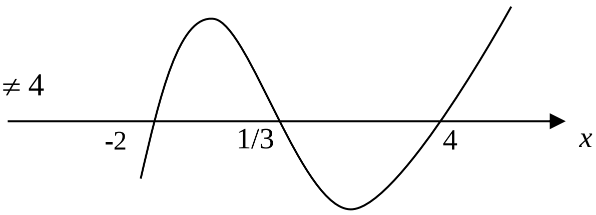
2016 A-Level H2 Maths Paper 1 Solutions

1

$$\frac{4x^2+4x-14}{x-4}-(x+3)=\frac{4x^2+4x-14-(x-4)(x+3)}{x-4}$$
$$=\frac{4x^2+4x-14-\left(x^2-x-12\right)}{x-4}$$
$$=\frac{3x^2+5x-2}{x-4}$$
$$=\frac{(3x-1)(x+2)}{x-4}$$

$$\frac{4x^2+4x-14}{x-4}<x+3, \qquad x\neq 4$$
$$\frac{4x^2+4x-14}{x-4}-(x+3)<0$$
$$\frac{(3x-1)(x+2)}{x-4}<0$$
$$(3x-1)(x+2)(x-4)<0, \qquad x\neq 4$$



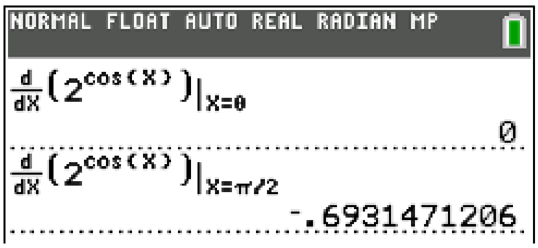
$x<-2$ or $\frac{1}{3}<x<4$

2(i)

$y=2^{\cos x}$

When $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

When $x=\frac{\pi}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-0.693$ (3 s.f.)



2(ii)

When $x=0$, $y=2^{\cos 0}=2$

Tangent at (0,2): $y=2$ ---(1)

When $x=\frac{\pi}{2}$, $y=2^{\cos\left(\frac{\pi}{2}\right)}=2^0=1$

Tangent at $\left(\frac{\pi}{2},1\right)$: $y-1=-0.693\left(x-\frac{\pi}{2}\right)$

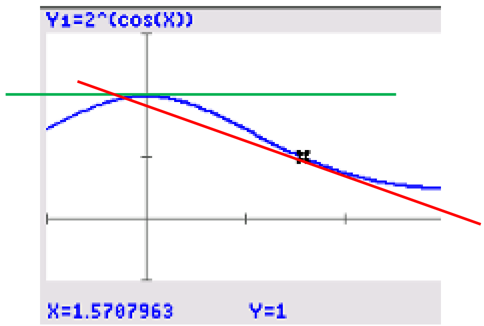
$y=-0.693x+2.09$ ---(2)

Substitute (1) into (2):

$2=-0.693x+2.09$

$x=0.128$

$\therefore$ The tangents meet at $(0.128, 2)$.



3

$$y = x^4 \to y = \overbrace{k(x-l)^4 + m}^{\mathrm{f}(x)}$$

By the way (question doesn't ask for it), this is the sequence of transformations:

$$y = x^4 \xrightarrow[\text{in the positive } x\text{-direction}]{\text{translate } l \text{ units}} y = (x-l)^4$$

$$\xrightarrow[\text{parallel to } y\text{-axis}]{\text{scale by factor } k} y = k(x-l)^4 \xrightarrow[\text{in the positive } y\text{-direction}]{\text{translate } m \text{ units}} y = k(x-l)^4 + m$$

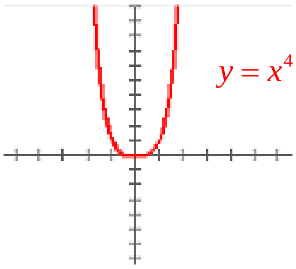
$(0,0) \to (l,m)$

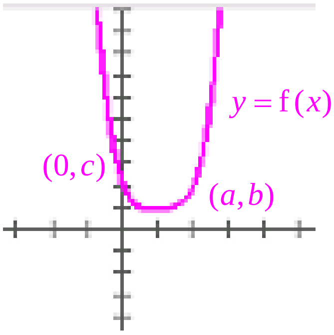
$\therefore (l,m) = (a,b)$, so $l = a$, $m = b$.

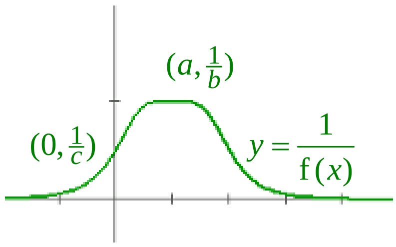
$(0,c)$ lies on $y = \mathrm{f}(x)$ graph

$\Rightarrow c = k(0-l)^4 + m$

$$k = \frac{c-m}{l^4} = \frac{c-b}{a^4}$$







4(i)

Given that the for the
Arithmetic series: first term = a, common difference = d,
Geometric series: first term = b, common ratio, r,
a, b, d, r are non-zero,

Aim: Eliminate a, then eliminate d and b, leaving r.

$a + 3d = br^4 \quad ---(1)$

$a + 8d = br^7 \quad ---(2)$

$a + 11d = br^{14} \quad ---(3)$

$(2)-(1)$: $5d = br^7 - br^4$

$\qquad 5d = br^4(r^3 - 1) \quad ---(4)$

$(3)-(2)$: $3d = br^{14} - br^7$

$\qquad 3d = br^7(r^7 - 1) \quad ---(5)$

$$\frac{(5)}{(4)}: \quad \frac{3}{5} = \frac{br^7(r^7-1)}{br^4(r^3-1)}$$

$$\frac{3}{5} = \frac{r^3(r^7-1)}{(r^3-1)}$$

$$3r^3 - 3 = 5r^{10} - 5r^3$$

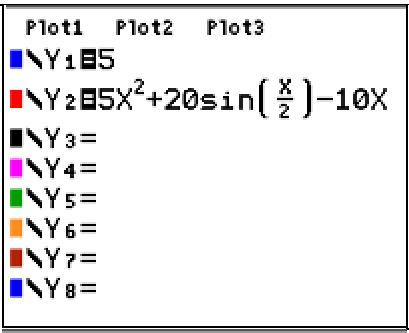
$$5r^{10} - 8r^3 + 3 = 0$$

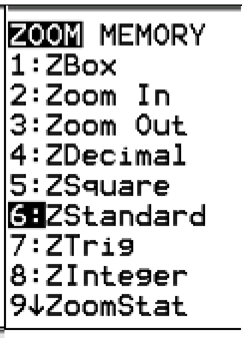
Using GC, $r = 0.74$ or $r = 1$ (reject since $|r|<1$)

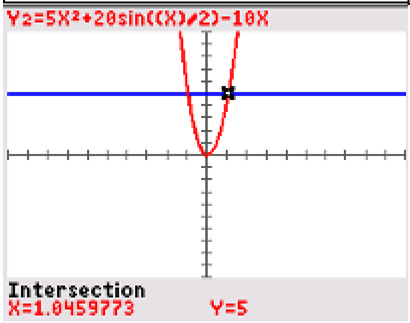
4(ii)	Required sum $= u_{n+1} + u_{n+2} + \ldots$ $= (u_1 + u_2 + \ldots) - (u_1 + u_2 + \ldots + u_n)$ $= S_\infty - S_n$ $= \dfrac{b}{1-r} - \dfrac{b(1-r^n)}{1-r}$ $= \dfrac{br^n}{1-r}$ $= \dfrac{b(0.74)^n}{0.26}$ or $3.85b(0.74)^n$
5(i)	$\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$ $\mathbf{u}+\mathbf{v} = \begin{pmatrix} 2+a \\ -1 \\ 2+b \end{pmatrix}, \quad \mathbf{u}-\mathbf{v} = \begin{pmatrix} 2-a \\ -1 \\ 2-b \end{pmatrix}$ $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = \begin{pmatrix} 2+a \\ -1 \\ 2+b \end{pmatrix} \times \begin{pmatrix} 2-a \\ -1 \\ 2-b \end{pmatrix} = \begin{pmatrix} 2b \\ 4b-4a \\ -2a \end{pmatrix}$ As usual, we do a dot product check to confirm this vector's correctness.
5(ii)	As the **i**- and **k**- components are equal, $2b = -2a$ i.e. $b = -a$ $\therefore (\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = \begin{pmatrix} -2a \\ -8a \\ -2a \end{pmatrix} = -2a \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ We can read it again: Yes, it's **i** and **k**, not **i** and **j**. $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ is a vector of length 1, given that it is a unit vector. $\Rightarrow \left\| -2a \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \right\| = 1$ $\lvert -2 \rvert \lvert a \rvert \left\| \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \right\| = 1$ $2\lvert a \rvert \sqrt{1^2+4^2+1^2} = 1$ $2\lvert a \rvert \sqrt{18} = 1$ $\lvert a \rvert = \frac{1}{2\sqrt{18}} = \frac{1}{6\sqrt{2}}$ $a = \pm\frac{1}{6\sqrt{2}}$ (or $\pm\frac{\sqrt{2}}{12}$)

5(iii)	$(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v})=0$ $\mathbf{u.u}-\mathbf{v.v}=0$ $\lvert\mathbf{u}\rvert^2-\lvert\mathbf{v}\rvert^2=0$ $\lvert\mathbf{v}\rvert^2=\lvert\mathbf{u}\rvert^2=2^2+(-1)^2+2^2=9$ $\lvert\mathbf{v}\rvert=3$
7(a)	$(-1+5\text{i})^2+(-1-8\text{i})(-1+5\text{i})+(-17+7\text{i})=1-10\text{i}-25+1+3\text{i}+40-17+7\text{i}$ $=-10\text{i}+10\text{i}-23+23=0$ $\therefore -1+5\text{i}$ is a root of the equation $w^2+(-1-8\text{i})w+(-17+7\text{i})=0$. Let the second root be $p+q\text{i}$. Sum of roots $=1+8\text{i}$ $(-1+5\text{i})+(p+q\text{i})=1+8\text{i}$ $p+q\text{i}=2+3\text{i}$
7(b)	As all the coefficients of the equation are real, $1+a\text{i}$ is a root $\Rightarrow$ $1-a\text{i}$ is a root. Let the third root be $m\in\mathbb{R}$. $z^3-5z^2+16z+k=[z-(1+a\text{i})][z-(1-a\text{i})](z-m)$ $=[z^2-2z+(1+a^2)](z-m)$ Comparing coefficients of z^2: $-5=-m-2\Rightarrow m=3$ z: $16=1+a^2+2m=1+a^2+6$ $a^2=9\Rightarrow a=3$ $(\because a>0)$ z^0: $k=-m(1+a^2)=-3(1+3^2)=-30$ If we try to use sum and product of roots (i.e. similar method to (a)), we find it doesn't work as well this time – there are **3 unknowns** to be found.
8(i)	Given $y=\text{f}(x)=\tan(ax+b)$ $---(1)$ $\text{f}'(x)=a\sec^2(ax+b)$ $=a[1+\tan^2(ax+b)]$ $=a(1+y^2)=a+ay^2$ (shown!) $\text{f}''(x)=2ay.\text{f}'(x)=2ay(a+ay^2)=2a^2(y+y^3)$ $\text{f}'''(x)=2a^2(1+3y^2).\text{f}'(x)=2a^3(1+y^2)(1+3y^2)$
8(ii)	Substitute $b=\frac{\pi}{4}$ into (1): $y=\text{f}(x)=\tan\left(ax+\frac{\pi}{4}\right)$ $\text{f}(0)=\tan\frac{\pi}{4}=1$ $\text{f}'(0)=a+a(1)^2=2a$ $\text{f}''(0)=2a^2(1+1^3)=4a^2$ $\text{f}'''(0)=2a^3(1+1^2)[1+3(1^2)]=16a^3$ $\therefore \text{f}(x)=\text{f}(0)+x\text{f}'(0)+\frac{x^2}{2!}\text{f}''(0)+\frac{x^3}{3!}\text{f}'''(0)+\ldots$ $\approx 1+2ax+2a^2x^2+\frac{8}{3}a^3x^3$

8(iii)	Substitute $a=2$ and $b=0$ into (1): $y=\mathrm{f}(x)=\tan(2x)$ $\mathrm{f}(0)=0$ $\mathrm{f}'(0)=2+2(0^2)=2$ $\mathrm{f}''(0)=2(2^2)(0+0^3)=0$ $\mathrm{f}'''(0)=2(2^3)(1+0^2)[1+3(0^2)]=16$ $\therefore \tan 2x \approx 2x+16\left(\frac{x^3}{3!}\right)=2x+\frac{8}{3}x^3$ Just as we made use of part (i) to do part (ii), we should ask ourselves if we can make use of earlier parts to do part (iii). **Did you?**
9(i)(a)	Let x be the distance the stone has fallen through the water at t seconds. Let $y=\frac{\mathrm{d}x}{\mathrm{d}t}$ ---(1) $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ ---(2) Substitute (1) & (2) into the given D.E.: $\frac{\mathrm{d}y}{\mathrm{d}t}+2y=10 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=10-2y$ (shown!)
9(i)(b)	$\int \frac{1}{10-2y}\,\mathrm{d}y=\int 1\,\mathrm{d}t$ $-\frac{1}{2}\ln\|10-2y\|=t+C$ where C is an arbitrary constant $\ln\|10-2y\|=-2t-2C$ $10-2y=\pm\mathrm{e}^{-2t-2C}=B\mathrm{e}^{-2t}$ where $B=\pm\mathrm{e}^{-2C}$ When $t=0$, $y=\frac{\mathrm{d}x}{\mathrm{d}t}=0 \Rightarrow B=10$ $10-2y=10\mathrm{e}^{-2t}$ $y=5-5\mathrm{e}^{-2t}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=5-5\mathrm{e}^{-2t}$ $x=5t+\frac{5}{2}\mathrm{e}^{-2t}+A$ where A is an arbitrary constant When $t=0$, $x=0 \Rightarrow A=-\frac{5}{2}$ $\therefore x=5t+\frac{5}{2}\mathrm{e}^{-2t}-\frac{5}{2}$
9(ii)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=10-5\sin\frac{1}{2}t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=10t+10\cos\frac{1}{2}t+C$ ---(1) Be methodical: Integrate twice first, then sub in the initial conditions. $x=5t^2+20\sin\frac{1}{2}t+Ct+D$ ---(2) (C, D are arbitrary constants) Sub $t=0, \frac{\mathrm{d}x}{\mathrm{d}t}=0$ into (1): $C=-10$ Sub $t=0, x=0$ into (2): $D=0$ $\therefore x=5t^2+20\sin\frac{1}{2}t-10t$
9(iii)	The stone reaches the bottom of the pond when $x=5$. For model in **(i)**: $5=5t+\frac{5}{2}\mathrm{e}^{-2t}-\frac{5}{2} \Rightarrow t=1.47$ (2 d.p.) Don't. Waste. Time. For model in **(ii)**: $5=5t^2+20\sin\frac{1}{2}t-10t \Rightarrow t=1.05$ (2 d.p.)







10(a)(i)

Let $y = \mathrm{f}(x) = 1 + \sqrt{x}$, for $x \in \mathbb{R}, x \geq 0$.

$\sqrt{x} = y - 1$

$\mathrm{f}^{-1}(y) = x = (y-1)^2$

$\mathrm{f}^{-1}(x) = (x-1)^2, \;\; x \in \mathbb{R}, x \geq 1$

10(a)(ii)

If $\mathrm{ff}(x) = x$, then $\mathrm{f}(1+\sqrt{x}) = x$.

$$1 + \sqrt{1+\sqrt{x}} = x \;\; ---(1)$$

$$\sqrt{1+\sqrt{x}} = x - 1$$

$$1 + \sqrt{x} = (x-1)^2 = x^2 - 2x + 1$$

$$\sqrt{x} = x^2 - 2x$$

$$x = (x^2 - 2x)^2 = x^4 - 4x^3 + 4x^2$$

$$\therefore x^4 - 4x^3 + 4x^2 - x = 0 \quad \text{(shown!)}$$

By GC, $x = 0, 0.383, 1$ or 2.62.

Substitute each value into (1) to check:

When $x = 0, 0.383, 1$, LHS $\neq$ RHS.

$\therefore$ They do not satisfy $\mathrm{ff}(x) = x$.

(Note: LHS > 1 for all $x \in \mathbb{R}$)

When $x = 2.62$, LHS = RHS.

$\therefore$ $x = 2.62$ is the only solution to $\mathrm{ff}(x) = x$.

As usual, we check if any value(s) needs to be rejected. Also, how many values is the question looking for?

As $x = 2.62$ is inside both D_{f} and $D_{\mathrm{f}^{-1}}$, it satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

10(b)(i)

A good approach for this question is to draw a table like in the Sec Sch days. This question is a test of perseverance & focus ☺

For a start…

n	0	1	2	3	4	5	6	7	8	9	10	11	12
g(n)	1	2	4										

At the end:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
g(n)	1	2	4	5	6	7	7	8	8	9	9	10	9

Good news: Each pair of numbers are consecutive integers! (Can you see why?)

g(4) = 6, g(7) = 8, g(12) = 9

10(b)(ii)

g(5) = g(6) = 7, so g is not one-one and g^{-1} does not exist.

11(i)(a)

When $a = 0$,

$$p:\ \mathbf{r}=\begin{pmatrix}1\\-3\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}0\\4\\-2\end{pmatrix},\lambda,\mu\in\mathbb{R}$$

$$l:\ \mathbf{r}=\begin{pmatrix}-1\\0\\1\end{pmatrix}+t\begin{pmatrix}-2\\1\\2\end{pmatrix},t\in\mathbb{R}$$

As $\begin{pmatrix}-2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}=-2+2=0$ and $\begin{pmatrix}-2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\4\\-2\end{pmatrix}=4-4=0$,

l is perpendicular to two vectors parallel to the plane p.

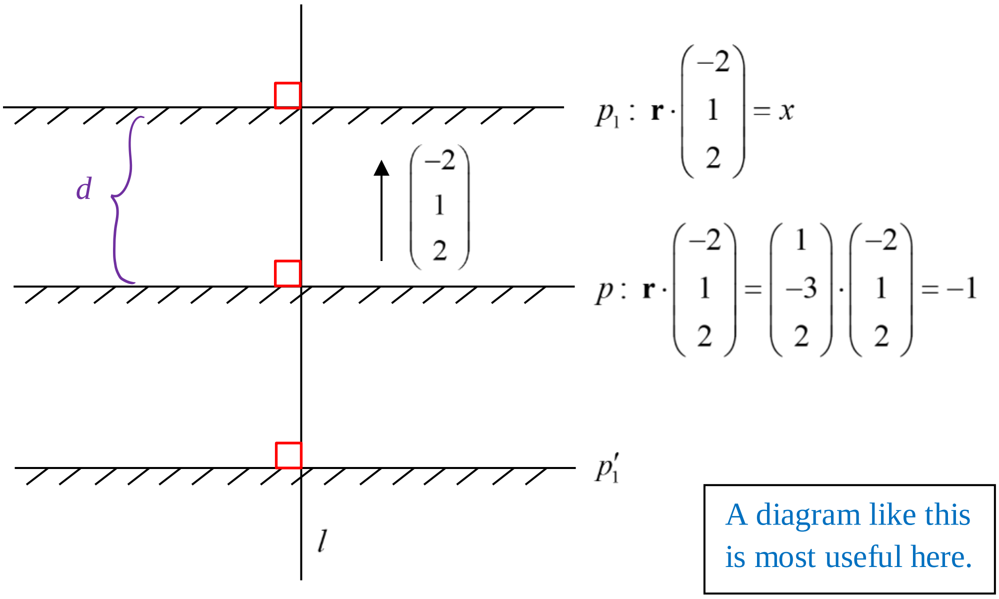
$\therefore$ l is perpendicular to p. (shown!)

When l and p meet,

$$\begin{pmatrix}1+\lambda+0\mu\\-3+2\lambda+4\mu\\2+0\lambda-2\mu\end{pmatrix}=\begin{pmatrix}-1-2t\\t\\1+2t\end{pmatrix}\Rightarrow\begin{cases}\lambda+0\mu+2t=-2\ ---(1)\\2\lambda+4\mu-t=3\ ---(2)\\0\lambda-2\mu-2t=-1\ ---(3)\end{cases}$$

Solving (1), (2), (3) simultaneously, $\lambda=-\frac{8}{9},\ \mu=\frac{19}{18},\ t=-\frac{5}{9}$

11(i)(b)



A diagram like this is most useful here.

The required planes, p_1 and p_1', must be parallel to p.

$$d = 12 \Rightarrow \frac{|x-(-1)|}{\left|\begin{pmatrix}-2\\1\\2\end{pmatrix}\right|} = 12 \Rightarrow \frac{|x+1|}{3} = 12$$

$$|x+1| = 36 \Rightarrow x+1 = \pm 36 \Rightarrow x = -35 \text{ or } 37$$

$\therefore$ The planes have equations $-2x+y+2z=-37$ and $-2x+y+2z=35$.

11(ii)

For l not to meet p at a unique point, l has to be parallel to p
(either do not intersect or intersect at infinity many points)
i.e. l is perpendicular to the normal vector of p.

A normal to p = $\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}a\\4\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\4-2a\end{pmatrix}$

$$\therefore \begin{pmatrix}-2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\4-2a\end{pmatrix} = 0$$

$$8+2+8-4a = 0$$

$$a = 4.5$$