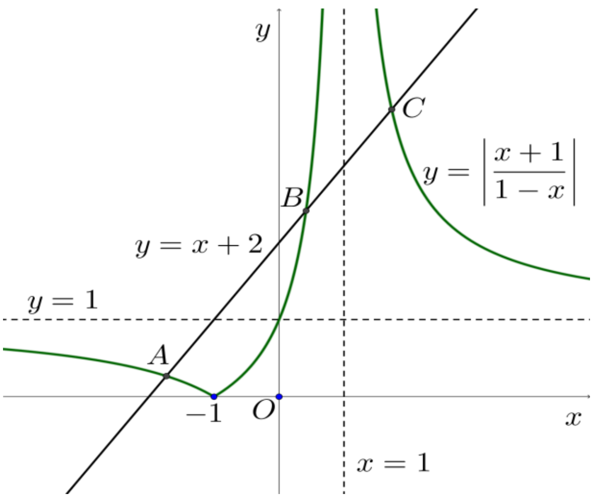
A Level 2015 P1 Solutions (edited for New H2 Math Syllabus 9758)

<table>
<tr><th>Qn</th><th colspan="2">Suggested Solutions</th></tr>
<tr><td>1(i)</td><td colspan="2">$C: y=\dfrac{a}{x^2}+bx+c$

$\left.\begin{aligned} -2.4&=\tfrac{1}{1.6^2}a+1.6b+c \quad ---(1)\\ 3.6&=\tfrac{1}{(-0.7)^2}a+(-0.7)b+c \quad ---(2)\\ \frac{\mathrm{d}y}{\mathrm{d}x}&=-\frac{2a}{x^3}+b \Rightarrow 2=-2a+b \quad ---(3)\end{aligned}\right\}$

Solving simultaneously,
$a=-3.59345\approx-3.593$
$b=-5.18691\approx-5.187$
$c=7.30274\approx7.303$</td></tr>
<tr><td>1(ii)</td><td colspan="2">When C crosses the x-axis, $y=\dfrac{a}{x^2}+bx+c=0$
By GC, $x=-0.589$ (3 d.p.)</td></tr>
<tr><td>1(iii)</td><td colspan="2">$y=-\dfrac{3.593}{x^2}-5.187x+7.303$
As $x\to\pm\infty$,
$y\to-5.187x+7.303$
$\therefore$ The other asymptote is the line $y=-5.187x+7.303$.</td></tr>
<tr><td>2(i)</td><td>
</td><td>(ii) By GC,
$x_A=-1.73$, $x_B=0.414$,
$x_C=1.73$ (3 s.f.)

From the graph, $\left|\dfrac{x+1}{1-x}\right|<x+2$
when
$-1.73<x<0.414$ or $x>1.73$.</td></tr>
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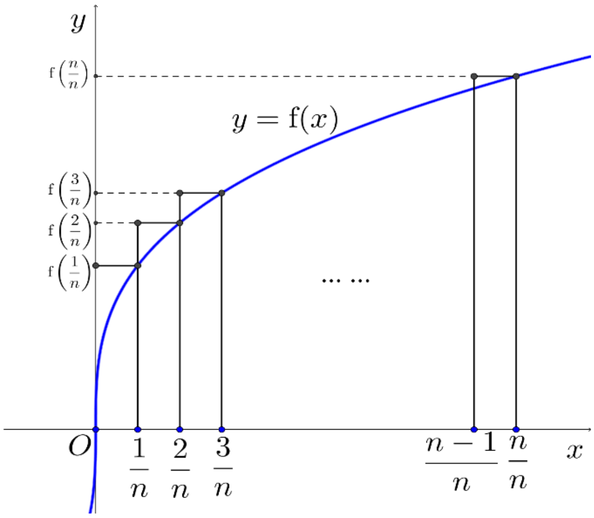
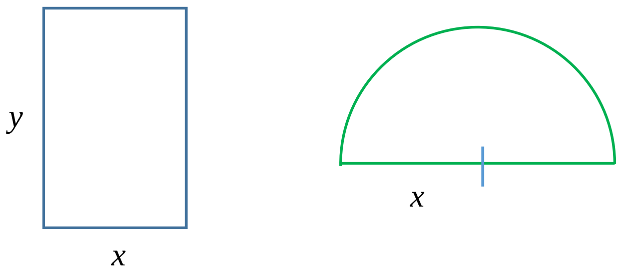
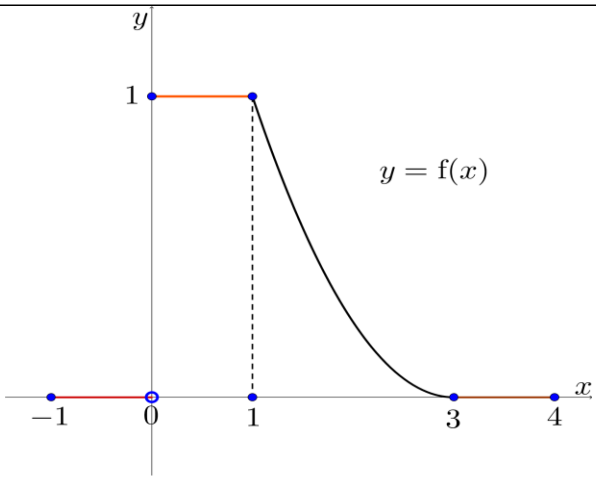
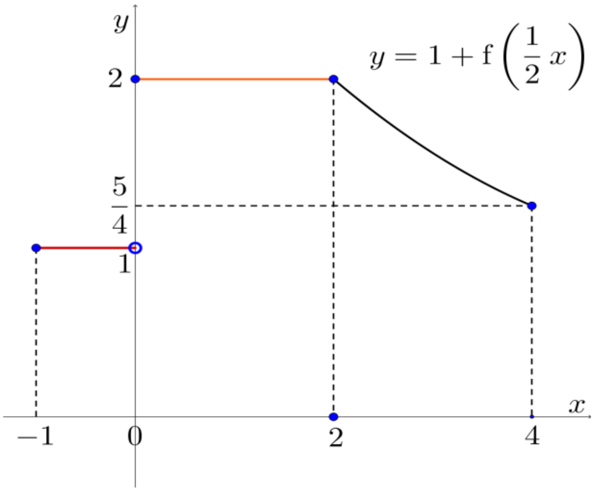
3(i)	 $\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+...+f\left(\frac{n}{n}\right)\right\}$ is the sum of areas of the n rectangles in the sketch. As $n \to \infty$, the space between the top of the rectangles and the curve $y=f(x)$ decreases to 0, so that this sum approaches the area under the curve. $\therefore \lim\limits_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+...+f\left(\frac{n}{n}\right)\right\}=\int_0^1 f(x)\,dx$
3(ii)	$\lim\limits_{x\to\infty}\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+...+\sqrt[3]{n}}{\sqrt[3]{n}}\right)$ $=\lim\limits_{x\to\infty}\frac{1}{n}\left(\sqrt[3]{\frac{1}{n}}+\sqrt[3]{\frac{2}{n}}+...+\sqrt[3]{\frac{n}{n}}\right)$ $=\lim\limits_{x\to\infty}\frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+...+f\left(\frac{n}{n}\right)\right)$ where $f(x)=x^{\frac{1}{3}}$ $=\int_0^1 f(x)\,dx$ $=\int_0^1 x^{\frac{1}{3}}\,dx=\frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^1=\frac{3}{4}$
4	 Total perimeter of shapes, $d=(2x+2y)+(\pi x+2x)=(4+\pi)x+2y$ $y=\frac{d-(4+\pi)x}{2}\quad ---(1)$ Total area of shapes, $A=xy+\frac{1}{2}\pi x^2\quad ---(2)$ Sub (1) into (2):

	$A=\dfrac{dx-\left(4+\pi\right)x^2}{2}+\dfrac{1}{2}\pi x^2$ $=\dfrac{d}{2}x-2x^2$ Let $\dfrac{\mathrm{d}A}{\mathrm{d}x}=\dfrac{d}{2}-4x=0$. $x=\dfrac{d}{8}$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}=-4<0$ when $x=\dfrac{d}{8}$. $\therefore$ The area is maximum when $x=\dfrac{d}{8}$. Maximum area $=\left(\dfrac{d}{2}\right)\left(\dfrac{d}{8}\right)-2\left(\dfrac{d}{8}\right)^2=\dfrac{1}{32}d^2$ (shown!)

5(i)	$y = x^2 \xrightarrow{\text{Translate } \binom{3}{0}} y = (x-3)^2 \xrightarrow[\text{parallel to the } y\text{-axis}]{\text{Scaling by factor } \frac{1}{4}} y = \frac{1}{4}(x-3)^2$ The graph of $y = x^2$ undergoes a translation of 3 units in the positive direction parallel to the *x*-axis, followed by a stretch of ¼ units parallel to the *y*-axis.
5(ii)	
5(iii)	
6(i)	Using standard series from MF26, $\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \dots$ $= 2x - 2x^2 + \frac{8}{3}x^3 - \dots$
6(ii)	$ax(1+bx)^c = ax\left(1 + c(bx) + \frac{c(c-1)}{2!}(bx)^2 + \frac{c(c-1)(c-2)}{3!}(bx)^3 + \dots\right)$ $= ax + abcx^2 + \frac{ab^2c(c-1)}{2}x^3 + \frac{ab^3(c)(c-1)(c-2)}{6}x^4 \dots$ Comparing coefficients of x: $a = 2$ x^2: $2bc = -2 \Rightarrow bc = -1$ x^3: $b^2c(c-1) = \frac{8}{3}$

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$(bc)^2-(bc)b=\frac{8}{3}$

$1+b=\frac{8}{3}$

or

$-b(c-1)=\frac{8}{3}$

$-bc+b=\frac{8}{3}$

$1+b=\frac{8}{3}$

$\therefore b=\frac{5}{3},\ c=-\frac{3}{5}$

Coefficient of x^4 in $ax(1+bx)^c$ is $\dfrac{(2)\left(\frac{5}{3}\right)^3\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)\left(-\frac{13}{5}\right)}{6}=-\dfrac{104}{27}$

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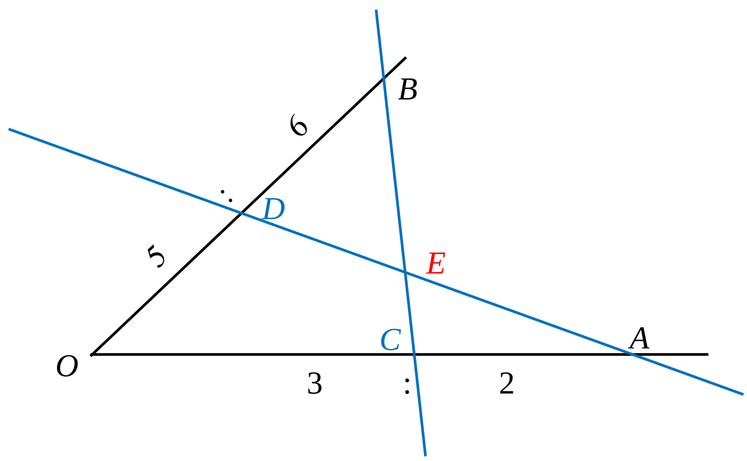
<tr><td>7(i)</td><td>



$\overrightarrow{OC}=\frac{3}{5}\mathbf{a}$, $\overrightarrow{OD}=\frac{5}{11}\mathbf{b}$

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<tr><td>7(ii)</td><td>

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\frac{3}{5}\mathbf{a}-\mathbf{b}$

Line BC: $\mathbf{r}=\overrightarrow{OB}+\lambda\overrightarrow{BC}=\mathbf{b}+\lambda\left(\frac{3}{5}\mathbf{a}-\mathbf{b}\right),\ \lambda\in\mathbb{R}$

$\mathbf{r}=\frac{3}{5}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$ ---(1) (shown!)

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\frac{5}{11}\mathbf{b}-\mathbf{a}$

Line AD: $\mathbf{r}=\mathbf{a}+\mu\left(\frac{5}{11}\mathbf{b}-\mathbf{a}\right)$

$\mathbf{r}=(1-\mu)\mathbf{a}+\frac{5}{11}\mu\,\mathbf{b},\ \mu\in\mathbb{R}$ ---(2)

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<tr><td>7(iii)</td><td>

When the lines BC and AD meet, (1) = (2)

$\frac{3}{5}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}=(1-\mu)\mathbf{a}+\frac{5}{11}\mu\,\mathbf{b}$

$\left(\frac{3}{5}\lambda-1+\mu\right)\mathbf{a}=\left(\frac{5}{11}\mu-1+\lambda\right)\mathbf{b}$

As $\mathbf{a}$ and $\mathbf{b}$ are non-parallel,

$\frac{3}{5}\lambda+\mu-1=0$ ---(3) and $\lambda+\frac{5}{11}\mu-1=0$ ---(4)

Solving (3) and (4) simultaneously, $\lambda=\frac{3}{4},\ \mu=\frac{11}{20}$

Sub into (1): $\overrightarrow{OE}=\frac{3}{5}\left(\frac{3}{4}\right)\mathbf{a}+\left(1-\frac{3}{4}\right)\mathbf{b}=\frac{9}{20}\mathbf{a}+\frac{1}{4}\mathbf{b}$

$\overrightarrow{AE}=\overrightarrow{OE}-\overrightarrow{OA}=-\frac{11}{20}\mathbf{a}+\frac{1}{4}\mathbf{b}$

$\overrightarrow{ED}=\overrightarrow{OD}-\overrightarrow{OE}=-\frac{9}{20}\mathbf{a}+\frac{9}{44}\mathbf{b}$

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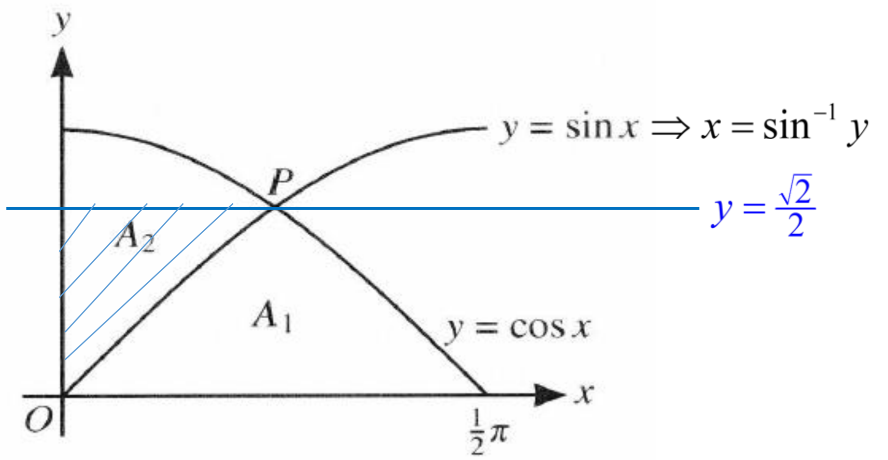
	$\overrightarrow{AE} = \frac{11}{9}\overrightarrow{ED}$ $\therefore\ AE : ED = 11:9$
8(i)	For athlete A, the lap timings $T, T+2, T+4, \ldots$ form an A.P. with first term T and common difference 2. Time taken for 50 laps $= \frac{50}{2}\left[2T+49(2)\right] = 25(2T+98)$ $1\frac{1}{2}$ hours = $1\frac{1}{2}\times 3600 = 5400s$; $1\frac{3}{4}$ hours = $1\frac{3}{4}\times 3600 = 6300s$. To meet the required time interval, $5400 \le 25(2T+98) \le 6300$ $216 \le 2T+98 \le 252$ $118 \le 2T \le 154$ $59 \le T \le 77$
8(ii)	For Athlete B, the lap timings $t, 1.02t, 1.02^2 t, \ldots$ form a G.P. with first term t and common ratio 1.02. Time taken for 50 laps $= \dfrac{t\left((1.02)^{50}-1\right)}{1.02-1} = 50t\left(1.02^{50}-1\right)$ To meet the required time interval, $5400 \le 50t\left(1.02^{50}-1\right) \le 6300$ $108 \le t\left(1.02^{50}-1\right) \le 126$ $63.845 \le t \le 74.486$ $63.9 \le t \le 74.4$ (3 s.f.) $\therefore$ Set of values of t = $\{t \in \mathbb{R} : 63.9 \le t \le 74.4\}$.
8(iii)	If each athlete takes $1\frac{1}{2}$ hours, then $t = 63.845$ and T = 59 [from parts (i) & (ii)]. On their 50th laps, Athlete A takes $T+49(2) = 157s$. Athlete B takes $1.02^{49}t = 168.47s$. Time difference $= \lvert 157-168.47 \rvert = 11s$ (nearest second)
9(a)	Given $\dfrac{w^2}{w^*}$ is purely imaginary,

	$\frac{w^2}{w^*} = k\mathrm{i}$ for some $k \in \mathbb{R}$ $w^2 = w^* k\mathrm{i}$ $(a+\mathrm{i}b)^2 = (a-\mathrm{i}b)k\mathrm{i}$ $a^2 - b^2 + 2ab\mathrm{i} = bk + ak\mathrm{i}$ Comparing real and imaginary parts, $a^2 - b^2 = bk \;\; ---(1)$ $\quad 2ab = ak \Rightarrow k = 2b \;\; ---(2)$ Sub (2) into (1): $a^2 - b^2 = 2b^2 \Rightarrow a^2 = 3b^2 \Rightarrow b^2 = \frac{1}{3}a^2 \Rightarrow b = \pm\frac{a}{\sqrt{3}}$ $\therefore w = a \pm \mathrm{i}\frac{a}{\sqrt{3}}$
10(i)	Area of A_1 + Area of A_2 $= \int_0^{\frac{\pi}{2}} \cos x \,\mathrm{d}x$ $= [\sin x]_0^{\frac{\pi}{2}}$ $= \sin\frac{\pi}{2} - \sin 0$ $= 1 \;\; ---(1)$ Area of A_1 $= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \,\mathrm{d}x$ (by symmetry) $= 2[\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= 2\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$ $= 2\left(1 - \frac{\sqrt{2}}{2}\right)$ $= 2 - \sqrt{2} \;\; ---(2)$ The working of someone who forgot how to integrate sin *x*. (1) – (2): $A_2 = 1 - (2-\sqrt{2}) = \sqrt{2} - 1 \;\; ---(3)$ (2) ÷ (3): $\frac{A_1}{A_2} = \frac{2-\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}\left(\sqrt{2}-1\right)}{\sqrt{2}-1} = \sqrt{2}$
10(ii)	 The shaded region is the region described. Rotating it 2π radians about the *y*-axis gives the volume $\int_0^{\frac{1}{2}\sqrt{2}} \pi x^2 \,\mathrm{d}y = \int_0^{\frac{1}{2}\sqrt{2}} \pi\left(\sin^{-1} y\right)^2 \mathrm{d}y$ (shown).

10 (iii)	$y = \sin u \Rightarrow u = \sin^{-1} y$ When y = 0, $u = \sin^{-1} 0 = 0$; $y = \frac{\sqrt{2}}{2}$, $u = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$. $\therefore \int_0^{\frac{\sqrt{2}}{2}} (\sin^{-1} y)^2 \, dy = \pi\int_0^{\frac{\pi}{4}} (u)^2 \left(\frac{dy}{du}\right) du$ $= \pi\int_0^{\frac{\pi}{4}} u^2 \cos u \, du$ (shown) Exact volume, $V = \pi\int_0^{\frac{\pi}{4}} u^2 \cos u \, du$ $= \pi\left\{\left[u^2 \sin u\right]_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}} (u)(\sin u) du\right\}$ Integration by Parts: $w = u^2$ $\frac{dv}{du} = \cos u$ $\frac{dw}{du} = 2u$ $v = \sin u$ Now $\left[u^2 \sin u\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{16}\sin\frac{\pi}{4} - 0$ $= \frac{\pi^2\sqrt{2}}{32}$ and $w = u$ $\frac{dv}{du} = \sin u$ $\frac{dw}{du} = 1$ $v = -\cos u$ $\int_0^{\frac{\pi}{4}} (u)(\sin u) du = \left[-u\cos u\right]_0^{\frac{\pi}{4}} + -\int_0^{\frac{\pi}{4}} (-\cos u) du$ $= \left(-\frac{\pi}{4} \times \frac{\sqrt{2}}{2}\right) + \left[\sin u\right]_0^{\frac{\pi}{4}}$ $= -\frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2}$ $\therefore V = \pi\left[\frac{\pi^2\sqrt{2}}{32} - 2\left(-\frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2}\right)\right]$ $= (\frac{\pi^3\sqrt{2}}{32} + \frac{\pi^2\sqrt{2}}{4} - \pi\sqrt{2})$ units^3
11(i)	$x = \sin^3\theta, \quad y = 3\sin^2\theta\cos\theta, \quad 0 \le \theta \le \frac{\pi}{2}$ $\frac{dx}{d\theta} = 3\sin^2\theta\cos\theta$, $\frac{dy}{d\theta} = (6\sin\theta\cos\theta)\cos\theta + 3\sin^2\theta(-\sin\theta)$ $= 3\sin\theta\left[2\cos^2\theta - \sin^2\theta\right]$ $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ $= \frac{3\sin\theta\left[2\cos^2\theta - \sin^2\theta\right]}{3\sin^2\theta\cos\theta}$ $= \frac{2\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$ $= 2\cot\theta - \tan\theta$ (shown) $---(1)$

11(ii)

$\dfrac{dy}{dx}=0 \Rightarrow 2\cot\theta-\tan\theta=0 \quad ---(2)$

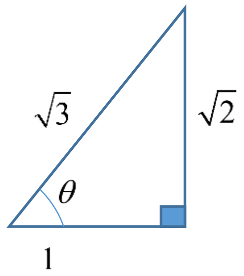
$(2)\times\tan\theta: \quad 2-\tan^2\theta=0$

$\tan^2\theta=2$

$\tan\theta=\sqrt{2} \quad (\tan\theta\ge 0 \text{ as } 0\le\theta\le\frac{\pi}{2})$

$\therefore$ C has a turning point when $\tan\theta=\sqrt{2}$.

From the diagram,



$x=\sin^3\theta \qquad\qquad y=3\sin^2\theta\cos\theta$

$=\left(\sqrt{\frac{2}{3}}\right)^3 \qquad\qquad =3\left(\sqrt{\frac{2}{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)$

$=\frac{2\sqrt{2}}{3\sqrt{3}} \qquad\qquad =\frac{2}{\sqrt{3}}$

$\therefore$ The turning point is $\left(\frac{2\sqrt{6}}{9},\frac{2\sqrt{3}}{3}\right)$.

Differentiating (1) w.r.t x,

$$\frac{d^2y}{dx^2}=\left(-2\operatorname{cosec}^2\theta-\sec^2\theta\right)\frac{d\theta}{dx}$$

$$=-\frac{2\operatorname{cosec}^2\theta+\sec^2\theta}{3\sin^2\theta\cos\theta}$$

As θ is acute,

$\operatorname{cosec}^2\theta$, $\sec^2\theta$, $\sin^2\theta$ and $\cos\theta$ are all positive

$$\frac{d^2y}{dx^2}=-\frac{6\sqrt{3}}{2}<0,$$

$\therefore$ The turning point is a maximum point.

11(iii)

$y=3\sin^2\theta\cos\theta=0$

$\Rightarrow \sin\theta=0 \text{ or } \cos\theta=0$

$\theta=0 \text{ or } \theta=\frac{\pi}{2} \quad (\text{as } 0\le\theta\le\frac{\pi}{2})$

$x=0 \text{ or } x=1$

$$\int y\,dx=\int y\left(\frac{dx}{d\theta}\right)d\theta$$

Bounded area

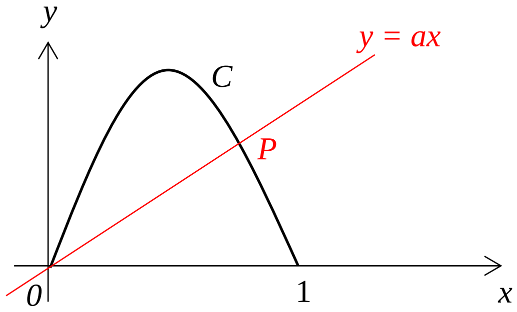
$=\int_0^1 y\,dx$

$=\int_0^{\frac{\pi}{2}}\left(3\sin^2\theta\cos\theta\right)\left(3\sin^2\theta\cos\theta\right)d\theta$

$=\int_0^{\frac{\pi}{2}}9\sin^4\theta\cos^2\theta\,d\theta \quad \text{(shown!)}$

$=0.884$ (3 d.p.)

Diagram for parts (iii) & (iv):



11(iv)	To find *P*, we can substitute the parametric equations of *C* into $y = ax$: $3\sin^2\theta\cos\theta = a\sin^3\theta$ $\sin^2\theta(3\cos\theta - a\sin\theta) = 0$ $\sin^2\theta = 0$ or $3\cos\theta = a\sin\theta$ $\theta = 0$ (rejected as $\theta = 0 \Rightarrow x = 0$) or $\tan\theta = \frac{3}{a}$ $\therefore \tan\theta = \frac{3}{a}$ (shown) For *P* to be the maximum point of *C*, $\tan\theta = \sqrt{2}$, using result from 11(ii) $\Rightarrow \frac{3}{a} = \sqrt{2}$ $\Rightarrow a = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$