Year 6 Mathematics: analysis and approaches
Higher Level
Paper 3
Preliminary Examinations

Monday 28 August 2023

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions on the answer sheets provided. Write your name and class on each answer sheet, and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is [**55 marks**].

Question	Marks
1	
2	

Name:____________________

Class:____________________

Index:____________________

	55

Answer **all** questions on the answer sheets provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to work with parametric equations. These are types of equations that employs an independent variable called a parameter (here denoted by θ) and in which dependent variables, x and y, are defined as continuous functions of the parameter and are not dependent on another existing variable.

Figure 1 shows the section ABCD of a mound at a skateboard park.

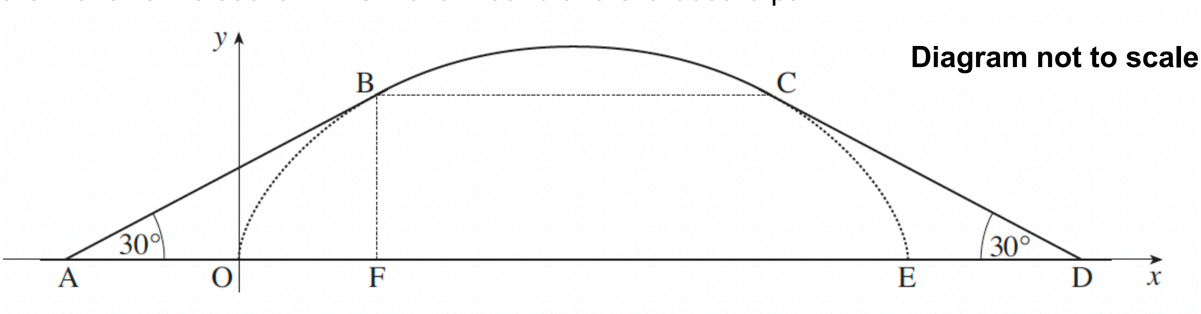

Figure 1

The section from B to C is part of the curve OBCE with parametric equations.

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta) \text{ for } 0 \le \theta \le 2\pi \text{ and } a > 0$$

where a is a constant.

(a) Find in terms of a,

(i) the length of the straight-line OE.

(ii) the maximum height of the mound. [4]

(b) (i) Find $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}$.

(ii) Hence find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. [4]

The straight-line sections AB and CD are inclined at 30° to the horizontal, and are tangents to the curve at B and C respectively. BC is parallel to the $x-$axis. BF is parallel to the $y-$axis.

(c) (i) Show that at the point B the parameter satisfies the equation

$$\sin\theta = \frac{1}{\sqrt{3}}(1-\cos\theta)$$

(ii) Find the value of θ that is a solution of this equation in degrees.

(iii) Show that $BF = 1.5a$.

(iv) Hence, find OF in terms of a. [5]

(This question continues on the following page)

(Question 1 continued)

(d) Find BC and AF in terms of a. [3]

(e) Given that the straight-line distance AD is 20 metres, calculate the value of a. [3]

Diagram not to scale.

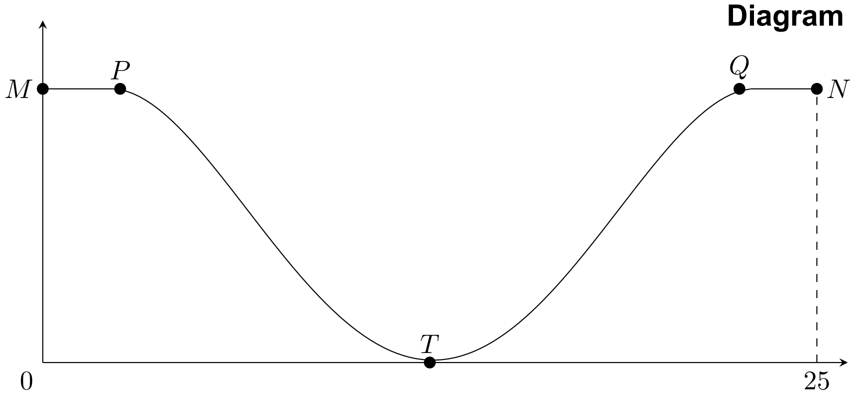

Figure 2

Another part of the skateboard park is shown in Figure 2. The cross-section MN is 25 metres, where MP and QN are horizontal and not equal in length. The section from P to Q is part of the curve with parametric equations.

$$x = 4(\theta - \sin\theta),\ y = 4(1 - \cos 2\theta) \text{ for } \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

The length of the dip PTQ can be calculated by the length of an arc formulae.

$$L_{PQ} = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

where a and b are the θ values at P and Q.

(f) Find the length of the dip PTQ. [5]

(g) Find the length of the track MPTQN from start to finish. [3]

2. [Maximum mark: 28]

This question asks you to explore the integral $\int_0^1 x^n (1-x)^n \mathrm{d}x$, where $n \geq 0$.

For $n \geq 0$, let

$$I_n = \int_0^1 x^n (1-x)^n \mathrm{d}x .$$

(a) (i) For $n \geq 1$, using integration by parts, show that

$$\int_0^1 x^{n-1} (1-x)^n \mathrm{d}x = \int_0^1 x^n (1-x)^{n-1} \mathrm{d}x . \qquad [4]$$

(ii) Deduce that

$$2\int_0^1 x^{n-1} (1-x)^n \mathrm{d}x = I_{n-1} . \qquad [3]$$

(b) (i) For $n \geq 1$, using integration by parts, show that

$$I_n = \frac{n}{n+1}\int_0^1 x^{n-1} (1-x)^{n+1} \mathrm{d}x . \qquad [4]$$

(ii) Using (a)(ii) and (b)(i), deduce that

$$I_n = \frac{n}{2(2n+1)} I_{n-1} . \qquad [3]$$

(c) Prove by induction that $I_n = \dfrac{(n!)^2}{(2n+1)!}$, $n \in \mathbb{Z}^+$. [9]

(d) By using the substitution $x = \sin^2\theta$, show that $I_{\frac{1}{2}} = \dfrac{\pi}{8}$. [5]