STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	2	-	7	1	0	1

ORA et LABORA

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES **15 August 2022**

HIGHER LEVEL **2 hours**

PAPER 1 **1400 – 1600 hrs**

Monday

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and your teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[110 marks]**.
- This question paper consists of **12** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/110

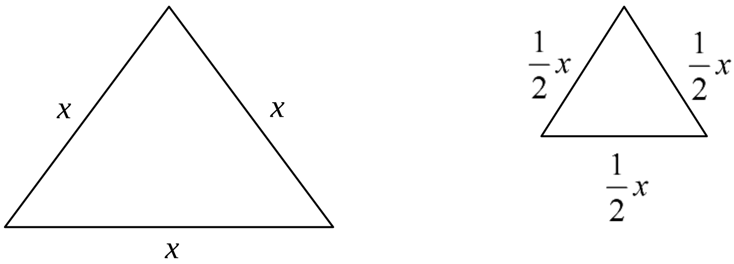
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (55 marks)

Answer **all** questions in the spaces provided.

1 **[Maximum mark: 5]**

Consider a sequence of equilateral triangles, where the length of a side of each triangle is half that of the previous triangle. The length of a side of Triangle 1 is x cm, and Triangles 1 and 2 are shown below.



Triangle 1 Triangle 2

(a) Find the area of Triangle 1 in terms of x. [2]

(b) The sequence of triangles continues to infinity. Given that the **total** area of all the triangles is $27\sqrt{3}$ cm^2, find the value of x. [3]

TURN OVER

2 **[Maximum mark: 7]**

Anthony cycles on the weekends for leisure. Anthony has an equal chance of cycling or not on any given Saturday. If Anthony cycles on Saturday, then he has a probability of 0.4 of not cycling on Sunday. If he does not cycle on Saturday, then he has a probability of 0.2 of not cycling on Sunday as well.

(a) Find the probability that Anthony cycles on exactly one day in a typical weekend. [2]

(b) Find the probability that Anthony cycled on exactly one day per weekend over all the four weekends in June. (You do not need to evaluate your answer.) [2]

(c) On a typical weekend, given that Anthony cycled on Sunday, find the probability that he also cycled on Saturday. [3]

……
……
……
……
……
……
……
……
……
……
……
……
……
……
……
……
……
……
……
……
……

TURN OVER

3 **[Maximum mark: 6]**

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^2 x\right)$. [2]

(b) Find the area bounded by the curve $y = \frac{\sin 2x}{\sqrt{2-\cos^2 x}}$, the x-axis, and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. [4]

TURN OVER

4 **[Maximum mark: 7]**

(a) Find the coefficient of x^4 in the expansion of $\left(x^2-1\right)^5$. [2]

(b) Hence or otherwise, find the term in x^4 in the expansion of $\left(x^2-1\right)^5\sqrt{1-x^2}$. [4]

(c) Write down the range of values of x for which the expansion in **(b)** is valid. [1]

TURN OVER

5 **[Maximum mark: 11]**

Consider the function $f(x) = \dfrac{x^2}{1+\ln x}$, defined on its maximal domain.

(a) Find the maximal domain of f. [3]

(b) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = \mathrm{e}$. [5]

(c) Find the range of values of x for which f is increasing. [3]

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

MORE SPACE IS AVAILABLE ON THE NEXT PAGE

TURN OVER

TURN OVER

6 **[Maximum mark: 12]**

Let $f_k(x) = kx^2 + x + k,\ x \in \mathbb{R}$, where k is a real constant.

(a) Find the range of values of k for which $f_k(x)$ is always positive. [4]

Let $k = 2$.

(b) Find the coordinates of the turning point of the graph of $y = f_2(x)$. [3]

(c) Sketch the graph of $y = \dfrac{1}{f_2(x) - 2}$, labelling clearly the coordinates of all turning point(s) and the equations of the asymptote(s). [5]

MORE SPACE IS AVAILABLE ON THE NEXT PAGE

TURN OVER

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

TURN OVER

7 **[Maximum mark: 7]**

Consider two skew lines $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mu \in \mathbb{R}$.

(a) Find a vector that is perpendicular to both lines. [3]

(b) Find the shortest distance between the two skew lines. [4]

TURN OVER

*Do **NOT** write solutions on this page.*

SECTION B (55 marks)

*Answer **all** questions on the writing paper provided.* ***Please start each question on a new page.***

8 **[Maximum mark: 20]**

Consider the equation $z^4 + 2z^2 - 4z + 8 = 0$, where $z \in \mathbb{C}$.

(a) Given that $z_1 = -1 + \sqrt{3}\mathrm{i}$ is a root, find all other roots of the equation. [5]

(b) Let z_2 be the root for which $\arg(z_2) = \dfrac{\pi}{4}$.

(i) Find $\dfrac{z_1}{z_2}$, leaving your answer in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$.

(ii) Consider the modulus and argument of z_1 and z_2 to find $\left|\dfrac{z_1}{z_2}\right|$ and show that $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{5\pi}{12}$.

(iii) With the results in (i) and (ii), determine the exact value of $\cos\left(\dfrac{5\pi}{12}\right)$. [11]

(c) Complex numbers z_1 and z_2 are represented by points A and B respectively on an Argand diagram.

(i) Sketch the Argand diagram with A and B, showing clearly the moduli and arguments of the complex numbers represented.

(ii) Show that the **square** of length AB is $8 - 2\sqrt{3}$ units^2. [4]

TURN OVER

*Do **NOT** write solutions on this page.*

9 [Maximum mark: 15]

(a) **(i)** Show that $\sum_{r=1}^{n}\left((r+1)^3 - r^3\right) = (n+1)^3 - 1$.

(ii) Hence, show that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. [7]

(b) A deck of cards is labelled with the numbers 1, 2, 3, … , *n* such that there are *r* cards labelled with the number *r*. That is, there is one card with the number 1, two cards with the number 2 and so on. A card is drawn randomly from the deck and *X* represents the number labelled on the card drawn.

(i) Show that $\mathrm{P}(X = r) = \frac{2r}{n(n+1)}$, where $1 \le r \le n$.

(ii) Find E(*X*) in terms of *n*.

(iii) Given that $n = 4$, find Var(*X*). [8]

10 [Maximum mark: 20]

Suppose $f_k(x) = x\mathrm{e}^{kx}$, where *k* is a non-zero real constant.

(a) **(i)** Find $f_k{}'(x)$.

(ii) Find the value of *x* for which $f_k{}'(x) = 0$.

(iii) Given that the stationary point of each of the graphs of $y = f_k(x)$, for all *k*, lies on the same straight line $y = mx + c$, determine the value of *m* and of *c*. [8]

(b) The n^{th} derivative of $f_k(x)$ is denoted by $f_k{}^{(n)}(x)$.

Prove by mathematical induction $f_k{}^{(n)}(x) = nk^{n-1}\mathrm{e}^{kx} + k^n x\mathrm{e}^{kx}$ for all $n \in \mathbb{Z}^+$. [7]

(c) Find the value of *k* if the coefficient of x^3 is equal to the coefficient of x^4 in the Maclaurin expansion of $f_k(x)$. [3]

(d) Describe geometrically an ordered sequence of transformations that map the graph of $y = f_1(x)$ to the graph of $y = f_2(x)$. [2]

End of Paper