STUDENT NAME: _____

TEACHER NAME: _____

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	2	-	7	1	0	2

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 2

Friday

19 August 2022

2 hours

1400 – 1600 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
  - **Section A:** Answer all questions showing working and answers in the spaces provided in the exam paper.
  - **Section B:** Answer all questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[110 marks]**.
- Number of printed pages = **14**.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/110

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (53 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 6]

The graph of the function $h(x) = \log_7(x-a)+b$ passes through the points $(0,1)$ and $\left(e^3, 1+\log_7 2\right)$.

Find the value of a and the value of b.

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2 [Maximum mark: 7]

For $\frac{\pi}{4} < x < \frac{3\pi}{4}$, find the value of x for which the sum to infinity

$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \cdots$$

equals to $\tan x$.

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3 [Maximum mark: 7]

The functions f and g are defined by

$$f(x) = \sqrt{-x^2 - 2x + 15}, \text{ for } -5 \le x \le -1$$

$$g(x) = -x + c, \text{ for } x, c \in \mathbb{R}.$$

(a) Find $f^{-1}(2)$ [4]

(b) Find the range of f. [1]

(c) Given that $(g \circ f)(x) \ge 0$ for $-5 \le x \le -1$, determine the set of possible values for c. [2]

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4 [Maximum mark: 6]

Alex and Bobby have each been given \$1000 to save for university.

Alex invests his money in an account that pays a nominal annual interest rate of 2.35% **compounded half-yearly.**

(a) Calculate the amount Alex will have in his account after 5 years. Give your answer correct to 2 decimal places. [3]

Bobby wants to invest his money in an account such that his investment will increase to 2 times the initial amount in 5 years. Assume the account pays a nominal annual interest of p% **compounded quarterly**.

(b) Determine the value of p. [3]

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5 [Maximum mark: 5]

Six vectors $\mathbf{v_1}, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ are each chosen to be either $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ with equal probability, with each choice made independently. Find the probability that the sum $\mathbf{v_1} + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 + \mathbf{v}_6$ is equal to the vector $\begin{pmatrix} 10 \\ 8 \end{pmatrix}$.

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6 [Maximum mark: 11]

A particle P starts from point A and moves along a straight line so that its velocity, $v\,\text{ms}^{-1}$, after t seconds is given by $v(t) = 2\sin t + 1 - \text{e}^{\sin t}$ for $0 \le t \le 6$.

(a) Find the values of t when P is at rest. [3]

(b) Find the acceleration of P when it changes direction. [2]

(c) Write down the number of times that the acceleration of P is $0\,\text{ms}^{-2}$. [1]

(d) Find the total distance travelled by P from $t = 0$ to $t = 6$. [2]

(e) Explain why P passes through A again. [3]

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7 [Maximum mark: 11]

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{x}\right)=\frac{1}{x^2}-\frac{\ln x}{x^2}$. [2]

Hence, find $\int \frac{\ln x}{x^2}\mathrm{d}x$. [2]

(b) The region bounded by the curve $y=\frac{\ln x}{x}$ and the lines $x=1, x=\mathrm{e}, y=0$ is rotated through 2π radians about the x-axis.

Find the **exact** volume of the solid generated. [7]

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*Do **NOT** write solutions on this page.*

SECTION B (57 marks)

Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

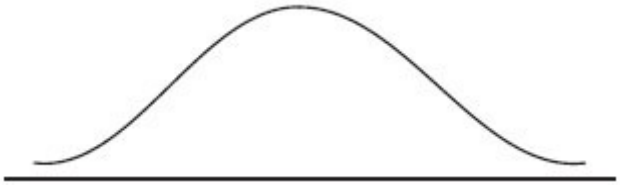
8 [Maximum Mark: 18]

A shop sells apples and pears. The masses, in grams, of apples and pears are normally distributed with means and standard deviations as shown in the following table.

	Mean	**Standard deviation**
Apple	204	6
Pear	150	σ

(a) Find the probability that a randomly chosen **apple** weighs between 186 grams and 222 grams. [2]

(b) Copy and complete the following normal distribution diagram, to represent the probability that the masses of the **apples** lie within three standard deviations of the mean, and shade the appropriate region. [2]



(c) Tom bought n **apples**. Find the least value of n such that the probability that more than 7 apples will each weigh between 198 grams and 210 grams is at least 0.85. [4]

(d) Three **pears** are chosen at random. Find the probability that one of them weighs less than the mean mass and each of the other two pears has a mass within one standard deviation of the mean mass. [4]

TURN OVER

*Do **NOT** write solutions on this page.*

The probability of a randomly chosen **pear** weighing between t and 158 grams, and the probability that a randomly chosen **pear** weighs less than t grams are each 0.35.

(e) Find the value of σ and of t. [4]

(f) In a random sample of 50 pears, find the most probable number of pears that each weighs less than t grams. [2]

9 [Maximum Mark:17]

Consider the planes

$$\prod_1 : \quad 2x+7y+5z=24$$
$$\prod_2 : \quad 3x-4y+\lambda z=\mu$$

and the line l passing through the point $\mathrm{A}(5,2,4)$ and point $\mathrm{B}(5,-1,3)$.

(a) Find a vector equation of line l. [1]

(b) The point C lies on l such that the foot of perpendicular of C onto $\prod_1$ has coordinates $(3,1,1)$. Find the coordinates of C. [5]

The line l does not intersect $\prod_2$.

(c) Show that $\lambda=12$ and find the possible values of μ. [4]

(d) Find the possible values of μ if the distance between $\prod_2$ and l is 2 units. [5]

(e) Find the acute angle between the planes $\prod_1$ and $\prod_2$. [2]

TURN OVER

*Do **NOT** write solutions on this page.*

10 [Maximum Mark: 22]

The population, P (in thousands), of a colony of ants on a small island can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$$

where t is the time measured in days and k and N are positive constants.

The constant N represents the maximum population of this colony of ants that the island can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{\mathrm{d}P}{\mathrm{d}t}$. [1]

(b) Show that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$. [4]

(c) Hence show that that the colony of ants grows the fastest when $P = \frac{N}{2}$. [6]

(d) Hence determine the corresponding fastest growth rate. [1]

Let P_0 being the initial population of the colony of ants.

(e) Show that the solution to the differential equation can be written in the form

$$kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right).$$ [8]

After 10 days the population is $5P_0$.

(f) Show that $k = \frac{1}{10}\ln(5.1)$ if $N = 205P_0$. [2]

End of Paper