STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	4	-	7	1	1	3

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2024

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 3

Wednesday

21 August 2024

1 hr

1300 – 1400 hrs/
1630 – 1730 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.
- Number of printed pages = **5.**

FOR MARKER USE ONLY:

Q1	Q2	TOTAL
		/55

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

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1. [Maximum mark: 24]

In this question, you will be exploring the Steiner inellipse, which is a unique ellipse that is tangential to the midpoints of a triangle.

Consider the cubic polynomial $p(z) = z^3 + z^2 - 2$.

It is given that z_1, z_2 and z_3 are the roots of the equation $p(z) = 0$, such that $-\pi < \arg z_1 < \arg z_2 < \arg z_3 < \pi$. In an Argand diagram, points P, Q and R represent the complex numbers z_1, z_2 and z_3 respectively.

(a) (i) Find z_1, z_2 and z_3.

(ii) Indicate P, Q and R on the Argand diagram. [5]

P, Q and R are joined together to form the triangle PQR. It is given that A is the midpoint of PR, B is the midpoint of QR, and C is the midpoint of PQ.

(b) Indicate A, B and C clearly on the same Argand diagram. [1]

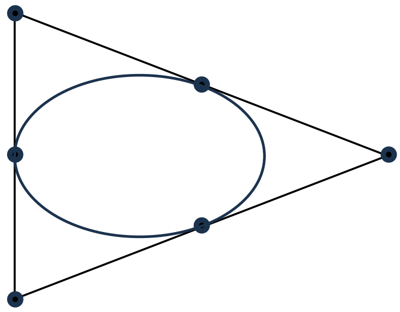
Let z_4 and z_5 be the roots of $p'(z) = 0$. Points H and K represent z_4 and z_5 respectively such that $|\mathrm{AH}| < |\mathrm{AK}|$.

(c) (i) Solve $p'(z) = 0$.

(ii) Indicate H and K on the same Argand diagram. [4]

Now, treat the Argand diagram as the real Cartesian *x*-*y* plane.

Marden's theorem states that the coordinates of H and K form the foci of the unique ellipse which is tangential to the triangle PQR at its midpoints A, B and C. This ellipse, known as the Steiner inellipse of triangle PQR, is illustrated in the following diagram.



(d) Find the coordinates of the midpoints A, B and C in your diagram. [2]

When H and K have coordinates (h,c) and (k,c), the geometrical centre (or centroid) of the ellipse is (m,c), where $m=\dfrac{h+k}{2}$.

The Steiner inellipse is then given by the following equation:

$$\frac{(x-m)^2}{a^2}+\frac{(y-c)^2}{b^2}=1, \text{ where } a^2-b^2=m^2.$$

(e) (i) Show that $a^2=\dfrac{4}{9}$.

(ii) Find the equation of this ellipse in the form above. [6]

(f) By considering a transformation of the equation in **part (e)(ii)** or otherwise, show that the equation of a Steiner inellipse of a triangle with vertices $\mathrm{X}(-1,-2)$, $\mathrm{Y}(1,0)$ and $\mathrm{Z}(-1,2)$ is given by:

$$3\left(x+\frac{1}{3}\right)^2+y^2=\frac{4}{3}$$ [2]

(g) Find the gradient of the ellipse in **part (f)** at the points $(0,1)$ and $(0,-1)$.

Deduce $\angle\mathrm{XYZ}$. [4]

[Turn Over]

2. [Maximum mark: 31]

In this question, you will be investigating Euler's and modified Euler's method.

The function $y = y(x)$, where $x > 0$, $y > 0$, satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y \quad -(*)$$

and $y(x_0) = y_0, \quad x \geq x_0$.

(a) Find the exact solution $y(x)$ in terms of x, x_0 and y_0. [4]

(b) Using **Euler's method** with step size $\frac{x - x_0}{n}$, where $n \in \mathbb{Z}^+$,

show that the estimate of $y(x)$ after n steps is given by

$$y_n = \left[1 + 2\left(\frac{x - x_0}{n}\right)\right]^n y_0.$$ [2]

Let $v = \left(1 + \frac{w}{n}\right)^n$.

(c) By considering $\ln v = n \ln\left(1 + \frac{w}{n}\right)$, use l'Hôpital's rule to find $\lim\limits_{n \to \infty} \ln v$. [4]

(d) Deduce that $\left(1 + \frac{w}{n}\right)^n \to \mathrm{e}^w$ as $n \to \infty$. [1]

(e) By considering the result obtained in **part (d)**, show that the Euler's estimate converges to the exact solution $y(x)$ in **part (a).** [2]

Modified Euler's method is given by $y_n = y_{n-1} + \frac{h}{2}\left[\mathrm{f}(x_{n-1}, y_{n-1}) + \mathrm{f}(x_n, y_n)\right]$,

where y_n is the value of y when $x = x_n$, $\mathrm{f}(x, y) = \frac{\mathrm{d}y}{\mathrm{d}x}$, and h is the step size.

(f) Using **modified Euler's method** with step size $h = \frac{x - x_0}{n}$, where $n \in \mathbb{Z}^+$,

show that the estimate of $y(x)$ after n steps is given by

$$y_n = \left(\frac{1 + \frac{x - x_0}{n}}{1 - \frac{x - x_0}{n}}\right)^n y_0.$$ [3]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y \quad -(*)$ for which $y = 1$ when $x = 0$.

(g) Find the exact value of of y when $x = 0.5$. [2]

(h) Without computing the approximate value of y, explain why the Euler's method in **part (b)** gives an under-estimation of the true value of y when $x = 0.5$. [1]

(i) Let $h = 0.1$.

Find an approximate value of y when $x = 0.5$ using

(i) the result obtained in **part (b)**;

(ii) the result obtained in **part (f)**.

[Leave your answers to 3 decimal places.]

State which of the methods: the Euler's method or the modified Euler's method, gives a better estimate of e . [4]

(j) Determine the minimum value of n such that $|\, y_n - \mathrm{e} \,| < 10^{-3}$ when $x = 0.5$ for Euler's method and modified Euler's method respectively. [8]

End of Paper