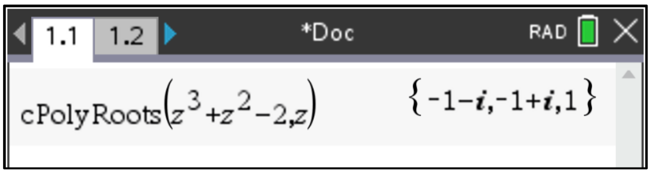
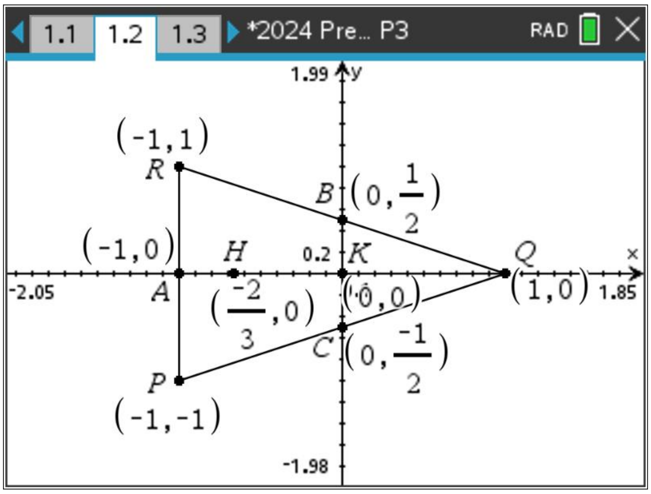
Qn	Suggested Solutions	Marks
1	**Complex polynomials, conics, calculus**	**[Max mark: 24]**
(a)(i)	Solve $p(z)=z^3+z^2-2=0$ **<u>Method 1: GDC</u>**  **<u>Method 2: Analytical</u>** $p(1)=1+1-2=0,\ \therefore (z-1)$ is a factor. $p(z)=(z-1)(z^2+2z+2)$ (by inspection/long division etc) $\therefore p(z)=0$ $\Rightarrow z=1 \quad \text{or} \quad z=\dfrac{-2\pm\sqrt{2^2-4(1)(2)}}{2}$ $=-1\pm \text{i}$ Since $-\pi<\arg z_1<\arg z_2<\arg z_3<\pi$, $z_1=-1-\text{i}$, $z_2=1$, $z_3=-1+\text{i}$.	(M1) **OR** (M1) A1 A1 A1
(a)(ii) **(b)** **(c)(ii)**		(a)(ii) A1 (label P, Q, R in the correct order) (b) A1 (label A, B, C) (c)(ii) A1 (label H, K)
(c)(i)	$p(z)=z^3+z^2-2$ $p'(z)=3z^2+2z$ Solving $p'(z)=0$, $z(3z+2)=0$ $\therefore z=0 \quad \text{or} \quad z=-\dfrac{2}{3}$	M1 A1 A1
(d)	Coordinates of the midpoints: $\text{A}(-1,0),\text{B}\left(0,\dfrac{1}{2}\right),\text{C}\left(0,-\dfrac{1}{2}\right)$	(M1)A1 (or labelled in diagram)

(e)(i)	Since $(h,c)=\left(-\frac{2}{3},0\right)$ and $(k,c)=(0,0)$, the centroid $(m,c)=\left(\frac{1}{2}\left(-\frac{2}{3}+0\right),0\right)=\left(-\frac{1}{3},0\right)$. Therefore, the equation of the ellipse is $\frac{\left(x+\frac{1}{3}\right)^2}{a^2}+\frac{y^2}{b^2}=1$ .... (*) Subst $\mathrm{A}(-1,0)$ into (*), we have $\frac{\left(-\frac{2}{3}\right)^2}{a^2}+0=1$ $\Rightarrow \quad a^2=\frac{4}{9}\ (\text{shown})$ **OR** The ellipse's semi-major axis, $a=\mathrm{AM}=\frac{2}{3}$, where M denotes the centroid. $\therefore a^2=\frac{4}{9}$.	A1 (seen anywhere) M1 (subst $(-1,0)$) A1 AG **OR** M1A1 ($a=\mathrm{AM}=\frac{2}{3}$) AG
(ii)	**<u>Method 1: Using a 2nd set of coordinates</u>** Subst $\left(0,\frac{1}{2}\right)$ into (*), we have $\frac{\left(\frac{1}{3}\right)^2}{\frac{4}{9}}+\frac{\left(\frac{1}{2}\right)^2}{b^2}=1 \quad \Rightarrow \quad \frac{1}{4}+\frac{1}{4b^2}=1 \quad \Rightarrow \quad b^2=\frac{1}{3}$ **<u>Method 2: Using $\boldsymbol{a^2-b^2=m^2}$</u>** Subst $a^2=\frac{4}{9}$, $m=\frac{1}{3}$ from above, we have $b^2=a^2-m^2$ $\quad =\frac{4}{9}-\left(\frac{1}{3}\right)^2=\frac{1}{3}$ *By any method, or a more tedious sequence of substitutions:* Solving, $a^2=\frac{4}{9}$, $b^2=\frac{1}{3}$. (*There is no need to find* a,b.) Hence, the equation of the ellipse is $\frac{\left(x+\frac{1}{3}\right)^2}{\frac{4}{9}}+\frac{y^2}{\frac{1}{3}}=1$ (accept equivalent forms)	 M1 (subst another coordinates A1 **OR** M1 (using $a^2-b^2=m^2$) A1 A1

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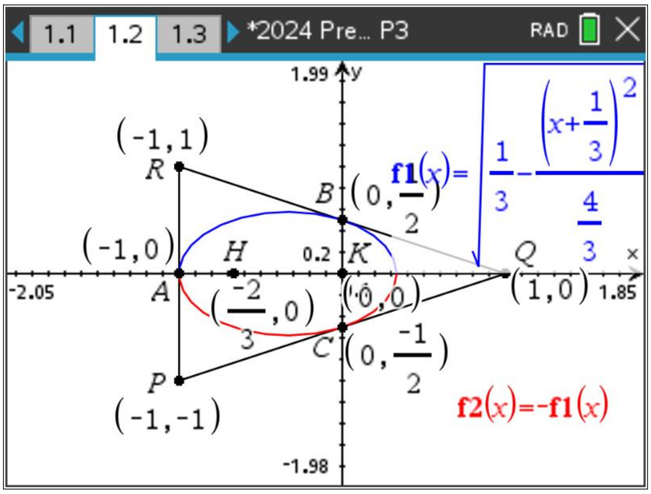

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<td>(Sketch of the ellipse – not required but always good to check answer, time permitting)</td>
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<td>(f)</td>
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By functions transformation, observe that the vertices are formed by a vertical stretch of factor 2.

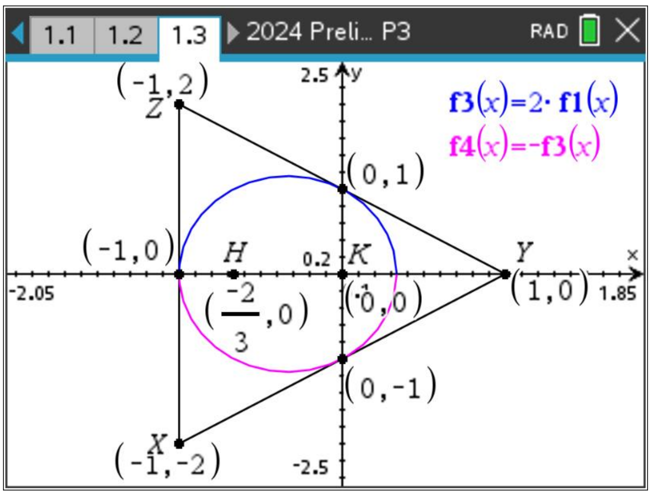
Original ellipse: $$\frac{\left(x+\frac{1}{3}\right)^2}{\frac{4}{9}}+\frac{y^2}{\frac{1}{3}}=1$$

For a vertical stretch of factor 2,

Replacing y by $\frac{1}{2}y$: $$\frac{\left(x+\frac{1}{3}\right)^2}{\frac{4}{9}}+\frac{\left(\frac{1}{2}y\right)^2}{\frac{1}{3}}=1$$

That is: $$\frac{\left(x+\frac{1}{3}\right)^2}{\frac{4}{9}}+\frac{y^2}{\frac{4}{3}}=1$$

Simplifying: $3\left(x+\frac{1}{3}\right)^2+y^2=\frac{4}{3}$ (shown)



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<td>(M1) Attempt to do a vertical stretch

A1

AG

(Sketch confirms the vertical stretch of factor 2, time permitting)</td>
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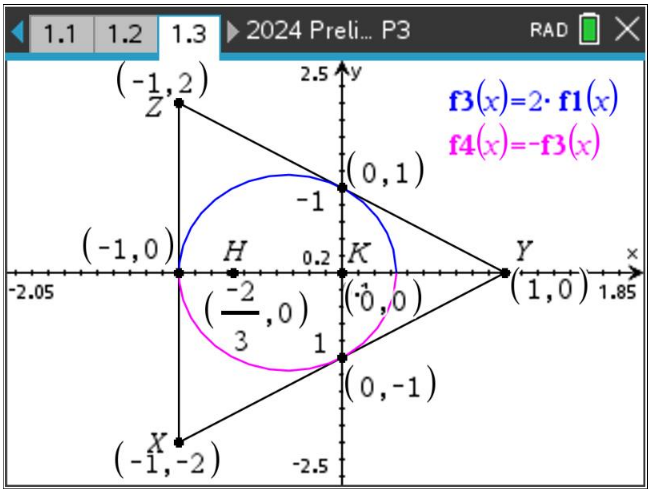
(g)	**<u>Method 1: Finding the gradient of the tangent</u>**	
		M1 A1 A1 **OR**
	Observe that ZY is a tangent to the ellipse at midpoint $(0,1)$.	M1
	Gradient at $(0,1)$ $=m_{ZY}=\dfrac{0-2}{1-(-1)}=-1$.	A1
	Similarly, gradient at $(0,-1)$ $=m_{XY}=\dfrac{0-(-2)}{1-(-1)}=1$.	A1
	<u>Method 2: GDC graph using gradient of appropriate functions</u>	**OR**
	$3\left(x+\dfrac{1}{3}\right)^2+y^2=\dfrac{4}{3}$ $y=\pm\sqrt{\dfrac{4}{3}-3\left(x+\dfrac{1}{3}\right)^2}$	M1
	From GDC, numerical derivative, $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\|_{(0,1)}=-1$	A1
	$\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\|_{(0,-1)}=1$	A1
	$\dfrac{d}{dx}\left(\sqrt{\dfrac{4}{3}-3\cdot\left(x+\dfrac{1}{3}\right)^2}\right)\|x=0 \qquad -1$ $\dfrac{d}{dx}\left(-\sqrt{\dfrac{4}{3}-3\cdot\left(x+\dfrac{1}{3}\right)^2}\right)\|x=0 \qquad 1$	

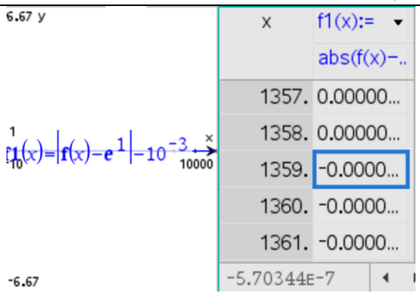
	Method 3: Implicit Differentiation	**OR**
	Differentiating implicitly w.r.t. x, $\frac{\mathrm{d}}{\mathrm{d}x}\left(3\left(x+\frac{1}{3}\right)^2+y^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4}{3}\right)$	M1
	$6\left(x+\frac{1}{3}\right)+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{y}\left(x+\frac{1}{3}\right)$	
	$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{(0,1)}=-\frac{3}{1}\left(0+\frac{1}{3}\right)=-1$, $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{(0,-1)}=-\frac{3}{-1}\left(0+\frac{1}{3}\right)=1$	A1A1
	By whichever method: Since $m_{\mathrm{XY}}\cdot m_{\mathrm{ZX}}=(1)(-1)=-1$, the tangents are perpendicular.	A1
	OR	**OR**
	$m_{\mathrm{XY}}=\ 1 \quad \Rightarrow \quad \angle\mathrm{XYO}=45°$ $m_{\mathrm{ZY}}=-1 \quad \Rightarrow \quad \angle\mathrm{ZYO}=45°$ Hence, $\angle\mathrm{XYZ}=45°+45°=90°$.	A1

2	**DE, Euler's and modified Euler's method**	**[Max mark: 31]**
(a)	$\int_{y_0}^{y} \frac{1}{y}\, \mathrm{d}y = \int_{x_0}^{x} 2\, \mathrm{d}x$ $\Rightarrow \ln y - \ln y_0 = 2(x - x_0)$ $\quad$ $\left[\text{No need for } \ln\lvert y\rvert \text{ as } y > 0\right]$ $\Rightarrow \ln \frac{y}{y_0} = 2(x - x_0)$ $\Rightarrow \frac{y}{y_0} = \mathrm{e}^{2(x - x_0)}$ $\Rightarrow y = y_0\, \mathrm{e}^{2(x - x_0)}$	M1A1 M1 A1
	Alternatively, $\int \frac{1}{y}\, \mathrm{d}y = \int 2\, \mathrm{d}x$ $\Rightarrow \ln y = 2x + c$ $\quad$ $\left[\text{No need for } \ln\lvert y\rvert \text{ as } y > 0\right]$ $\Rightarrow y = \mathrm{e}^{2x+c} = \mathrm{e}^{c}\, \mathrm{e}^{2x} = A\mathrm{e}^{2x}$ At (x_0, y_0), $y_0 = A\mathrm{e}^{2x_0}$ $A = y_0 \mathrm{e}^{-2x_0}$ $\Rightarrow y = y_0\, \mathrm{e}^{2(x - x_0)}$	M1 A1 M1 A1
(b)	Apply Euler's method: $y_n = y_{n-1} + h\mathrm{f}(x_{n-1}, y_{n-1})$ with $\mathrm{f}(x, y) = 2y,\ h = \frac{x - x_0}{n}$ $y_n = y_{n-1} + h(2y_{n-1})$ $= (1 + 2h)y_{n-1}$ $= (1 + 2h)^2 y_{n-2}$ $\vdots$ $= (1 + 2h)^n y_0$ i.e $y_n = \left[1 + 2\left(\frac{x - x_0}{n}\right)\right]^n y_0$ $\quad$ (shown)	M1 (apply Euler's method with correct $\mathrm{f}(x_{n-1}, y_{n-1})$) A1 $(1 + 2h)y_{n-1}$ AG
	Alternatively, $y_1 = y_0 + h(2y_0) = (1 + 2h)y_0$ $y_2 = y_1 + h(2y_1) = (1 + 2h)y_0 + 2h(1 + 2h)y_0 = (1 + 2h)(1 + 2h)y_0 = (1 + 2h)^2 y_0$ $\vdots$ $y_n = (1 + 2h)^n y_0$ i.e $y_n = \left[1 + 2\left(\frac{x - x_0}{n}\right)\right]^n y_0$ $\quad$ (shown)	M1 A1 AG

(c)	$\ln v = n\ln\left(1+\frac{w}{n}\right) = \frac{\ln\left(1+\frac{w}{n}\right)}{\frac{1}{n}}$	M1
	$\lim_{n\to\infty}\ln v = \lim_{n\to\infty}\frac{\ln\left(1+\frac{w}{n}\right)}{\frac{1}{n}} \qquad \left(\frac{0}{0}\right)$	M1
	$= \lim_{n\to\infty}\frac{\left(\frac{1}{1+\frac{w}{n}}\right)\left(-\frac{w}{n^2}\right)}{-\frac{1}{n^2}}$	A1
	$= \lim_{n\to\infty}\left(\frac{w}{1+\frac{w}{n}}\right)$ $= w$	A1
(d)	$\lim_{n\to\infty}\ln v = w \Rightarrow \lim_{n\to\infty} v = \mathrm{e}^w$ Since $v=\left(1+\frac{w}{n}\right)^n$, i.e. $\lim_{n\to\infty}\left(1+\frac{w}{n}\right)^n = \mathrm{e}^w$ i.e. $\left(1+\frac{w}{n}\right)^n \to \mathrm{e}^w$ as $n\to\infty$. (deduced)	R1 AG
(e)	Let $w=2(x-x_0)$. $\therefore \lim_{n\to\infty}\left(1+\frac{2(x-x_0)}{n}\right)^n = \mathrm{e}^{2(x-x_0)}$	A1
	Since $y_n = \left[1+2\left(\frac{x-x_0}{n}\right)\right]^n y_0$, hence, $\lim_{n\to\infty} y_n = \mathrm{e}^{2(x-x_0)}\, y_0 = y(x)$ (shown)	R1

(f)	Apply the Modified Euler's Method: $y_n = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n)\right]$ with $f(x,y) = 2y,\ h = \frac{x - x_0}{n}$ $y_n = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n)\right]$ $y_n = y_{n-1} + \frac{h}{2}(2y_{n-1} + 2y_n)$ $= (1+h)y_{n-1} + hy_n$ $(1-h)y_n = (1+h)y_{n-1}$ $y_n = \left(\frac{1+h}{1-h}\right)y_{n-1}$ $y_{n-1} = y_{n-2} + \frac{h}{2}(2y_{n-2} + 2y_{n-1})$ $= (1+h)y_{n-2} + h\,y_{n-1}$ $(1-h)y_{n-1} = (1+h)y_{n-2}$ $y_{n-1} = \left(\frac{1+h}{1-h}\right)y_{n-2}$ $y_n = \left(\frac{1+h}{1-h}\right)y_{n-1} = \left(\frac{1+h}{1-h}\right)^2 y_{n-2}$ $\vdots$ $y_n = \left(\frac{1+h}{1-h}\right)^n y_0$ i.e. $y_n = \left(\dfrac{1+\frac{x-x_0}{n}}{1-\frac{x-x_0}{n}}\right)^n y_0$ (shown)	M1 (with correct $f(x_{n-1}, y_{n-1}) = 2y_{n-1},$ $f(x_n, y_n) = 2y_n$) A1 M1 AG
(g)	$\frac{dy}{dx} = 2y$ $-(*)$ for which $y=1$ when $x=0$. When $x_0 = 0,\ y_0 = 1$, the exact solution is $y = y_0\,e^{2(x-x_0)}$ i.e. $y = e^{2x}$ when $x = 0.5, y = e$.	M1 A1
(h)	The graph of $y = e^{2x}$ is concave upwards. OR $\frac{d^2y}{dx^2} > 0$ OR $\frac{dy}{dx}$ increases Hence the tangent drawn using Euler's method gives an <u>under-estimate</u> of the true value of y.	R1

(i)	If $h=0.1$, then $n=5$. **(i) <u>Euler's Method</u>** $y_n=\left[1+2\left(\frac{0.5-0}{n}\right)\right]^n=\left(1+\frac{1}{n}\right)^n$ $y_5=1.2^5=2.488\ (3\text{d.p.})$ $(1.2)^5$ 2.48832 **(ii) <u>Modified Euler's Method</u>** $y_n=\left(\frac{1+\frac{x-x_0}{n}}{1-\frac{x-x_0}{n}}\right)^n y_0$. $y_n=\left(\frac{1+\frac{0.5-0}{n}}{1-\frac{0.5-0}{n}}\right)^n y_0 \Rightarrow y_n=\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right)^n$ $y_5=\left(\frac{1.1}{0.9}\right)^5=2.727\ (3\text{d.p.})$ $\left(\frac{1.1}{0.9}\right)^5$ 2.72741282664	M1 A1 A1
	Modified Euler's Method	A1
(j)	**<u>Euler's Method</u>** $y_n=\left[1+2\left(\frac{0.5-0}{n}\right)\right]^n=\left(1+\frac{1}{n}\right)^n$ $\lvert y_n-\text{e}\rvert<10^{-3}$ $\Rightarrow\left\lvert\left(1+\frac{1}{n}\right)^n-\text{e}\right\rvert<10^{-3}$ By GDC (table), When $n=1358$, $\left\lvert\left(1+\frac{1}{n}\right)^n-\text{e}\right\rvert-10^{-3}>0$ When $n=1359$, $\left\lvert\left(1+\frac{1}{n}\right)^n-\text{e}\right\rvert-10^{-3}<0$ Minimum $n=1359$ $f(x):=\left(1+\frac{1}{x}\right)^x$	M1 A1 M1 A1



Modified Euler's Method

$$y_n=\left(\frac{1+\frac{0.5-0}{n}}{1-\frac{0.5-0}{n}}\right)^n=\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right)^n$$ M1

$$|\,y_n-\mathrm{e}\,|<10^{-3}$$

$$\Rightarrow\left|\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right)^n-\mathrm{e}\right|<10^{-3}$$ A1

By GDC (table),

When $n=15$, $\left|\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right)^n-\mathrm{e}\right|-10^{-3}>0$ M1

When $n=16$, $\left|\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right)^n-\mathrm{e}\right|-10^{-3}<0$

Minimum $n=16$ A1

$$h(x):=\left(\frac{1+\frac{1}{2\cdot x}}{1-\frac{1}{2\cdot x}}\right)^x$$