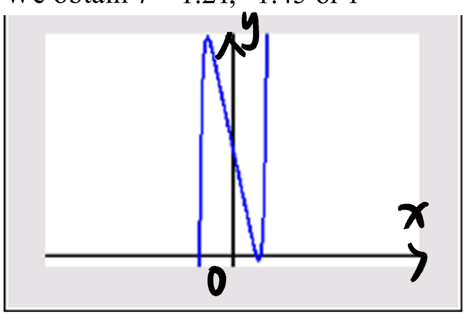
2017 H2 Mathematics Paper 2 Pure Math (Suggested Solutions)

<table>
<tr><td>1(i)</td><td>
Substitute $x=\frac{3}{t}, y=2t$ into $y=2x$
$2t=2\left(\frac{3}{t}\right)$
$t^2=3$
$t=\pm\sqrt{3}$
when $t=\sqrt{3}, x=\sqrt{3}, y=2\sqrt{3} \quad \therefore A\left(\sqrt{3},2\sqrt{3}\right)$
when $t=-\sqrt{3}, x=-\sqrt{3}, y=-2\sqrt{3} \quad \therefore B\left(-\sqrt{3},-2\sqrt{3}\right)$
Length of AB = $\sqrt{\left(\sqrt{3}+\sqrt{3}\right)^2+\left(2\sqrt{3}+2\sqrt{3}\right)^2}=\sqrt{60}=2\sqrt{15}$ units
</td></tr>
<tr><td>1(ii)</td><td>
At $P\left(\frac{3}{p},2p\right)\Rightarrow t=p$
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}=2\div\left(\frac{-3}{t^2}\right)=-\frac{2t^2}{3}$
Gradient of tangent at P = $-\frac{2p^2}{3}$
Equation of tangent at P:
$\left(y-2p\right)=-\frac{2p^2}{3}\left(x-\frac{3}{p}\right)$
$y=-\frac{2p^2}{3}x+4p$

At D : let y = 0
$x=\frac{6}{p} \quad \therefore$ coordinate of $D\left(\frac{6}{p},0\right)$
At E : let x = 0
$y=4p \quad \therefore$ coordinate of $E\left(0,4p\right)$

Coordinate of F= $\left(\frac{\frac{6}{p}+0}{2},\frac{4p+0}{2}\right)=\left(\frac{3}{p},2p\right)$
</td></tr>
</table>

	Let $x=\dfrac{3}{p}\quad\cdots(1)$ $y=2p\quad\cdots(2)$ from (2), $p=\dfrac{y}{2}\cdots(3)$ substitute (3) into (1) $x=\dfrac{6}{y}$ $\therefore xy=6$ The Cartesian equation of the curve traced by F as p varies is given by $xy = 6$ (or any equivalent form).
2(i)	For the given arithmetic progression, $a=3, S_{13}=156$ $\dfrac{13}{2}[6+12d]=156$ $d=1.5$
(ii)	For the given geometric progression, $a=3, S_{13}=156$ $\dfrac{3(r^{13}-1)}{r-1}=156$ $r^{13}-1=52r-52$ $r^{13}-52r+51=0\quad$ (shown) Using GC to sketch the graph of $y=x^{13}-52x+51$ and solve for the values of x-intercepts. We obtain $r=1.21,-1.45$ or 1  If $r = 1$, all the terms of the GP will be the same resulting $S_{13}\neq 156$. Hence, $r\neq 1$ even though $r=1$ is a root of the equation. $\therefore$Possible values of the common ratio are 1.21, -1.45.

(iii)

Since $r > 0, r = 1.210024$

$3(1.210024)^{n-1} > 100[3+(n-1)1.5]$

$3(1.210024)^{n-1} > 150+150n$

$3(1.210024)^{n-1} - 150 - 150n > 0$

Using GC to solve,

n	$3(1.210024)^{n-1} - 150 - 150n$
41	-149.9 < 0
42	991.74 > 0
43	2404.7 > 0

$\therefore$ The smallest possible value of $n = 42$.

3(a)

(i) $y = \mathrm{f}(2x)$ results in a scaling of the graph of y = f(x) with scale factor $\frac{1}{2}$ parallel to x-axis

$$(a,0) \xrightarrow{\mathrm{f}(2x)} \left(\frac{a}{2},0\right)$$

coordinates of x intercept: $\left(\frac{a}{2},0\right)$

coordinates of y intercept: $(0,b)$

(ii) $y = \mathrm{f}(x-1)$ results in a translation of the graph of y = f(x) by 1 unit in the positive direction of the x-axis

$$(a,0) \xrightarrow{\mathrm{f}(x-1)} (a+1,0)$$

$$(0,b) \xrightarrow{\mathrm{f}(x-1)} (1,b)$$

$\therefore$coordinates of y intercept: not possible to find

(iii) $y = \mathrm{f}(2x-1)$ results in a translation of the graph of y = f(x) by 1 unit in the positive direction of the x-axis, followed by scaling with scale factor $\frac{1}{2}$ parallel to the x-axis.

$$(a,0) \xrightarrow{\mathrm{f}(2x-1)} \left(\frac{a+1}{2},0\right)$$

$$(0,b) \xrightarrow{\mathrm{f}(2x-1)} \left(\frac{1}{2},b\right)$$

coordinates of x intercept: $\left(\frac{a+1}{2},0\right)$

coordinates of y intercept: not possible.

	(iv) $y=\mathrm{f}^{-1}(x) \Rightarrow$ reflection of the graph of $y=\mathrm{f}(x)$ about the line $y=x$ $(a,0) \xrightarrow{\mathrm{f}^{-1}(x)} (0,a)$ $(0,b) \xrightarrow{\mathrm{f}^{-1}(x)} (b,0)$ coordinates of y intercept: $(0,a)$ coordinates of x intercept: $(b,0)$
(b) (i)	$a=1$ When $a=1$, the value of $\mathrm{g}(1)$ will be undefined. Hence, $\mathrm{g}(x)$ will not be a function. Thus, $a=1$ has to be excluded from the domain of g.
(b) (ii)	$\mathrm{g}^2(x)=\mathrm{gg}(x)=\mathrm{g}\left(1-\frac{1}{1-x}\right)=\mathrm{g}\left(\frac{-x}{1-x}\right)$ $=\mathrm{g}\left(\frac{x}{x-1}\right)$ $=1-\frac{1}{1-\left(\frac{x}{x-1}\right)}$ $=1-\frac{x-1}{-1}$ $=x, \quad x\in\mathbb{R}, x\neq 1$ Hence g is self-inverse $\Rightarrow g(x)=g^{-1}(x)$ $\therefore \mathrm{g}^{-1}: x\to 1-\frac{1}{1-x}, x\in\mathbb{R}, x\neq 1$. Alternatively, To find $\mathrm{g}^{-1}(x)$, Let $y=\mathrm{g}(x)$ $y=1-\frac{1}{1-x}$ $1-y=\frac{1}{1-x}$ $x=1-\frac{1}{1-y}$ $\mathrm{g}^{-1}: x\to 1-\frac{1}{1-x}, x\in\mathbb{R}, x\neq 1$

<table>
<tr>
<td>(b)
(iiii)</td>
<td>

$\mathrm{g}^2(b)=\mathrm{g}^{-1}(b)$

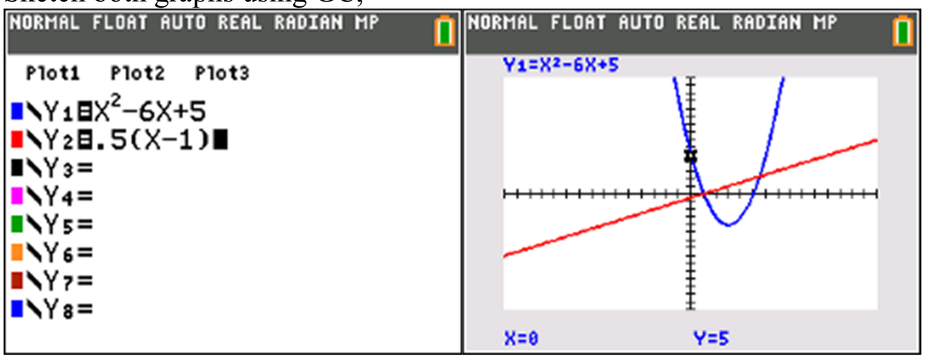
$b=1-\dfrac{1}{1-b}$

$(1-b)^2=1$

$b=0$ or $b=2$

</td>
</tr>
<tr>
<td>4(a)</td>
<td>

Sketch both graphs using GC,



Intersections points: (1, 0) and (5.5, 2.25)

Area of the plate = area enclosed

$$=\int_1^{5.5}\left(\frac{x-1}{2}-\left(x^2-6x+5\right)\right)\mathrm{d}x$$

$$=15.1875\text{ units}^2\text{(exact)}$$

</td>
</tr>
<tr>
<td>(b)
(i)</td>
<td>

Volume of the container

$$=\pi\times\int_0^1\left(\frac{\sqrt{y}}{a-y^2}\right)^2\mathrm{d}y$$

$$=\pi\times\int_0^1\left(\frac{y}{\left(a-y^2\right)^2}\right)\mathrm{d}y$$

$$=-\frac{\pi}{2}\times\int_0^1\left(-2y\left(a-y^2\right)^{-2}\right)\mathrm{d}y$$

$$=\frac{\pi}{2}\left[\frac{1}{a-y^2}\right]_0^1=\frac{\pi}{2}\frac{1}{a(a-1)}=\frac{\pi}{2a(a-1)}\text{ units}^3$$

</td>
</tr>
<tr>
<td>(b)
(ii)</td>
<td>

$\dfrac{\pi}{2b(b-1)}=4\times\dfrac{\pi}{2a(a-1)}$

$a^2-a=4b^2-4b$

$4b^2-4b+a-a^2=0$

</td>
</tr>
</table>

	$b=\dfrac{4\pm\sqrt{(-4)^2-4(4)(a-a^2)}}{2(4)}$ $=\dfrac{4\pm\sqrt{(-4)^2-4(4)(a-a^2)}}{2(4)}$ $=\dfrac{1}{2}\pm\dfrac{1}{2}\sqrt{1-a+a^2}$ $=\dfrac{1}{2}\left(1\pm\sqrt{1-a+a^2}\right)$ Since $a>1$, and volume has to be formed in the same way, $\therefore\ b>1$ $\sqrt{1-a+a^2}=\sqrt{\left(a-\dfrac{1}{2}\right)^2+\dfrac{3}{4}}>1 \quad \because \left(a-\dfrac{1}{2}\right)^2>\dfrac{1}{4}$ Since $b>1$, $\therefore b=\dfrac{1}{2}\left(1+\sqrt{1-a+a^2}\right)$
5(i)	The possible values of T are 2, 3, 4 and 5 $P(T=2)=P(RR)=\dfrac{6}{9}\times\dfrac{5}{8}=\dfrac{5}{12}$ $P(T=3)=P(RYR \text{ or } YRR)=\left(\dfrac{6}{9}\times\dfrac{3}{8}\times 2!\right)\times\dfrac{5}{7}=\dfrac{5}{14}$ since the first two counters can be RY or YR. $P(T=4)=P(RYYR)$ $=\left(\dfrac{6}{9}\times\dfrac{3}{8}\times\dfrac{2}{7}\times\dfrac{3!}{2!}\right)\times\dfrac{5}{6}=\dfrac{5}{28}$ The first 3 balls can be in any order but last one must be R. $P(T=5)=P(RYYYR)$ $=\left(\dfrac{6}{9}\times\dfrac{3}{8}\times\dfrac{2}{7}\times\dfrac{1}{6}\times\dfrac{4!}{3!}\right)\times\dfrac{5}{5}=\dfrac{1}{21}$ The first 4 balls can be in any order but last one must be R.
(ii)	$E(T)=\sum_{\text{all } t} t\, P(T=t)$ $=2\times\dfrac{5}{12}+3\times\dfrac{5}{14}+4\times\dfrac{5}{28}+5\times\dfrac{1}{21}$ $=\dfrac{20}{7}=2\dfrac{6}{7}$

$$E(T^2) = \sum_{\text{all } t} t^2 P(T = t)$$

$$= 2^2 \times \frac{5}{12} + 3^2 \times \frac{5}{14} + 4^2 \times \frac{5}{28} + 5^2 \times \frac{1}{21} = \frac{125}{14}$$

$$\text{Var}(T) = \text{E}(T^2) - [\text{E}(T)]^2$$

$$= -[6]^2$$

$$= \frac{125}{14} - \left(\frac{20}{7}\right)^2 = \frac{75}{98}$$

(iii)

$$\text{P}(T \geq 4) = \text{P}(T = 4) + \text{P}(T = 5) = \frac{19}{84}$$

Let X denote the number of games, out of 15, that Lee takes out at least 4 counters out of the bag

$$X \sim B\left(15, \frac{19}{84}\right)$$

$$P(X \geq 5) = 1 - P(X \leq 4)$$

$$= 0.23791$$

$$\approx 0.238 \text{ (to 3 s.f)}$$

6

(i) Since the 4 cards in each family must be next to each other, number of ways to arrange 4 cards within each family = $(4!)^5$.

$\therefore$ Number of possible ways $= 5! \times (4!)^5$

$= 955514880$

(ii) Let R, B, G, Y and O be Red, Blue, Green, Yellow and Orange
Let M, F, D, S be Mother, Father, Daughter, Son

$M_R\ D_R\ S_R$	$F_R\ F_G\ F_Y\ F_O\ F_B$	$M_B\ D_B\ S_B$	X X X X X X X X X
3!	Fixed 3! Fixed $\underbrace{\qquad\qquad}_{2!}$	3!	The remaining 9 cards

no. of ways for arranging the $M_R\ D_R\ S_R$ = 3!
no. of ways for arranging the $M_B\ D_B\ S_B$ = 3!

no. of ways for arranging the $F_G\ F_Y\ F_O$ = 3!

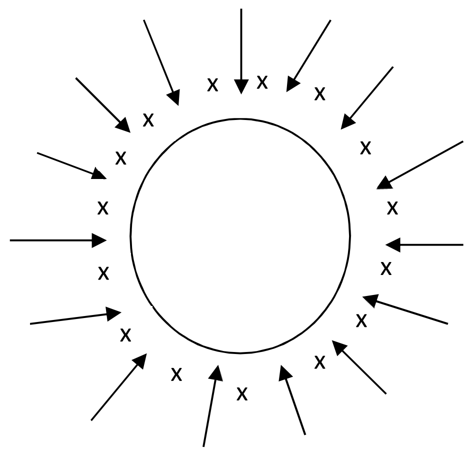
no. of ways for arranging the big unit and the remaining 9 cards = 10!

no. of ways for arranging the Red and Blue families at the ends of the big unit = 2

$$\text{Number of possible ways} = (3!)^3 \times 10! \times 2$$
$$= 1567641600$$

(iii) No. of ways to arrange the non-fathers in a circle = $(15-1)!$

No. of ways to slot in the fathers = ${}^{15}P_5$



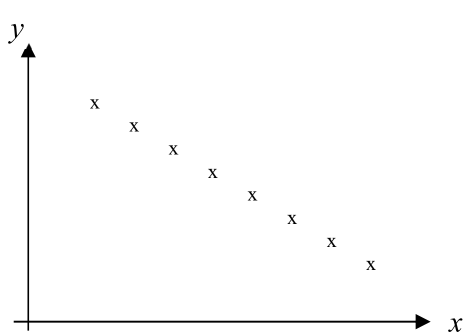
$$\therefore \text{Required probability} = \frac{(15-1)! \times {}^{15}P_5}{(20-1)!}$$
$$= \frac{1001}{3876} \text{ or } 0.258$$

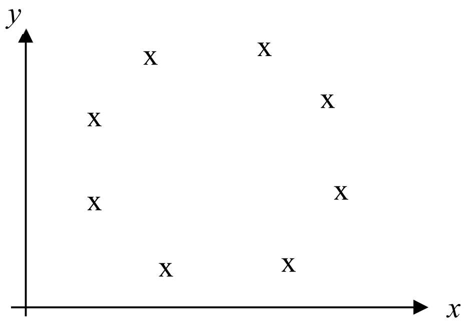
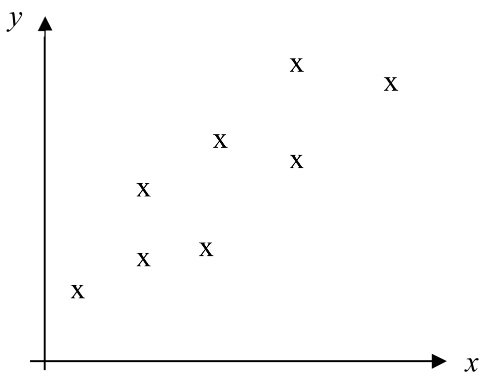
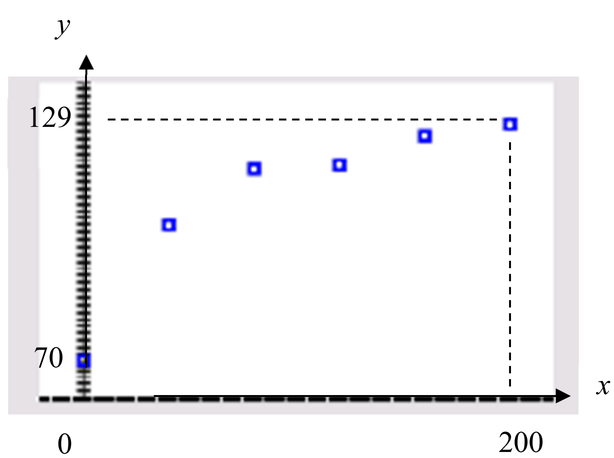
7 (i) A random sample is a sample drawn in such a way that each biscuit bar in the population has an equal chance of being selected as a member of the sample and each biscuit bar must be selected independently.

(ii)) Let X be the random variable" the mass of certain type of biscuit bar, in grams and μ be the population mean mass of biscuit bars, in grams".

Unbiased estimate of population mean,

$$\bar{x} = \frac{\sum(x-32)}{40} + 32 = \frac{-7.7}{40} + 32 = 31.8075 \approx 31.8$$

	Unbiased estimate of population variance, $s^2 = \frac{1}{39}[\sum(x-32)^2 - \frac{\left(\sum(x-32)\right)^2}{40}]$ $= \frac{1}{39}[11.05 - \frac{(-7.7)^2}{40}]$ $= 0.24533 = 0.245$ (3 s.f) (iii) Test H$_0$: $\mu = 32$ Against H$_1$: $\mu \neq 32$ at the 1 % of significance Under H$_0$, Since n = 40 is large, by Central Limit Theorem $\overline{X} \sim N(32, \frac{0.24533}{40})$ approximately Using a two tailed z-test, $\overline{x} = 31.8$ gives rise to z_{calc} = -2.46 and p- value = 0.0140. Since p-value > 0.01, we do not reject H$_0$ and conclude that at the 1% level of significance. there is insufficient evidence that the mean mass of biscuit bars is not 32 grams. (iv) The sample size, n = 40 is sufficiently large for the distribution of the sample mean of the masses of the biscuit bars to be approximately normally distributed by Central Limit Theorem, hence the production manager does not need to know anything about the population distribution of the masses of the biscuit bars.
8 (a)	***Not tested in 2020 A levels Exams*** (i)  (ii)

	 (iii) 
8 (b)	 Model D is appropriate (ii) Using GC, a = 4.18 b = 74.0 r = 0.981 (iii) The r value is close to 1 which suggest a strong positive linear correlation between the average yields of corn and the amount of fertiliser applied. Also, x = 189 is within the given data range (interpolation).

9

(i)
The probability that a light being faulty is constant at 0.08 for every light.
The event that the light is faulty is independent of another light being faulty.

(ii) Let X be the random variable "the number of faulty lights in a box of 12".

$$X \sim \mathrm{B}(12, 0.08)$$

$$\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 0.632$$

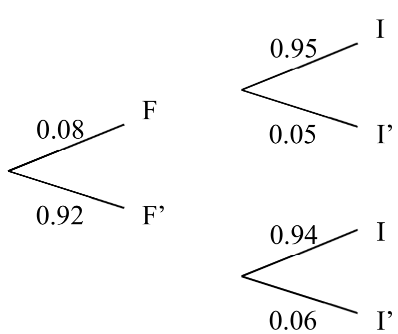
(iii) Required Probability= $[\mathrm{P}(X \geq 1)]^{20} = 1.04 \times 10^{-4}$

(iv) Let Y be the random variable" number of faulty lights in a carton of 240".

$$Y \sim \mathrm{B}(240, 0.08)$$

$$\mathrm{P}(Y \geq 20) = 1 - \mathrm{P}(Y \leq 19) = 0.458$$

(v) The answer to part (iv) is greater than the answer to part (iii) since the events in (iii) are a subset of events in (iv).



Legend: F: Faulty, I: Correctly Identified

(vi)

$$\mathrm{P}(\text{not faulty} \mid \text{identified faulty}) = \frac{\mathrm{P}(\text{not faulty and identified as faulty})}{\mathrm{P}(\text{identified as faulty})}$$

$$= \frac{\mathrm{P}(F'I')}{\mathrm{P}(FI) + \mathrm{P}(F'I')}$$

$$= \frac{0.92 \times 0.06}{0.92 \times 0.06 + 0.08 \times 0.95}$$

$$= \frac{69}{164} \text{ or } 0.421$$

	(vii) $\mathrm{P}\big(\text{Correctly identified lights as faulty or not faulty}\big)$ $= \mathrm{P}(\text{faulty and correctly identified}) + \mathrm{P}(\text{not faulty and correctly identified})$ $= 0.08 \times 0.95 + 0.92 \times 0.94$ $= 0.941$ (viii) From (vii), the quick test seem to be 94% accurate. However, from (vi), we understand that out of the number of lights identified faulty, 42.1% of them will be incorrectly identified. As such, the quick test is not worthwhile, since light identified as faulty are probably discarded. Moreover, there is cost involved in administering a test which needs to be considered.
10	(i) Let X be the random variable "the mass of a sphere, in grams". $X \sim \mathrm{N}(20, 0.5^2)$ $\mathrm{P}(X > 20.2) = 0.345$ (ii) Let Y be the random variable "the mass of a coated sphere, in grams". $Y = 1.1X \sim \mathrm{N}(1.1 \times 20, 1.1^2 \times 0.5^2)$ $Y \sim \mathrm{N}(22, 0.55^2)$ $\mathrm{P}(21.5 < Y < 22.45) = 0.612$ (iii) Let M be the random variable "the mass of a bar, in grams". $M \sim \mathrm{N}(\mu, \sigma^2)$ where μ and σ are the mean and standard deviation of the masses of metal bars respectively. $Z = \dfrac{M-\mu}{\sigma} \sim N(0,1)$ $\mathrm{P}(M > 12.2) = 0.6 \Rightarrow P(Z > \dfrac{12.2-\mu}{\sigma}) = 0.6$ $\dfrac{12.2-\mu}{\sigma} = -0.25335$ ---(1) $\mathrm{P}(M < 12) = 0.25 \Rightarrow \mathrm{P}(\mathrm{Z} < \dfrac{12-\mu}{\sigma}) = 0.25$ $\dfrac{12-\mu}{\sigma} = -0.67449$ ---(2) From equation (1): $-0.25335\sigma + \mu = 12.2$ ---(3) Frome equation (2): $\mu - 0.67449\sigma = 12$ ---(4) Solving (3) and (4) mean mass, $\mu = 12.3$ g standard deviation, $\sigma = 0.475$ g

	(iv) $Y_1 + Y_2 + M \sim \mathrm{N}(56.3203, 0.83063)$ $\mathrm{E}(Y_1 + Y_2 + M) = 2(22) + 12.3 = 56.320$ $\mathrm{Var}(Y_1 + Y_2 + M) = 2(0.55^2) + 0.475^2 = 0.83063$ $\mathrm{P}(Y_1 + Y_2 + M > k) = 0.75$ Using G.C $k = 55.7$