STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	2	-	7	1	0	3

ORA et LABORA

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 3

Monday

22 August 2022

1 hour

0800 – 0900 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- Answer all the questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.
- This question paper consists of **4** printed pages including the cover sheet.

FOR MARKER USE ONLY:

Q1	Q2	TOTAL
		/55

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

....

1. [Maximum mark: 25]

In this question you will investigate the Gamma function and some of its properties using calculus and methods of proofs.

The Gamma function, $\Gamma(n)$, is defined as

$$\Gamma(n) = \int_0^\infty t^{n-1}\mathrm{e}^{-t}\,\mathrm{d}t,$$

for any number $n \in \mathbb{R}$ except non-positive integers. (E.g., $n \neq 0, -1, -2, \ldots$)

To approximate $\Gamma(n)$ using technology for some value of n, replace "∞" by "999999".

(a) Find the value of $\Gamma(1)$, of $\Gamma(2)$, of $\Gamma(3)$, of $\Gamma(4)$, of $\Gamma(5)$ and of $\Gamma(6)$. [3]

(b) Using a factorial, formulate a conjecture on the value of $\Gamma(n)$ **for any positive integer $\boldsymbol{n}$.** [1]

Consider $f(x) = x^n\mathrm{e}^{-x}$, **for any $\boldsymbol{n > 0}$.**

(c) By sketching the graph of $y = f(x)$, deduce what happens to $f(x)$ as $x \to \infty$. [2]

(d) Using integration by parts, show that $\Gamma(n+1) = n\,\Gamma(n)$, **for any $\boldsymbol{n > 0}$.** [3]

(e) Hence, using mathematical induction, prove your conjecture in (b). [5]

Consider $\Gamma\left(\frac{1}{2}\right)$.

(f) By using the substitution $t = \frac{1}{2}y^2$, express $\Gamma\left(\frac{1}{2}\right)$ as an integral in terms of y. [5]

The probability density function of the random variable $Z \sim \mathrm{N}(0,1)$ is given by

$$f(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}.$$

(g) Deduce, with justification, the value of $\displaystyle\int_0^\infty f(z)\,\mathrm{d}z$. [2]

(h) Hence, show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [2]

(i) Hence, find the exact value of $\Gamma\left(\frac{3}{2}\right)$. [2]

2. [Maximum mark: 30]

In this question you will investigate the geometrical properties of complex numbers and roots of unity.

Consider $z = 1 + \sqrt{2} + \mathrm{i}$, where $\mathrm{i}^2 = -1$.

(a) Find the exact value of z^2, leaving your answer in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. [2]

(b) Find the exact value of $\arg z^2$. [2]

(c) Find the exact value of $\arg z$. [2]

(d) Hence, find the exact value of $\tan\frac{\pi}{8}$. [2]

(e) For any two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$,

(i) show that $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

(ii) Hence, interpret $|z_1 - z_2|$ **geometrically**. [2]

Let $\omega = \frac{z}{|z|}$.

(f) Show that $\left|\omega^k\right| = 1$, for any integer k. [1]

(g) Sketch the 16 points representing the complex numbers ω^k, for $k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm7, 8$, in an Argand diagram. [2]

(h) The points in the Argand diagram represented by the complex numbers ω^{-5}, ω^{-1}, ω^3 and ω^7 form a square. Find the area of this square. [2]

(i) Find the exact value of $\left|\omega - \frac{1}{\omega}\right|$ and of $\left|\omega^2 - \frac{1}{\omega^2}\right|$. [5]

(j) Show that $\left|\omega - \frac{1}{\omega}\right|\left|\omega^3 - \frac{1}{\omega^3}\right| = \sqrt{2}$. [4]

(This question continues on the following page)

(Question 2 continued)

Define

$$\prod_{k=1}^{n} a_k = a_1 \times a_2 \times \cdots \times a_n.$$

(k) Copy and complete the table below: [6]

n	4	7	10
$\displaystyle\prod_{k=1}^{n} \left\lvert \omega^k - \frac{1}{\omega^k} \right\rvert =$			

End of Paper