrm{E}(X)$ either by using the formula or by symmetry, if possible. Verify your answer by using the GC.

(a)

x	1	3	5
$\mathrm{P}(X = x)$	1/8	1/2	3/8

$$\mathrm{E}(X) = (1)\left(\frac{1}{8}\right) + (3)\left(\frac{1}{2}\right) + (5)\left(\frac{3}{8}\right) = 3.5$$

(b)

x	-4	-2	0	2	4
$\mathrm{P}(X = x)$	1/8	1/4	1/4	1/4	1/8

By symmetry, $\mathrm{E}(X) = 0$

(c)

x	1	2	3	4
$\mathrm{P}(X = x)$	1/4	1/4	1/4	1/4

By symmetry, $\mathrm{E}(X) = \dfrac{2+3}{2} = 2.5$

2 - Mode

- The **most likely value** of the DRV
- $P(x = m) \geq P(X = x)$, for all other values of x and m is the mode

Suppose the probability distribution of X is as follows.

x	1	2	5	10
$\mathrm{P}(X=x)$	0.15	0.3	0.3	0.25

Since $\mathrm{P}(X=2)$ and $\mathrm{P}(X=5)$ are both the largest in the distribution, the **mode of X** are 2 and 5.

---

3 - Variance

A - Definition

- It's the measurement of the amount of spread of the distribution's possible values with respect to the mean
  - If it's **large**, the possible values **deviate a lot** from the mean
  - If it's **small**, the possible values **are close** to the mean

B - Formulae

- $Var(X) = E[(X - \mu)]^2$
- $Var(X) = \sum_{all\,x} [(x - \mu)^2 \times P(X = x)]$

★ $Var(X) = E(X^2) - [E(X)]^2$

C - Properties

- $var(X) \geq 0$
- $Var(a) = 0$
- $Var(aX) = a^2\,Var(X)$
- $Var(aX + b) = a^2\,Var(X)$

---

4 - Standard Deviation

- It's the positive square root of X's variance
  - $\sigma = \sqrt{Var(X)} \Rightarrow \sqrt{E(X)^2 - [E(X)]^2}$

C - Probability Distribution

1 - Definition

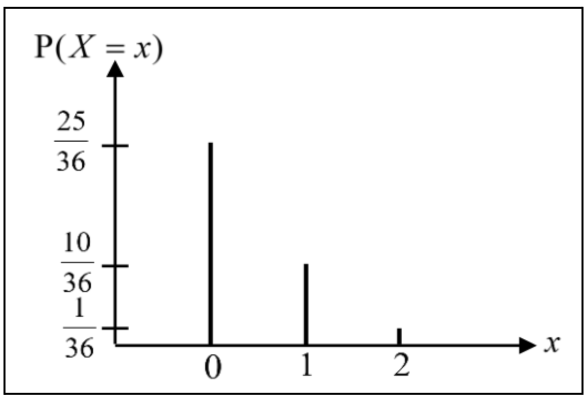
- It's a **graphical visualisation** that **shows the probabilities for all of the outcomes** that the DRV can have for each input x

---

2 - Properties

- $0 \leq P(X = x) \leq 1$ [The probability of each outcome **must be between** 0 and 1]
- $\sum_{all\,x} P(X = x) = 1$ [The sum of all the outcomes' probabilities **must add up** to 1]

x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$



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3 - Function Form

- The probability distribution function (**p.d.f**) $f(x)$ is the function responsible for allocating all of the outcomes' probabilities

$$f(x) = P(X = x) = \begin{cases} \frac{25}{36} & \text{if } x = 0 \\ \frac{10}{36} & \text{if } x = 1 \\ \frac{1}{36} & \text{if } x = 2 \end{cases}$$

4 - Use Case

A - Definition

- A *Binomial distribution* is a type of **discrete probability distribution** that models the probability of getting x number of successes in an experiment X that has a fixed number of independent trials n, and each trial has an equal probability of success p with only 2 outcomes (success or failure)
    - $x = 0$ means no successes & all failures
        - The binomial distribution will tell you the probability of getting all failures in n trials
    - $x = n$ means all successes & no failures
        - The binomial distribution will tell you the probability of getting all successes in n trials

★ $X \sim B(n, p)$

B - Criteria

- **FIST**
    - **F**ixed number of trials
    - **I**ndependent trials
    - **S**ame probability of success for all trials
    - **T**wo mutually exclusive outcome per trial [success or failure]

C - Formulae

1 - Probability Distribution Function

- $P(X = x) = f(x) = \binom{n}{x} p^{x}(1 - p)^{n-x}\ ,\ x = 0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ \dots\ ,\ n$ **[MF27]**

A fair die is rolled 20 times. Let Y be the random variable denoting the number of ‘sixes’ obtained. Find the probability of obtaining 4 ‘sixes’.

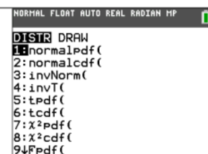
<u>Solution</u>
Let Y be r.v. denoting the number of “sixes” obtained out of 20 tosses.

$$Y \sim \mathrm{B}\left(20, \frac{1}{6}\right)$$

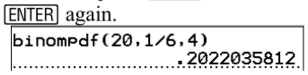
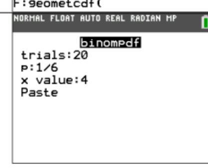
Always remember to define the random variable (r.v.) used, together with its distribution.

$$\begin{aligned} \text{P(obtaining 4 'sixes')} &= \mathrm{P}(Y = 4) \\ &= 0.202 \text{ (3 s.f.)} \end{aligned}$$

<u>GC steps</u>

<table>
<tr><td><u>Step 1</u>
Start with the home screen.
Use “DISTR” by pressing $\boxed{\text{2nd}}\boxed{\text{VARS}}$.</td><td>
</td></tr>
<tr><td><u>Step 2</u>
Scroll up to select binompdf(.

Press $\boxed{\text{ENTER}}$.</td><td>
</td></tr>
<tr><td><u>Step 3</u>
Key in the values for n, p and x respectively.

Go to Paste and press $\boxed{\text{ENTER}}$.
Press $\boxed{\text{ENTER}}$ again.

</td><td>
</td></tr>
</table>

2 - Cumulative Distribution Function

- The **sum** of all the probabilities up to a certain point k , $k \leq n$

★ $P(X \leq x) = \sum_{k=0}^{x} P(X = k)$

★ The $binomcdf$ function of the GC only accepts the form $P(X \leq x)$, so convert all required probabilities into this form in order to use this function

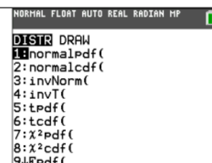
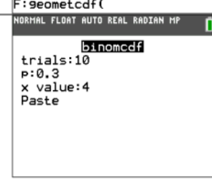
A man shoots at a target 10 times. The probability that he will hit the target each time is 0.3. Let W be the random variable denoting the number of times he hits the target. Assuming the result of all shots is independent of one another, find the probability of him hitting the target at most four times.

Solution

$$W \sim \mathrm{B}(10, 0.3)$$

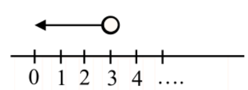
P(man hits the target at most 4 times) $= \mathrm{P}(W \leq 4)$
$= 0.850$ (3 s.f.)

GC Steps

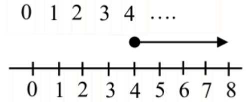
Step 1 Start with the home screen. Use "DISTR" by pressing [2nd][VARS].	
Step 2 Scroll up to select **binomcdf(**. Press [ENTER].	
Step 3 Key in the values for n, p and x respectively. Go to **Paste** and press [ENTER]. Press [ENTER] again. binomcdf(10,0.3,4) .8497316674	

Suppose $X \sim \mathrm{B}(8, 0.15)$, then

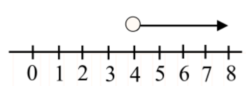
(a) $\mathrm{P}(X < 3) = \mathrm{P}(X \leq 2)$
$= 0.895$



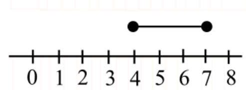
(b) $\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3)$
$= 0.0214$



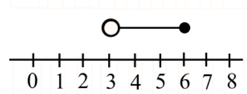
(c) $\mathrm{P}(X > 4) = 1 - \mathrm{P}(X \leq 4)$
$= 0.00285$



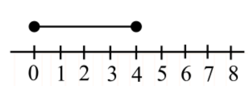
(d) $\mathrm{P}(4 \leq X \leq 7) = \mathrm{P}(X \leq 7) - \mathrm{P}(X \leq 3)$
$= 0.0214$



(e) $\mathrm{P}(3 < X \leq 6) = \mathrm{P}(X \leq 6) - \mathrm{P}(X \leq 3)$
$= 0.0213$



(f) $\mathrm{P}(|X - 2| \leq 2) = \mathrm{P}(-2 \leq X - 2 \leq 2)$
$= \mathrm{P}(0 \leq X \leq 4)$
$= \mathrm{P}(X \leq 4)$
$= 0.997$



D - Measurements

1 - Expectation

- $E(X) = np$

---

2 - Mode

1. Use the $binompdf$ function in the GC to display the probability distribution
2. Find the x value that has the largest probability in the Y_1 column
3. List down 3 probabilities with the largest one in the middle
4. Say that the middle probability is the mode

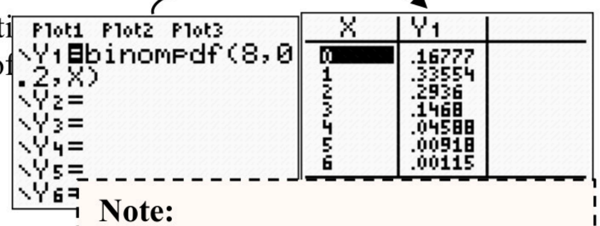
Samples, each of 8 articles, are taken at random from a large consignment in which 20% of articles are defective. Find the number of defective articles that is most likely to occur in a single sample.

Press $\boxed{\text{2nd}}\boxed{\text{GRAPH}}$ *to view this list of results.*

<u>Solution</u>

Let X be the random variable denoting the number of defective articles out of 8.

$$X \sim \mathrm{B}\,(8, 0.2)$$



Using GC,

$$\begin{cases} \mathrm{P}(X=0)=0.168 \\ \mathrm{P}(X=1)=0.336 \\ \mathrm{P}(X=2)=0.294 \end{cases}$$

Note:
It can be proven that the mode of the distribution is near the mean *np*.
i.e., the mode is <u>**near**</u> $8 \times 0.2 = 1.6$.

In the presentation, we need to list down 3 probabilities, with the largest probability in the middle. Thus is to show that P(X = 1) has the largest probability.

Thus, the number of defective articles that is most likely to occur (i.e. mode of the distribution) is 1.

---

3 - Variance

- $Var(X) = np(1 - p)$

---

4 - Standard Deviation

$\sigma = \sqrt{Var(x)} \Rightarrow \sqrt{np(1 - p)}$

E - Finding the unknowns

1 - The number of fixed trials

To find *n* and *r*, given the probability

Example 8:

Given that $X \sim \mathrm{B}(n, 0.2)$ and $\mathrm{P}(X \geq 1) > 0.68$, find the least value of n.

<u>Solution</u>

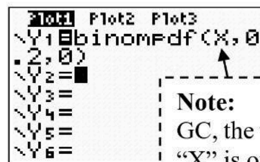
Method 1 (Using GC):

$\mathrm{P}(X \geq 1) > 0.68$

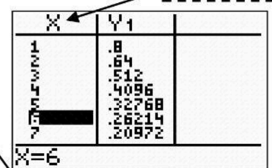
$1 - \mathrm{P}(X = 0) > 0.68$

$\mathrm{P}(X = 0) < 1 - 0.68$

$\mathrm{P}(X = 0) < 0.32$



Note: In the GC, the variable "X" is our "*n*".



From GC,

$$\left.\begin{array}{l} \text{When } n = 5,\ \mathrm{P}(X = 0) = 0.32768 > 0.32 \\ \text{When } n = 6,\ \mathrm{P}(X = 0) = 0.26214 < 0.32 \\ \text{When } n = 7,\ \mathrm{P}(X = 0) = 0.20972 < 0.32 \end{array}\right]$$

$\therefore\ n \geq 6$

Thus the least value of n is 6.

We need to list down the probabilities for 3 appropriate values of n in order to derive an inequality for n.

Method 2:

$$\mathrm{P}(X \geq 1) > 0.68$$

$$1 - \mathrm{P}(X = 0) > 0.68$$

$$\mathrm{P}(X = 0) < 1 - 0.68$$

$$\binom{n}{0}(0.2)^0(0.8)^n < 0.32$$

$$(0.8)^n < 0.32$$

$$n\ln(0.8) < \ln(0.32)$$

$$n > \frac{\ln 0.32}{\ln 0.8}$$

$$n > 5.106$$

Note:
Need to change the inequality sign since $0.8 < 1$, thus $\ln 0.8$ is negative.

Since n is a positive integer, the least value of n is 6.

2 - The probability of success

3.3 To find p, the probability of success

Example 10:

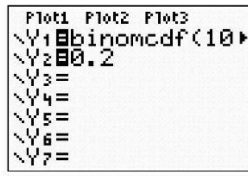
Video 16B.9

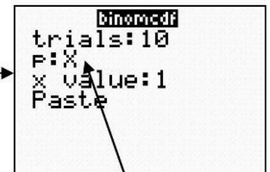
The random variable X has the binomial distribution B(10, p) such that $0 < p < 0.3$. Given that $P(X = 0 \text{ or } 1) = 0.2$, find p correct to 2 decimal places.

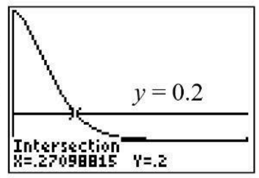
Solution

Method 1: Sketch **graphs** using GC

$$P(X = 0 \text{ or } 1) = 0.2$$
$$P(X = 0) + P(X = 1) = 0.2$$
$$P(X \le 1) = 0.2$$







Note: In the GC, the variable "X" is our "p".

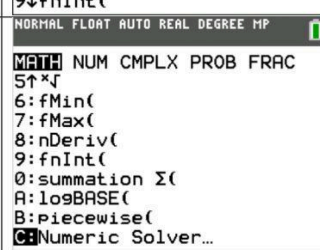
Note:
1. In your presentation, the graph needs to be drawn to show the examiner how you obtain the answer for p.
2. The value of p is the x-coordinate of the point of intersection.

Thus $p = 0.27$ (2d.p.)

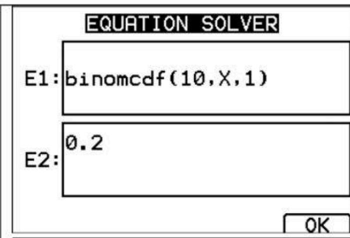
Note: Since p is any real value between 0 and 1, i.e. $0 < p < 1$, we do not use "table" in the GC to find p. Thus, we use a graphical method. Adjust the "Window" setting to: Xmin = 0, Xmax = 1, Ymin = –0.1, Ymax = 1 so that the graph is visible.

Method 2: Use **Numeric Solver**

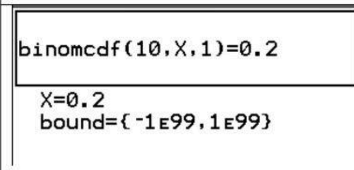
$$P(X \le 1) = 0.2$$

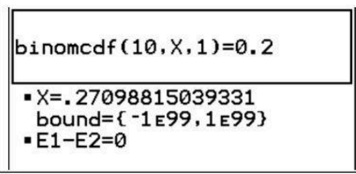
Step 1 Press $\boxed{\text{MATH}}$.	
Step 2 Scroll to **C: Numeric Solver** and press $\boxed{\text{ENTER}}$.	

<table>
<tr>
<td><u>Step 3</u>
Enter the both sides of the equation in E1 and E2 as shown on the right.

and press OK.</td>
<td>
</td>
</tr>
<tr>
<td><u>Step 4</u>
Make a guess for p,

Say, 0.2
(since $0 < p < 0.3$)

and press SOLVE.</td>
<td>


</td>
</tr>
</table>

Using GC, $p = 0.27$ (2 d.p.)

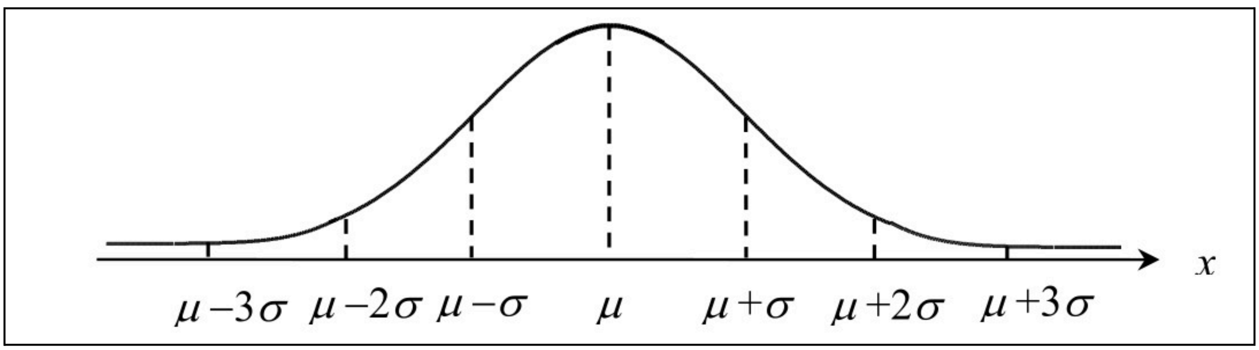
4 - Normal Distribution

A - Definition

1. A random variable X is said to be **continuous** if its **possible values** are **infinite** and **uncountable**
   - ➢ Hence, it takes on a **range of values**

2. Many C.R.V follow the normal distribution
   - It is a **bell-shaped curve** that models how data tends to **group around the mean** μ, with **probabilities tapering off symmetrically** as they move further away from the mean, in **standard deviations** σ
   - ★ $X \sim N(\mu, \sigma^2)$
     - ➢ $kX \sim N(k\mu\,,\ k^2\sigma^2)$
     - ➢ $aX \pm b \sim N(a\mu \pm b\,,\ a^2\sigma^2)$
     - ➢ $aX \pm bY \sim (a\mu_X \pm b\mu_Y\,,\ a^2(\sigma_X)^2 + b^2(\sigma_Y)^2)$
     - ➢ $(X_1 + X_2 + X_3 + \ \dots\ + X_n) \sim N(n\mu, n\sigma^2)$

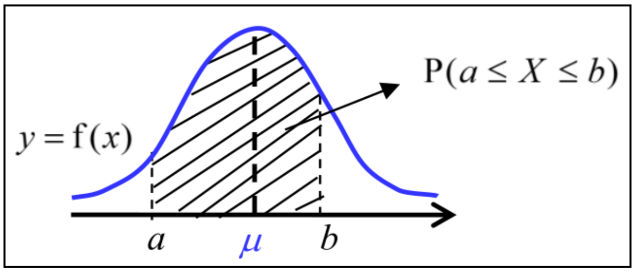
3. Its P.D.F is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\ ,\ x \in R$
   - ➢ Use the $normalpdf()$ function in the GC to sketch the curve



- ★ It's **symmetrical** about the line $x = \mu$
- ★ Its **non-stationary point of inflexions** are at $x = \mu \pm \sigma$
- ★ μ = **mean** = **median** = **mode**

B - Properties

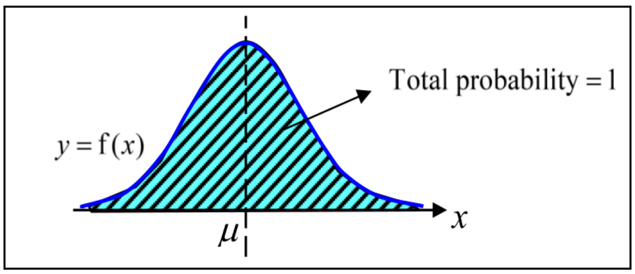
- The **area under the curve** gives the probability **between 2 bounds** a and b
  - $P(u - k\sigma < X < u + k\sigma)$ gives the probability than X is within k standard deviations from the mean
  - ★ The **signs** of the comparison operators **do not** matter



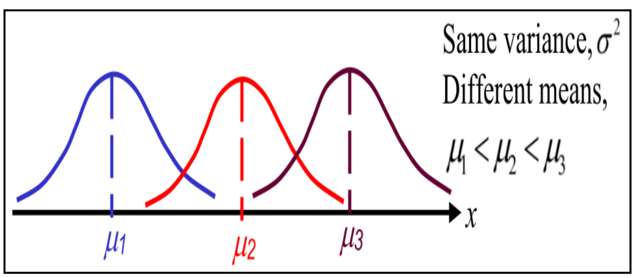
- To find it, use the $normalcdf\ (a\ ,\ b\ ,\ \mu\ ,\ \sigma)$ function

---

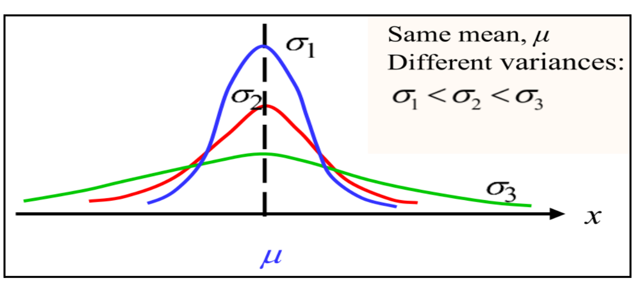
- The **total area under the curve** is 1 as it represents the sum of all probabilities



- The **mean** affects the **location** of the curve



- The **variance** affects the **narrowness** of the curve
  - **Smaller** variances mean the **probabilities** are **nearer** to the **mean**, hence leading to a **smaller spread** and a **narrower curve**
  - **Larger** variances mean the **probabilities** are **further away** from the **mean**, hence leading to a **bigger spread** and a **wider curve**

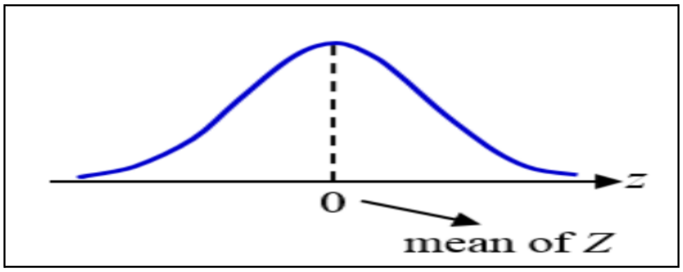


C - Standardization

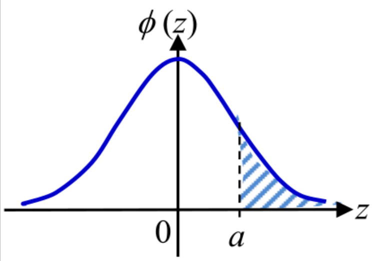
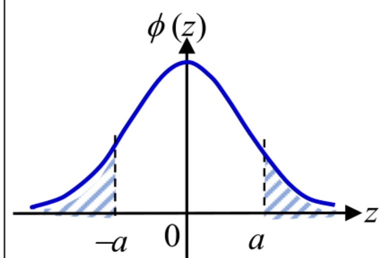
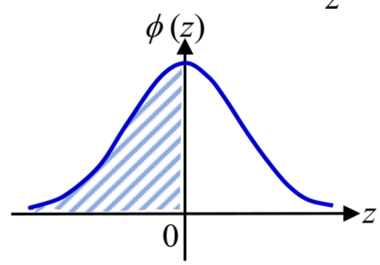
1 - Definition

- It's the **conversion** of a **general** normal distribution curve of mean μ, variance σ^2, and is symmetrical to the line $X = \mu$, to a **standard** normal curve of mean 0, variance 1, and is symmetrical about the line $z = \mu = 0$

★ $Z = \frac{X-\mu}{\sigma}$, $Z \sim N(0, 1)$



B - Properties

(a) $\mathrm{P}(Z>a)=1-\mathrm{P}(Z<a)$ 	(b) $\mathrm{P}(Z>a)=\mathrm{P}(Z<-a)$ 	(c) $\mathrm{P}(Z<0)=\mathrm{P}(Z>0)=\frac{1}{2}$ 

C - Finding The Boundary

- If $P(X \leq a) = p$ is given
  - ★ $invNorm(p, \mu, \sigma, LEFT)$
- If $P(X \geq a) = p$ is given
  - ★ $invNorm(p, \mu, \sigma, RIGHT)$
- If $P(\mu - a \leq X \leq \mu + a) = p$ is given
  - ★ $invNorm(p, \mu, \sigma, CENTER)$
  - ★ If **another boundary** b is involved, it will **need to be rewritten** to fit the first 2 forms

$$\mathrm{P}(15<X<b)=0.842$$
$$\mathrm{P}(X<b)-P(X\leq 15)=0.842$$
$$\mathrm{P}(X<b)=0.842+P(X\leq 15)$$
$$=0.842+0.11507$$
$$=0.95707$$

From GC, $b = 22.3$ (3 s.f.)

5 - Sampling

A - Definition

- Sampling is used to **estimate population parameters**

- Taking **random samples** from a **population** makes it easier to conduct analysis as the **population** could be too big, expensive, or time consuming to do a thorough analysis, hence making it impossible. Also, sampling can also protect the population when measurement or testing could be destructive.

- It is mostly **used** for **random variables that are normally distributed** for **any sample size** n

- It can also be used for **approximating random variables that are not normally distributed** if its **sample size** n is **greater than or equal to 30**, via the Central Limit Theorem

---

- Population
  - It's the **entire set of possible observations** of a random variable X

- Sample
  - It's a **portion of the population**, consisting of n i**ndependent observations** of the random variable X

- Random Sample
  - A **sample** drawn from the population is said to be **random** if each element of the population
    - Has an **equal chance** of being selected
    - Each selection is **independent** of other selections

- Unbiased Estimator
  - The **random variable** T used to **estimate** the unknown **population parameter** θ
    - **Examples** of population parameters are **population mean** μ and **population variance** σ^2
  - T is said to be **unbiased** if $E(T) = \theta$
  - Denoted using capital letters
    - $\bar{X}$ is the unbiased estimator for the population μ

- Unbiased Estimate
  - The **numerical output** from its corresponding **unbiased estimator** for a **particular sample**
  - Denoted using small letters
    - $\bar{x}$ is an unbiased estimate of the population mean μ

B - Formulae

1 - Sum

- $T = \sum_{i=1}^{n} X_i$
  - The distribution of T is the **sampling distribution of the sample sum**
    - $T \sim N(n\mu, n\sigma^2)$

---

2 - Expectation

- $\bar{X} = \frac{Sum}{n}$
  - $\bar{X}$ is also known as the **unbiased estimate** of the population mean μ
    - $E(\bar{X}) = \mu$
  - The distribution of $\bar{X}$ is the **sampling distribution of the sample expectation**
    - $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

---

3 - Variance

★ Biased

- $Var(\bar{X}) = \frac{\sigma^2}{n}$
- $Var(\bar{X}) = \frac{\sum(x-\bar{x})^2}{n}$

★ Unbiased **[MF27]**

- $s^2 = \frac{n}{n-1}\left(\frac{\sum(x-\bar{x})^2}{n}\right)$

6 - Hypothesis Testing

A - Definition

- It is a formal statistical procedure in which a dataset is used to evaluate the **null** hypothesis H_0 against the **alternative** hypothesis H_1, for **normal** distributions for $n \in Z^+$ OR for **non-normal** distributions for $n > 30$
- The **decision** to **reject** or **not to reject** H_0 is based on comparing the **p-value** to a pre-specified **significance level** α and seeing if it is **smaller** or **greater** than it

---

  - H_0 is the the **current claim** being tested, and is stated in terms of the population parameter
  - H_1 is the **claim** that we are **trying to find evidence for**

  - A **one-tail** hypothesis test is to check if there is an **increase**/**decrease** in the population mean μ, hence μ has been **underestimated**/**overestimated**

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0\,,\ \mu < \mu_0$$

  - A **two-tail** hypothesis test is to check if there is a **change** in the population mean

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

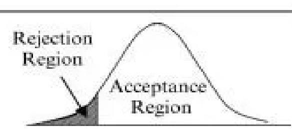
  - α is the probability of incorrectly rejecting a true H_0

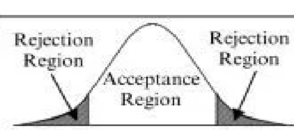
  - It separates the **critical** (rejection) region and the **non-critical** (acceptance) region
    - The **critical** region represents the **outcomes** that are **very unlikely**
    - The **non-critical** region represents the **outcomes** that are **very likely**

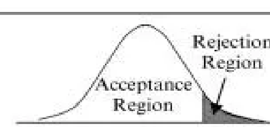
  - The **smaller** the **significance level**, the **stricter** the **test** becomes as the critical region becomes smaller, making a test result less likely to fall into that region

  - Hence, the stronger the evidence must be to reject H_0 as a very low bar has been set for the probability of the sample results to occur by chance

One-Tailed Test (Left Tail)	Two-Tailed Test	One-Tailed Test (Right Tail)
$H_0 : \mu_X = \mu_0$ $H_1 : \mu_X < \mu_0$	$H_0 : \mu_X = \mu_0$ $H_1 : \mu_X \neq \mu_0$	$H_0 : \mu_X = \mu_0$ $H_1 : \mu_X > \mu_0$







- **p-value** is the probability of getting an outcome that is abnormal, assuming that H_0 is true
  - If it is **smaller** than a
    - It falls **inside** of the **critical** region
    - It means that **getting** this **outcome** would be **rare**
    - It indicates that the **observed data** is **unlikely** under H_0
    - This is **strong** evidence against H_0 , so **reject** it
  - If they are bigger than a
    - It falls **outside** of the **critical** region
    - It means that if H_0 g**etting** this **outcome** would be **common**
    - It indicates that the **observed data** is **consistent** under H_0
    - This is **weak** evidence against H_0 , so **do not reject** it

B - Method

1. Define the random variable X and any symbols used in the context (if necessary)

2. State H_0 and H_1

3. Specify the significance level a

4. Assuming that H_0 is true, determine the distribution $\bar{X}$, and indicate if the Central Limit Theorem was used

5. Calculate the $p - value$ via the $Z - Test$ function in the GC and compare it with a

   - Select $<$ OR $>$ to do a **one-tail** hypothesis test
   - Select $\neq$ for a **two-tail** hypothesis test

   - If $p - value \leq significance\ level$
     i. Reject H_0
   - If $p - value > significance\ level$
     i. Accept H_0

6. State the conclusion in the context of the question

C - Example

The lengths of metal bars produced by a particular machine are **normally distributed** with **mean** length **420 cm** and **standard deviation 12 cm**. The machine is serviced, after which a **sample of 100** bars gives a **mean** length of **423 cm**. Test, at the **5% significance level**, whether there is a **change** in the **mean** length of the bars produced by the machine, **assuming** that the **standard deviation** remains the **same**.

---

1 - Define the random variable X and any symbols used in the context (if necessary)

Let X be the random variable of the length, in cm, of a randomly chosen metal bar and μ cm be the population mean length of a metal bar.

$$X \sim N(420, 12^2)$$

---

2 - State H_0 and H_1

$$H_0 = 420$$
$$H_1 \neq 420$$

H_0 : There is no change to the mean length

H_1 : There is a change to the mean length

The goal is to find out if there is sufficient evidence to reject H_0 and accept H_1 by seeing if the **p-value** is **smaller** than the **significance level**

---

3 - Specify the significance level a

Testing at the 5% significance level

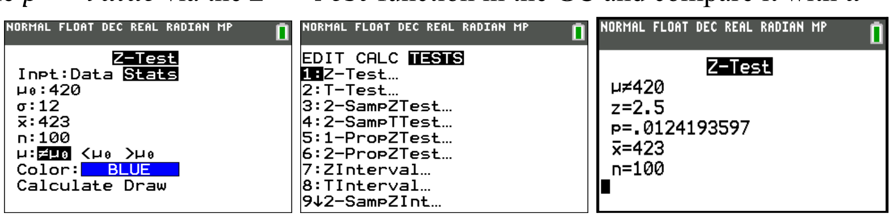
---

4 - Assuming that H_0 is true, determine the distribution $\bar{X}$, and indicate if the Central Limit Theorem was used

Under H_0, $\bar{X} \sim (420, \frac{12^2}{100})$

---

5 - Calculate the $p - value$ via the $Z - Test$ function in the GC and compare it with a



From the GC, p-value = 0.0124 < 0.05

---

6 - State the conclusion in the context of the question

Reject H_0 and conclude that there is sufficient evidence, at the 5% significance level, that there is a change in the mean length of the bars produced by the machine

7 - Correlation & Regression

A - Definition

- It is a **procedure** to **determine** if **2 variables** are **related** via the equation $y = a + bx$
    - **Positive** linear correlation moves from **bottom** to **top**
    - **Negative** linear correlation moves from **top** to **bottom**

    - **Curvilinear** Correlation **moves** as a **curve**
    - **No** correlation means there is no relationship, even a linear one, between x and y

- **Interpolation** is **estimating within** the data range
- **Extrapolation** is **estimating outside** of the data range
    - This should be done with caution as the relationship between x and y may not be linear outside of the given data range

- a is the $y - intercept$
- b is the $gradient$

B - Conversion To A Linear Equation

- To convert a non-linear equation to a linear equation

$$y = a + bx^2$$

- Assign the non-linear variable to a linear variable

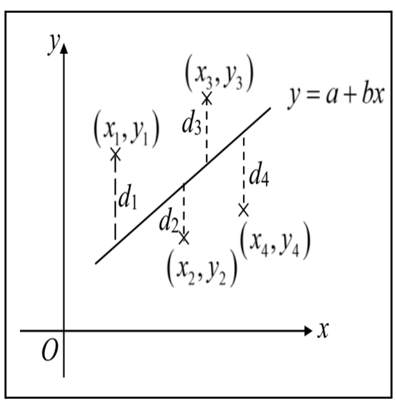
$$Let\ X = x^2$$

- Substitute it into the original equation

$$\therefore y = a + bX\ ,\ X = x^2$$

C - Regression Line

- It is a **line of best fit** when the sum of squares of the distances between the data points and the regression line is minimised



(x_n, y_n) are the data points

d_n are the vertical deviation from the regression line

D - PMCC

1 - Definition

- The **Product Moment Correlation Coefficient** measures the **strength** of the **linear relationship** between x and y

---

2 - Properties

- $-1 \leq r \leq 1$
  - $-\ ve$ sign : Negative Gradient
  - $+\ ve$ sign : Positive Gradient

- $r = \pm 1$: **All** data points **lie** exactly on a straight **line**
  - $r = -1$ **:** Perfect $-\ ve$ linear correlation
  - $r = 1$: Perfect $+\ ve$ linear correlation

- The **closer** r is to ± 1 , the **stronger** the **linear correlation**

- $r = 0$: No **Linear** Correlation

- r does not imply any causal link between x and y
  - If $r = 0.8$ for the sales of ice cream at beaches and the number of drowning incidents at the same beach in a particular period, it does not mean that buying ice cream will cause drowning incidents

---

3 - Factors For A Reliable Estimate

★ Interpolation was done

★ r is close to ± 1

D - Derivation

1. Within the $STAT$ menu in the GC
    - $L1$: x data
    - $L2$: y data

2. Calculate the required values using $LinReg(a + bx)$
    - $XList$: x data
    - $YList$: y data
    - $FreqList$: NA
    - $Store\ RegEQ$: NA

3. To find the **regression line** of y on x
    - $y = \{a\} + \{b\}x$

4. To find the **regression line** of x on y
    - $x = \{a\} + \{b\}y$
    - Make y the subject
        - $y = \frac{x-\{a\}}{\{b\}}$

5. To find the **PMCC**
    - Look at the value of r
    - Ensure $STAT\ DIAGNOSTICS$ is ON under $[MODE]$

- To get the **scatter plot**
    - Press $[Stat\ Plot]$ under $STAT$
    - Select $Plot$ 1 and turn it on
    - Input the required values
        - $Type$: 1^{st} option
        - $XList$: x data
        - $YList$: y data
        - $Mark$: $+$

E - Formulae

1 - Regression Line

- ★ A - Regression Line Of y on x
  - ➢ $y - \bar{y} = b(x - \bar{x})$, $b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$ **[MF27]**
- ★ B - Regression Line of x on y
  - ➢ $x - \bar{x} = d(y - \bar{y})$, $d = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$
- ★ Point Of Intersection of A and B
  - ○ $(\bar{x}, \bar{y})$, $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

---

2 - PMCC

Estimated ***product moment correlation coefficient*** for a sample, denoted by ***r***, is given by:

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\sum(x-\bar{x})^2\right\}\left\{\sum(y-\bar{y})^2\right\}}} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{n}\right)\left(\sum y^2 - \frac{(\sum y)^2}{n}\right)}}$$

(n is the sample size.)

- Use only when the raw data is not given

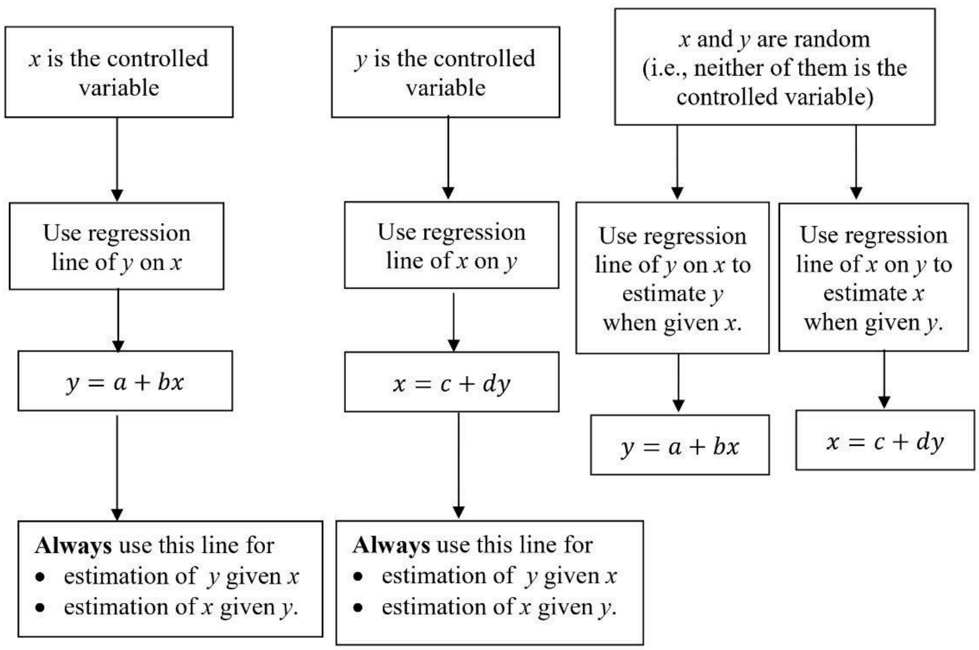
F - Chain Of Thoughts

3.5 Prediction/Estimation of Values

Video 20.6

In estimating the value of one variable when given the value of the other variable, a suitable regression line has to be chosen based on the following criteria.

A variable is said to be a controlled variable if it can be held constant during an experiment or research.



We use:

- The **regression line of y on x** if x is the controlled variable.
  (i) To estimate y when given x;
  (ii) To estimate x when given y.

- The **regression line of x on y** if y is the controlled variable.
  (i) To estimate y when given x;
  (ii) To estimate x when given y.

If **both variables are *random***, i.e., neither of them is the controlled variable, then we use

(i) the regression line of y on x to estimate y when given x,
(ii) the regression line of x on y to estimate x when given y.