Year 6 Mathematics: analysis and approaches
Higher level
Paper 2
Preliminary Examinations

Solution

Tuesday 30 August 2022

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your name and class on each answer sheet, and attach them to the examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks].**

Section A		Section B	
Question	**Marks**	**Question**	**Marks**
1		10	
2		11	
3		12	
4			
5			
6			
7			
8			
9			

Name: ____________________

Class: ____________________

Index: ____________________

$\frac{\quad}{110}$

18 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

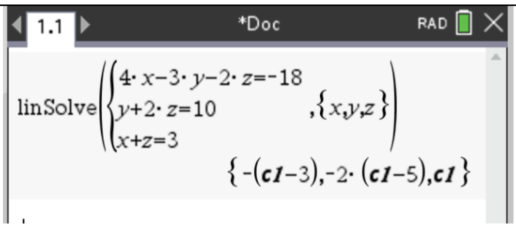
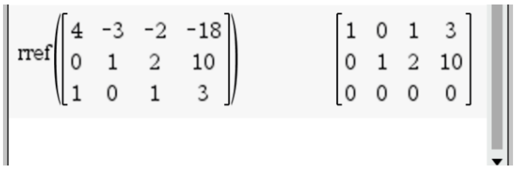
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

(a) Solve the system of linear equations:

$$\begin{aligned}4x-3y-2z&=-18\\y+2z&=10\\x+z&=3\end{aligned}$$ [3]

(b) Determine whether $(4,12,-1)$ is a solution to (a). [2]

(a)	 $\text{linSolve}\left(\begin{cases}4\cdot x-3\cdot y-2\cdot z=-18\\y+2\cdot z=10\\x+z=3\end{cases},\{x,y,z\}\right)$ $\{-(c1-3),-2\cdot(c1-5),c1\}$ Or  $\text{rref}\left(\begin{bmatrix}4&-3&-2&-18\\0&1&2&10\\1&0&1&3\end{bmatrix}\right)\quad\begin{bmatrix}1&0&1&3\\0&1&2&10\\0&0&0&0\end{bmatrix}$ $x=3-\lambda,\ y=10-2\lambda,\ z=\lambda\quad\lambda\in\mathbb{R}$
(b)	**Mtd 1** Yes. From (a), when $\lambda=-1$, $(x,y,z)=(4,12,-1)$ **Mtd 2** Let $x=3-\lambda=4\ \Rightarrow\ \lambda=-1$ Then $y=10-2(-1)=12$ and $z=\lambda=-1$. Thus $(4,12,-1)$ is a solution to (a).

2. [Maximum mark: 5]

The probability density function of the continuous random variable W is defined by

$$f(w)=\begin{cases}\dfrac{k}{w^2+25} & 0\le w\le 5,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Show that $k=\dfrac{20}{\pi}$. [3]

(b) Find $\mathrm{E}(W)$, the expected value of W . [2]

(a)	$\int_0^5 f(w)\,\mathrm{d}w=1$ $\int_0^5 \frac{k}{w^2+25}\,\mathrm{d}w=1$ $k[\frac{1}{5}\tan^{-1}(\frac{w}{5})]_0^5=1$ $\frac{1}{5}k[(\frac{\pi}{4})-0]=1$ $k=\frac{20}{\pi}$ (shown)
(b)	$\mathrm{E}(W)=\int_0^5 wf(w)\,\mathrm{d}w$ $=\int_0^5 w\frac{20}{\pi}\frac{1}{w^2+25}\,\mathrm{d}w$ $= 2.206356...$

3. [Maximum mark: 7]

A company manufactures washing detergent. They gave a number of identical coloured shirts to some customers and asked them to keep a record of how often the shirts were washed. After four months, the company went back to the customers and note down how many times the shirts had been washed, and the colour intensity of the shirt.

Number of washes, N	5	7	9	4	13	8	7
Colour Intensity, I	17	14.2	9	15.4	8.3	9.4	9

The relationship between N and I can be modelled by the regression line with equation $I = aN + b$.

(a) (i) Find the value of a and of b.

(ii) Write down the value of Pearson's product-moment correlation coefficient, r, for these data. [4]

(b) Interpret, in context, the value of a found in part (a)(i). [1]

A customer washed a shirt ten times.

(c) Use your regression line to estimate the colour intensity of the shirt. [2]

Q3(a)(i)	$a = -0.967$ $b = 19.1$
(ii)	–0.781
b	In every additional one wash, the colour intensity decreases/weaken by 0.967.
c	9.41

4. [Maximum mark: 5]

Consider the nine letters from the two words “DANCE” and “FILM”.

(a) Find the number of 9-letter arrangements which begins with “EA” in that order. [2]

(b) Four letters are selected at random.

Find the number of selections which contain more consonants than vowels. [3]

<table>
<tr><td>(a)</td><td>$\underbrace{\text{"EA"}}_{1\,way}\ \underbrace{\text{X X X X X X X}}_{7!\,ways}$

No. of arrangements $=1\times 7!=5040$ ways</td></tr>
<tr><td>(b)</td><td>6 consonants D, N, C, F, L, M
3 vowels A, E, I

<table>
<tr><td>Case</td><td>Selections</td></tr>
<tr><td>A) 3C 1V</td><td>$\binom{6}{3}\times\binom{3}{1}=20\times 3=60$</td></tr>
<tr><td>B) 4C 0V</td><td>$\binom{6}{4}\times\binom{3}{0}=15\times 1=15$</td></tr>
</table>

Total no. of selections = 60 + 15 = 75.</td></tr>
</table>

5. [Maximum mark: 6]

The probability of a bolt being faulty is 0.2. A random sample of 25 bolts is tested.

(a) Find the probability that there are more than 3 faulty bolts in the sample. [3]

These bolts are sold in bags of 25. Glenn buys 15 bags.

(b) Find the probability that exactly 8 of these bags contain more than 3 faulty bolts. [3]

Q5a	Let X be the number of faulty bolts. $X \sim B(25,0.2)$ $\mathrm{P}(\mathrm{X}>3)=1-P(X\le 3)$ $= 0.766$
b	Let Y represent the number of bags. $\mathrm{Y}\sim\mathrm{B}(15,0.766)$ $\mathrm{P}(Y=8)=0.0293$

6. [Maximum mark: 8]

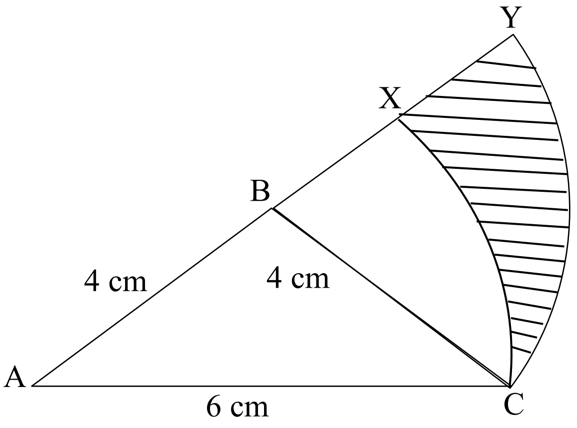


diagram not to scale

The diagram above shows an isosceles triangle ABC where AB = BC = 4 cm and AC = 6 cm. The arc XC is part of a circle with centre A and radius 6 cm. The arc YC is also part of another circle with centre B and radius 4 cm. Point A, B, X and Y are on a straight line.

(a) Show that angle CBY is 1.45 radians. [3]

(b) Find the perimeter of the shaded region. [5]

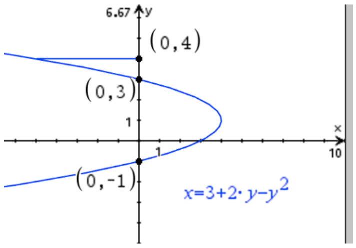
Q4a	$\angle ABC = \cos^{-1}\left(\frac{4^2+4^2-6^2}{2\times4\times4}\right)$ $= 1.696$ rad $\angle CBY = \pi - 1.696$ $= 1.45$ rad
b	$YC = 4(1.45) = 5.8$ $\angle BAC = \frac{1.45}{2} = 0.722734247$ rad $XC = 6(0.722734247) = 4.33641$ cm $XY = 4-2 = 2$ Perimeter $= 5.8 + 4.33641 + 2$ $= 12.13641$ cm $= 12.1$ cm

7. [Maximum mark: 5]

Find the exact area of the region enclosed by the curve with equation $x = 3 + 2y - y^2$ and the y-axis for $0 \le y \le 4$.

$x = 3 + 2y - y^2 = 0$

$\Rightarrow \; y = -1 \text{ or } 3$



Required area

$= \int_0^3 x \, \mathrm{d}y + \int_3^4 (-x) \, \mathrm{d}y$

$= \int_0^3 (3 + 2y - y^2) \mathrm{d}y + \int_3^4 (-3 - 2y + y^2) \mathrm{d}y$

$= [3y + y^2 - \frac{1}{3}y^3]_0^3 + [-3y - y^2 + \frac{1}{3}y^3]_3^4$

$= 9 + \frac{7}{3}$

$= \frac{34}{3} \text{ units}^2$

8. [Maximum mark: 7]

Let $f(x)=\dfrac{5-3x}{3-5x-2x^2}$.

(a) Express $f(x)$ into partial fractions. [3]

(b) Hence, or otherwise, expand $\dfrac{5-3x^2}{3-5x^2-2x^4}$ in ascending powers of x up to and including the term in x^4. [4]

(a)	$f(x)=\dfrac{5-3x}{3-5x-2x^2}=\dfrac{5-3x}{(3+x)(1-2x)}$ Let $\dfrac{5-3x}{(3+x)(1-2x)}=\dfrac{A}{3+x}+\dfrac{B}{1-2x}$ **Mtd 1 By cover-up rule** $A=\dfrac{5-3(-3)}{1-2(-3)}=2$ $B=\dfrac{5-3(\frac{1}{2})}{3+(\frac{1}{2})}=1$ **Mtd 2 Comparison method** $\dfrac{5-3x}{(3+x)(1-2x)}=\dfrac{A}{3+x}+\dfrac{B}{1-2x}=\dfrac{(A+3B)+(B-2A)x}{(3+x)(1-2x)}$ $\begin{cases} A+3B=5 \\ B-2A=-3 \end{cases} \Rightarrow A=2 \text{ and } B=1$ $f(x)=\dfrac{2}{3+x}+\dfrac{1}{1-2x}$
(b)	$\dfrac{5-3x^2}{3-5x^2-2x^4}$ $=\dfrac{5-3x^2}{(3+x^2)(1-2x^2)}$ $=\dfrac{2}{3+x^2}+\dfrac{1}{1-2x^2}$ $=\dfrac{2}{3}(1+\dfrac{x^2}{3})^{-1}+(1-2x^2)^{-1}$ $=\dfrac{2}{3}[1-\dfrac{x^2}{3}+(\dfrac{x^2}{3})^2-\left(\dfrac{x^2}{3}\right)^3+\cdots]+[1+2x^2+(2x^2)^2+\cdots]$ $=\frac{5}{3}+\frac{16}{9}x^2+\frac{110}{27}x^4+\cdots$(up to x^4) (=1.667 + 1.778 x + 4.074 x^4 +…)

9. [Maximum mark: 8]

A particle P moves along the $x-$axis with velocity $v\ \mathrm{ms}^{-1}$ at time t seconds given by

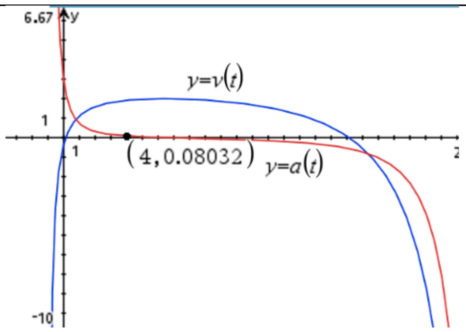
$$v(t)=2-\cot^2\left(\sqrt{\frac{t+1}{3}}\right) \quad \text{for } 0\le t\le 25.$$

(a) Find the acceleration of P when $t=4$. [2]

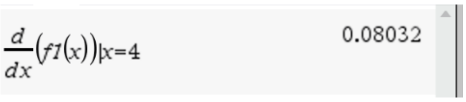
(b) The particle P is instantaneously at rest when $t=T_1$ and $t=T_2$ where $T_1<T_2$. Find T_1 and T_2. [2]

(c) Show that P passes the starting point twice and find the distance travelled by P at these instants. [4]

(a)



Or

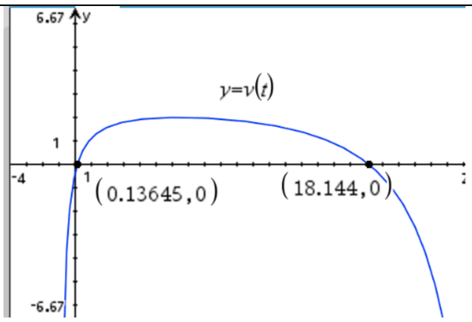


$\frac{d}{dx}(f1(x))|x=4$ 0.08032

Acceleration =

$0.0803\ \mathrm{ms}^{-2}$ (correct to 3s.f.)

(b)



T_1 = 0.136 s, T_2 = 18.1 s

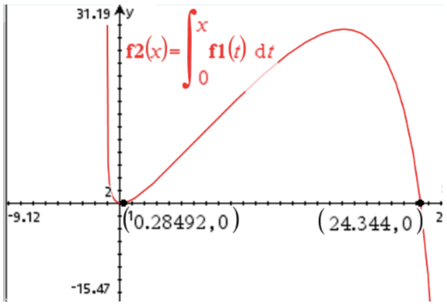
(c)

Displacement at time $=s(T)=\int_0^T v(t)\mathrm{d}t$

P passes starting point

⇒ displacement = 0.

Mtd 1 (Graphical)



Solving $s(T)=0$ gives

T = 0.28492 s or T = 24.344 s

Mtd 2 (nsolve)

nsolve($\int_0^T v(t)\mathrm{d}t=0,T,0.2$) gives T = 0.28492 s

nsolve($\int_0^T v(t)\mathrm{d}t=0,T,25$) gives T = 24.344 s

Distance travelled for $0\le T\le 0.28492$

$=\int_0^{0.28492}|v(t)|\mathrm{d}t$

$=0.04659\ \mathrm{m}$

Distance travelled for $0\le T\le 24.344$

$=\int_0^{24.344}|v(t)|\mathrm{d}t$

$=54.728\ \mathrm{m}$

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer sheets provided. Please start each question on a new page.

10. [Maximum mark: 14]

Joan walks to work from home via either route A or route B to go to her office.

Her travelling time via route A, in X minutes, is assumed to be normally distributed with mean of 38 and standard deviation of 7.

Find

(a) (i) $\mathrm{P}(X<45)$;

(ii) $\mathrm{P}(30<X<45)$. [4]

Her travelling time via route B, in Y minutes, is assumed to be normally distributed with mean of 41 and standard deviation of σ.

(b) Given that $\mathrm{P}(Y>45)=0.12$, calculate the value of σ. [3]

(c) If Joan leaves the house at 8 am to walk to work and she hopes to reach her office by 8.45 am, state with a reason, which route she should take. [2]

Joan works 20 days a month and she needs to report to office by 8.45 am.

Due to road works, she decides to take route A to work.

(d) Given that she reaches her office on time for the whole month, find the expected number of days that her travelling time is more than 30 minutes. [5]

Q9a(i)	$X \sim (38, 7^2)$ $P(X < 45) = 0.841$
(ii)	$P(30 < X < 45) = 0.715$
b	$z = 1.17499$ $1.17499 = \frac{45-41}{\sigma}$ $\sigma = 3.40$
c	Route A $\mathrm{P}(\mathrm{X} > 45) = 1 - 0.841 = 0.159$ Route B $\mathrm{P}(\mathrm{Y} > 45) = 0.12$ She will take route B as the probability of her taking more than 45 min is lower.
d	$\mathrm{P}(\mathrm{X}>30 \mid \mathrm{X} \leq 45) = \frac{\mathrm{P}(30<\mathrm{X}<45)}{\mathrm{P}(\mathrm{X} \leq 45)}$ $= \frac{0.714796}{0.841345}$ $= 0.850$ Expected number of days = 20(0.85018) = 17.0

Do **not** write solutions on this page.

11. [Maximum mark: 19]

From a $10\,\text{cm}$ by $20\,\text{cm}$ by $30\,\text{cm}$ acryllic block, a pyramid EPQR is cut out, where P, Q and R are the midpoints of [DE], [AE] and [EF] respectively.
Take $1\,\text{cm}$ along the x-, y- and z-axes to represent **i**, **j** and **k** respectively.

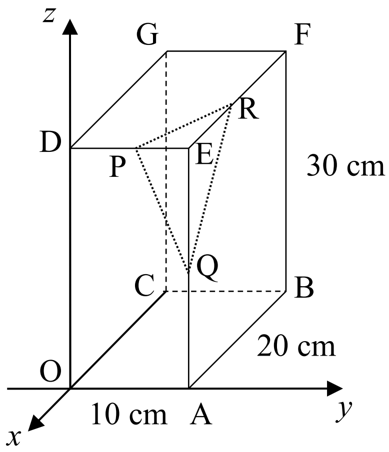


Diagram not to scale

(a) (i) Find the vector $\overrightarrow{\text{PQ}}$.

(ii) Show that $\overrightarrow{\text{PR}} = \begin{pmatrix} -10 \\ 5 \\ 0 \end{pmatrix}$. [3]

(b) Find the Cartesian equation of the plane PQR. [4]

(c) With PQR as the base, find the height of the pyramid EPQR. [5]

(d) Hence, or otherwise, find the volume of the pyramid EPQR. [3]

(e) Find the angle between the planes PQR and the plane ABFE.
Give your answer in the form $\arccos(m)$ where $m \in \mathbb{Q}$. [4]

(a)	$\overrightarrow{\text{OP}} = \begin{pmatrix} 0 \\ 5 \\ 30 \end{pmatrix}$, $\overrightarrow{\text{OQ}} = \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix}$, $\overrightarrow{\text{OR}} = \begin{pmatrix} -10 \\ 10 \\ 30 \end{pmatrix}$ (i) **Mtd 1** $\overrightarrow{\text{PQ}} = \overrightarrow{\text{PE}} + \overrightarrow{\text{EQ}} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -15 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -15 \end{pmatrix}$

Mtd 2

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 0 \\ 5 \\ -15 \end{pmatrix}$$

(ii) **Mtd 1**

$$\overrightarrow{PR} = \overrightarrow{PE} + \overrightarrow{ER} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 0 \end{pmatrix}$$

Mtd 2

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} -10 \\ 5 \\ 0 \end{pmatrix}$$

b

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} 0 \\ 5 \\ -15 \end{pmatrix} \times \begin{pmatrix} -10 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 75 \\ 150 \\ 50 \end{pmatrix} = 25\begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$$

Let normal vector ***n*** $= \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$.

Equation of plane PQR is given by

$$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{p}.\boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 30 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$$

$$3x + 6y + 2z = 0 + 30 + 60$$

$$3x + 6y + 2z = 90$$

c

Height of pyramid = shortest distance from E to plane PQR

Let foot of normal from E to plane PQR be T.

Step 1 Equation of EF is $\boldsymbol{r} = \overrightarrow{OE} + \lambda\,\boldsymbol{n}$

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 10 \\ 30 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$$

Step 2

Let $\overrightarrow{OT} = \begin{pmatrix} 3\lambda \\ 10 + 6\lambda \\ 30 + 2\lambda \end{pmatrix}$, where λ is to be found.

Step 3

T lies on plane PQR, so

$$3(3\lambda) + 6(10 + 6\lambda) + 2(30 + 2\lambda) = 90$$

$$49\lambda = -30$$

$$\lambda = -\frac{30}{49} \text{ (or } -0.612244...)$$

Step 4

$$\overrightarrow{OT} = \begin{pmatrix} -1.83673 \\ 6.32653 \\ 28.77551 \end{pmatrix} \text{ (or } \begin{pmatrix} -90/49 \\ 310/49 \\ 1410/49 \end{pmatrix})$$

$$\overrightarrow{ET} = \overrightarrow{OT} - \overrightarrow{OE} = \begin{pmatrix} -1.83673 \\ 6.32653 \\ 28.77551 \end{pmatrix} - \begin{pmatrix} 0 \\ 10 \\ 30 \end{pmatrix} = \begin{pmatrix} -1.83673 \\ -3.67347 \\ -1.22449 \end{pmatrix}$$

$$|\overrightarrow{ET}| = \sqrt{18.3673...} = 4.2857...$$

Height of pyramid = 4.29 cm (to 3 s.f.)

d

Area of triangle PQR = $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$= \frac{1}{2}\left\| \begin{pmatrix} 75 \\ 150 \\ 50 \end{pmatrix} \right\|$$

$$= 87.5 \text{ (or } \frac{175}{2}) \text{ cm}^2$$

Volume of pyramid PQR $= \frac{1}{3}(87.5)(4.2857...)$

$$= 125 \text{ cm}^3$$

e

Angle between planes PQR and ABFE

= angle between ***n*** and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$= \cos^{-1}\left(\frac{\begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{49}\,\sqrt{1}}\right)$$

$$= \cos^{-1}\left(\frac{6}{7}\right)$$

$$m = \frac{6}{7}$$

Do **not** write solutions on this page.

12. [Maximum mark: 21]

The curve C defined by the equation $\mathrm{e}^{x+y} = x - y$ has a stationary point A.

(a) Show that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{1+\mathrm{e}^{x+y}}$. [3]

(b) (i) Deduce that all the stationary points of C lie on the line $y = -x$.

(ii) Hence show that C has exactly one stationary point A and find the coordinates of A. [5]

It can be shown that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{1}{2}\left(1 - \frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

(c) Determine the nature of the turning point A. [2]

(d) (i) Explain, with working, why the curve C does not have any x – intercepts.

(ii) The line $y = k$ intersects the curve C. State the possible values of k.

(iii) Sketch the graph of C for $x \geq 0$, showing clearly all intercepts and asymptotes if any. [6]

(e) (i) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$.

(ii) Find the Maclaurin series of y up to the term in x^2 giving your answer correct to **3 significant figures**. [5]

(a)

$\mathrm{e}^{x+y} = x - y \quad \ldots\ldots \quad (1)$

Diffferentiate both sides wrt *x*:

$$\mathrm{e}^{x+y}(1+\frac{\mathrm{d}y}{\mathrm{d}x}) = 1 - \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-\mathrm{e}^{x+y}}{(1+\mathrm{e}^{x+y})}$$

$$1 + \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1-\mathrm{e}^{x+y}}{1+\mathrm{e}^{x+y}} = \frac{2}{1+\mathrm{e}^{x+y}} \quad \text{(shown)}$$

(b)

(i) For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

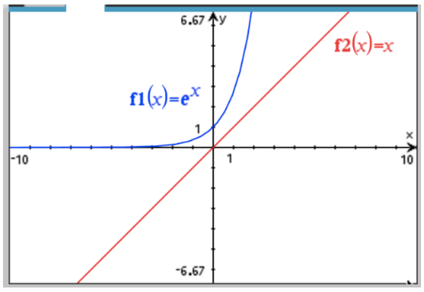
$$1 + 0 = \frac{2}{1+\mathrm{e}^{x+y}}$$

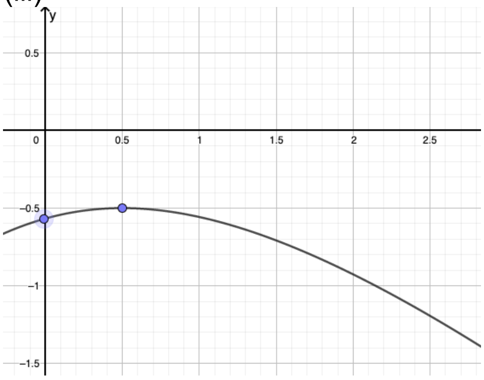
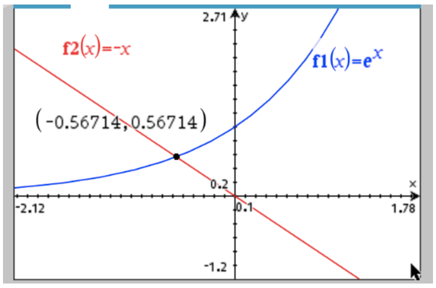
$$\mathrm{e}^{x+y} = 1$$

$$x + y = 0$$

$$y = -x$$

Hence all stationary points lie on $y = -x$.

(b)	(ii) Subst. into (1), we have $e^0 = x-(-x) \Rightarrow x = \frac{1}{2}$. $y = -x = -\frac{1}{2}$ Since there is only one pair of values for (x, y), therefore curve C has exactly one stationary point $A(\frac{1}{2}, -\frac{1}{2})$.
(c)	When $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -\frac{1}{2}\left(1-\frac{dy}{dx}\right)\left(1+\frac{dy}{dx}\right)^2$ $= -\frac{1}{2}(1-0)(1+0)^2$ $= -\frac{1}{2}$ < 0 By second derivative test, A is a maximum point.
(d)	(i) For *x*-intercepts, *y* = 0. (1) becomes $e^x = x$.  Since the graphs of $y = e^x$ and $y = x$ do not intersect, $e^x = x$ has no solution. Therefore, there is no *x*-intercept for curve *C*.
(d)	(ii) Curve *C* is continuous for $x \in \mathbb{R}$ and has only one maximum point A. Then A is the global maximum point. So $k \le -\frac{1}{2}$.

(d)	(iii) 
(e)	(i) When $x = 0$, $e^y = -y$.  $y = -0.56714...$ $\frac{dy}{dx} = 0.276207...$ (using (a))
(e)	(ii) When $x = 0$, $y = -0.56714...$ $\frac{dy}{dx} = 0.276207...$ $\frac{d^2y}{dx^2} = -0.589422...$ Maclaurin series of *y* is given by $y = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + ...$ $= -0.567 + 0.276x - 0.295x^2$ (to 3 s.f.)