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PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/02**
Paper 2

26 August 2025 Tuesday 2 hours 15 min

2025 SECONDARY FOUR EXPRESS / FIVE NORMAL (ACADEMIC)
PRELIMINARY EXAMINATIONS

WORKED SOLUTIONS

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{(n-r)!r!} = \dfrac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 The equation of a curve is $y = \dfrac{5x}{6-2x}$. Find the x-coordinates of the points at which the gradient of the curve is 7.5. [4]

$$y = \frac{5x}{6-2x}$$

$$\frac{dy}{dx} = \frac{(6-2x)(5) - 5x(-2)}{(6-2x)^2}$$

$$= \frac{30}{(6-2x)^2}$$

$$\frac{30}{(6-2x)^2} = 7.5$$

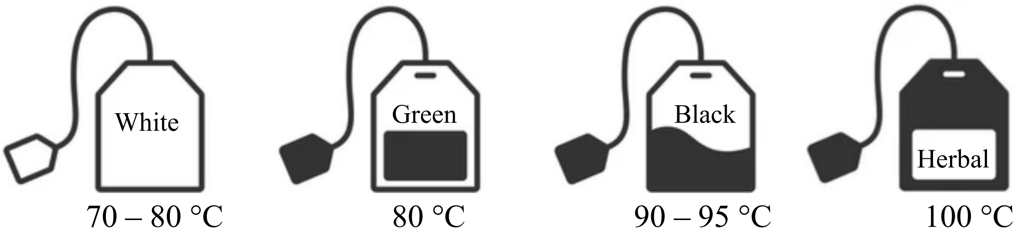
$$(6-2x)^2 = 4$$

$$6-2x = \pm 2$$

$$-2x = -4 \quad or \quad -8$$

$$x = 2 \quad or \quad 4$$

2 The temperature of water in a tea cup, T °C, t minutes after boiling water is poured into the empty cup can be modelled by the formula $T = 25 + 75\mathrm{e}^{-kt}$.



The diagram above shows the ideal brewing temperature for various teas at a particular teahouse. At this teahouse, green tea leaves are placed into the tea cup for brewing at the ideal temperature when $t = 6$. Determine whether the teahouse's standards for brewing tea would be adhered to if black tea leaves are placed into the teacup when $t = 2$. [4]

When $t = 6$, $T = 80$,

$$80 = 25 + 75e^{-6k}$$

$$\frac{55}{75} = e^{-6k}$$

$$\ln\frac{55}{75} = \ln e^{-6k}$$

$$k = -\frac{1}{6}\ln\frac{55}{75} = 0.0516925$$

Alternative:

$$\ln 55 = \ln\left(75e^{-6k}\right)$$

$$\ln 55 = \ln 75 - 6k$$

$$k = \frac{1}{6}(\ln 75 - \ln 55)$$

When $t = 2$, $T = 92.6$ °C

Since the temperature of the water would be within the ideal range of 90 – 95 °C, the standards for brewing black tea **would** be adhered to.

3 It is given that $f'(x) = 6\cos 3x - 12\sin 3x$ and $f\left(\frac{\pi}{3}\right) = 0$.

(a) Find f(x). [4]

$$f'(x) = 6\cos 3x - 12\sin 3x$$

$$f(x) = \frac{6\sin 3x}{3} + \frac{12\cos 3x}{3} + c$$

$$= 2\sin 3x + 4\cos 3x + c$$

$$f\left(\frac{\pi}{3}\right) = 0$$

$$2\sin\pi + 4\cos\pi + c = 0$$

$$c = 4$$

$$\therefore\ f(x) = 2\sin 3x + 4\cos 3x + 4$$

(b) Find the exact value of $f''\left(\frac{\pi}{12}\right)$. [2]

$$f''(x) = -18\sin 3x - 36\cos 3x$$

$$f''\left(\frac{\pi}{12}\right) = -18\sin\left(\frac{\pi}{4}\right) - 36\cos\left(\frac{\pi}{4}\right)$$

$$= -18\left(\frac{\sqrt{2}}{2}\right) - 36\left(\frac{\sqrt{2}}{2}\right)$$

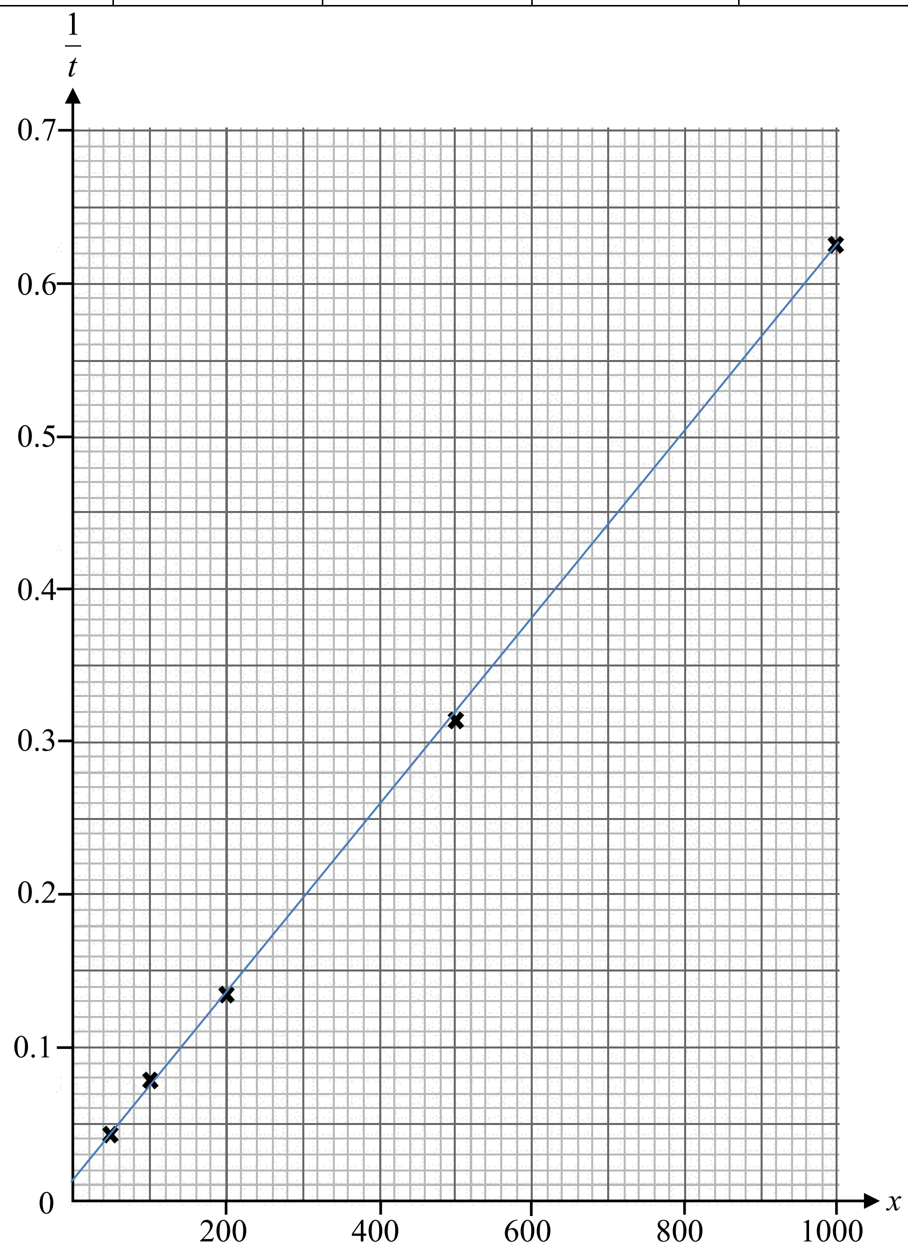
$$= -27\sqrt{2}$$

4 **(a)** The time taken, t in minutes, to download a particular gaming software is related to internet speed, x Mbps. The variables x and t are related by the formula $t = \frac{a}{b+x}$, where a and b are positive constants. The data below shows some measured values of x and t.

x (Mbps)	50	100	200	500	1000
t (minutes)	23	13	7.3	3.2	1.6

(i) Plot $\frac{1}{t}$ against x and draw a straight line graph. [2]

$\frac{1}{t}$	0.043	0.077	0.137	0.313	0.625



(ii) Use the graph to estimate the value of a and of b, correct to 2 significant figures. [4]

$$\frac{1}{t} = \frac{b}{a} + \frac{1}{a}x$$

$$\text{Gradient} = \frac{0.62 - 0.02}{1000} = 0.0006$$

$$\frac{1}{a} = 0.0006$$

$$a = \frac{1}{0.0006} = 1700 \ (2 \ s.f.)$$

$$\frac{b}{a} = 0.02$$

$$b = \frac{0.02}{0.0006} = 33 \ (2 \ s.f.)$$

(iii) Use the graph to estimate the time taken for John to download the gaming software if the speed of his internet is 240 Mbps. [2]

From the graph,

$$\frac{1}{t} = 0.16$$

$$t = 6.25 \text{ min}$$

(iv) Suggest an alternative set of variables to be plotted on each axis to draw a straight line to represent the formula. [1]

$$tb + xt = a$$

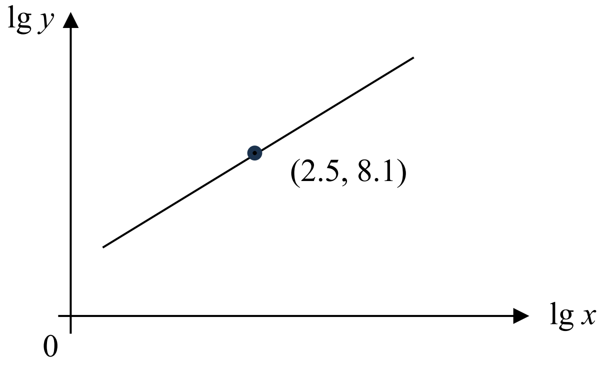
$$xt = -bt + a$$

An alternative would be to plot

xt against t **OR** t against xt

(b) The diagram shows part of a straight line graph drawn to represent the equation $y = px^q$.

Given that the line passes through the point (2.5, 8.1) and has a gradient of 3, find the value of p correct to the nearest integer. [3]



$y = px^q$

$\lg y = \lg p + q \lg x$

When $\lg y = 8.1$, $\lg x = 2.5$,

$8.1 = \lg p + 3(2.5)$

$\lg p = 0.6$

$p = 10^{0.6} = 4$ (to nearest integer)

5 **(a)** Find the range of values of the constant k for which $k\left(x^2+x\right)+2\left(x^2+x+1\right)=0$ has no real roots. [4]

$(k+2)x^2+(k+2)x+2=0$

Since there are no x-intercepts,

$(k+2)^2-4(k+2)(2)<0$

$k^2+4k+4-8k-16<0$

$k^2-4k-12<0$

$(k-6)(k+2)<0$

$\therefore\ -2<k<6$

(b) Hence, explain why $-\left(x^2+x\right)+2\left(x^2+x+1\right)$ is always positive. [2]

Since the **discriminant < 0 when *k* = –1** and the **coefficient of x^2**, i.e. 1, is **positive**,

$-\left(x^2+x\right)+2\left(x^2+x+1\right)$ is always positive

6 **(a)** By using a suitable substitution, or otherwise, solve the equation

$3\log_6 x = \log_x 3 + \log_x 2 + \log_5 25$. [4]

$$3\log_6 x = \log_x 3 + \log_x 2 + \log_5 25$$

$$3\log_6 x = \log_x 6 + 2$$

$$3\log_6 x = \frac{1}{\log_6 x} + 2$$

Let $y = \log_6 x$

$$3y = \frac{1}{y} + 2$$

$$3y^2 - 2y - 1 = 0$$

$$(3y+1)(y-1) = 0$$

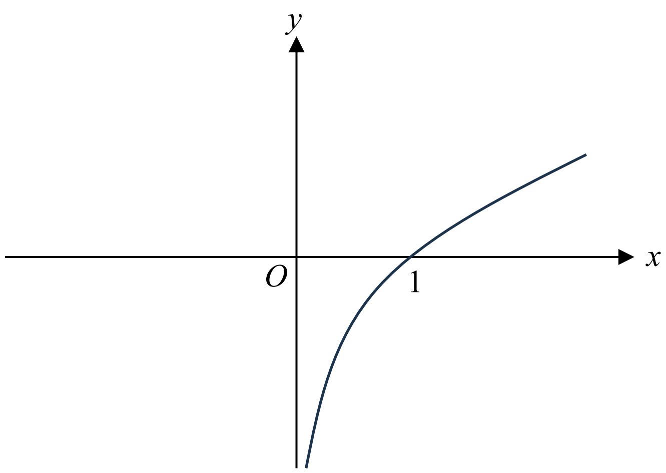
$$y = -\frac{1}{3} \quad or \quad y = 1$$

$$\log_6 x = -\frac{1}{3} \quad or \quad \log_6 x = 1$$

$$x = 6^{-\frac{1}{3}} \quad or \quad x = 6$$

$$= 0.550 \ (3s.f.)$$

(b) Sketch the graph of $y = \log_6 x$ on the given axes. Label any axial intercepts. [1]



(c) **(i)** It is given that the equation of a curve is $y = e^{2x+1} + 2e^{x+1} - 3e$.

Explain why the curve has no stationary point. [3]

$\frac{dy}{dx} = 2e^{2x+1} + 2e^{x+1}$

Since $e^{2x+1} > 0$ and $e^{x+1} > 0$ for all real values of x,

$2e^{2x+1} + 2e^{x+1} > 0$, $\frac{dy}{dx} \neq 0$

Therefore the curve has no stationary points

(ii) Find the x-intercept of the curve $y = e^{2x+1} + 2e^{x+1} - 3e$. [3]

$e^{2x} \cdot e + 2e^{x} \cdot e - 3e = 0$

$e^{2x} + 2e^{x} - 3 = 0$

$Let\ y = e^{x}$

$y^2 + 2y - 3 = 0$

$(y+3)(y-1) = 0$

$e^{x} = -3\ (rej) \quad or \quad e^{x} = 1$

$x = 0$

7 **(a)** Factorise $x^3 - 125$. [1]

$x^3 - 125 = (x-5)\left(x^2 + 5x + 25\right)$

(b) The expression $-2x^3 + 7x^2 + ax + b$, where a and b are integers, has a factor of $x - 5$ and a remainder of -11 when divided by $2x + 1$. Find the value of a and of b. [4]

Let $f(x) = -2x^3 + 7x^2 + ax + b$.

$f(5) = 0$

$-2(5)^3 + 7(5)^2 + a(5) + b = 0$

$5a + b = 75 \text{ ----- } (1)$

$f\left(-\frac{1}{2}\right) = -11$

$-2\left(-\frac{1}{2}\right)^3 + 7\left(-\frac{1}{2}\right)^2 + a\left(-\frac{1}{2}\right) + b = -11$

$2 - \frac{1}{2}a + b = -11$

$a = 2b + 26 \text{ ----- } (2)$

Subst (2) into (1):

$5(2b + 26) + b = 75$

$11b = -55$

$b = -5,\ a = 16$

(c) Hence, or otherwise, show that the equation $x^3 - 125 = -2x^3 + 7x^2 + ax + b$ has only one real root. [4]

$x^3 - 125 = -2x^3 + 7x^2 + 16x - 5$

$(x-5)\left(x^2 + 5x + 25\right) = (x-5)\left(-2x^2 - 3x + 1\right)$

$(x-5)\left(3x^2 + 8x + 24\right) = 0$

$x - 5 = 0 \quad or \quad 3x^2 + 8x + 24 = 0$

$x = 5 \qquad b^2 - 4ac = 8^2 - 4(3)(24)$

$= -224$

< 0

There is no real root for $3x^2 + 8x + 24 = 0$.

Therefore there is only one real root $x = 5$.

<u>Alternative</u>

$x^3 - 125 = -2x^3 + 7x^2 + 16x - 5$

$3x^3 - 7x^2 - 16x - 120 = 0$

$(x-5)\left(3x^2 + 8x + 24\right) = 0$

8 The equation of a curve is $y = e^{-2x}\tan x$.

(a) Show that $\dfrac{dy}{dx} = e^{-2x}(1 - \tan x)^2$. [3]

$$\frac{dy}{dx} = e^{-2x}\sec^2 x - 2e^{-2x}\tan x$$
$$= e^{-2x}\left(\tan^2 x + 1 - 2\tan x\right)$$
$$= e^{-2x}(\tan x - 1)^2$$

(b) Find, in terms of π, the x-coordinate of the stationary point for $0 < x < \dfrac{\pi}{2}$. [2]

$e^{-2x}(\tan x - 1)^2 = 0$

Since $e^{-2x} > 0$ for all real values of x,

$(\tan x - 1)^2 = 0$

$\tan x = 1$

$x = \dfrac{\pi}{4}$

(c) Explain why y is never decreasing. [2]

Since $e^{-2x} > 0$ and $(\tan x - 1)^2 \geq 0$ for all real values of x,

$\frac{dy}{dx} \geqslant 0$ therefore y is never decreasing.

(d) What does your answer to **part (c)** imply about the stationary point found in **part (b)**? Explain your answer. [2]

The stationary point at $x = \frac{\pi}{4}$ is a **point of inflexion.**

Since y is never decreasing, the **gradient of the curve must be positive (or dy/dx > 0) slightly to the left and right** of the stationary point.

9 **(a)** State

(i) the principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, [1]

$-60°$ or $-\frac{\pi}{3}$

(ii) the values between which the principal value of $\tan^{-1}x$ must lie, [1]

$-90° < \tan^{-1}x < 90°$

(iii) the range of values of k for which the equation $4\sin^2 x = k$ has a solution. [1]

$0 \leq k \leq 4$

(b) **(i)** Prove the identity $\cot x - \cot x \tan^2 x + \tan x = \dfrac{1+\cos 2x}{\sin 2x}$. [4]

$\cot x - \cot x \tan^2 x + \tan x$

$= \cot x - \dfrac{1}{\tan x}\tan^2 x + \tan x$

$= \dfrac{\cos x}{\sin x} - \tan x + \tan x$

$= \dfrac{2\cos^2 x}{2\sin x \cos x}$

$= \dfrac{1+\cos 2x}{\sin 2x}$

(ii) Hence solve the equation $\cot x - \cot x \tan^2 x + \tan x = \operatorname{cosec} 2x - 3$ for $0° \le x \le 180°$.

[5]

$\dfrac{1+\cos 2x}{\sin 2x} = \operatorname{cosec} 2x - 3$

$\dfrac{1+\cos 2x}{\sin 2x} = \dfrac{1}{\sin 2x} - 3$

$1+\cos 2x = 1 - 3\sin 2x$

$\tan 2x = -\dfrac{1}{3}$

$\tan \alpha = \dfrac{1}{3}$

$\alpha = \tan^{-1}\left(\dfrac{1}{3}\right) = 18.434°$

$2x = 180° - 18.434°,\ 360° - 18.434°$

$2x = 161.566°,\ 341.566°$

$x = 80.8°,\ 170.8°$

10 **(a)** Points $P(-2, 0)$, $Q(5, 1)$, $R(6, -6)$ and $S(-1, -7)$ lie on a circle.

PQ and SR are parallel chords of the same length.

Show that the centre of the circle is at $(2, -3)$. [2]

Centre of circle

$$= \left(\frac{-2+6}{2}, \frac{0-6}{2}\right)$$

$$= (2, -3)$$

OR

$$\text{Midpoint of } PQ = \left(\frac{-2+5}{2}, \frac{0+1}{2}\right)$$

$$= \left(\frac{3}{2}, \frac{1}{2}\right)$$

$$\text{Midpoint of } SR = \left(\frac{6-1}{2}, \frac{-6-7}{2}\right)$$

$$= \left(\frac{5}{2}, -\frac{13}{2}\right)$$

$$\text{Centre of circle} = \left(\frac{\frac{3}{2}+\frac{5}{2}}{2}, \frac{\frac{1}{2}-\frac{13}{2}}{2}\right)$$

$$= (2, -3)$$

(b) Find the equation of the circle. [3]

Radius

$$= \sqrt{(5-2)^2 + (1+3)^2}$$

$$= 5 \text{ units}$$

$$(x-2)^2 + (y+3)^2 = 25$$

OR

$$x^2 - 4x + y^2 + 6y - 12 = 0$$

(c) A second circle has centre C. It is given that the y-axis is the perpendicular bisector of CR and is also a tangent to the second circle.

(i) State the coordinates of C. [1]

Centre of second circle = (–6, –6)

(ii) Hence explain why the x-axis is a tangent to the second circle. [2]

Since the y-axis is a tangent to the second circle, the radius of the circle is 6 units. The highest point, H, of the second circle is (–6, 0).

Since the circle touches the x-axis at exactly one point at (–6, 0), the x-axis is a tangent to the second circle.

Or

Since CH is a vertical radius of the circle and H lies on the x-axis, the x-axis, which is horizontal, must be a tangent to the second circle (rad⊥tan)

11 A particle moves from point P towards point O. The speed of the particle, t seconds after passing point P, can be modelled by the formula $v = (k - 2t)^n$, where k and n are constants.

(a) Find an expression for the acceleration of the particle in terms of k, t and n. [1]

$a = -2n(k - 2t)^{n-1}$

The particle has a speed of 27 m/s when $t = 3.5$ and comes to instantaneous rest 4.8 metres to the right of point O when $t = 8$.

(b) Find the value of k and of n. [3]

When $t = 8$, $v = 0$,

$0 = (k - 16)^n$

$k = 16$

When $t = 3.5$, $v = 27$,

$27 = (16 - 7)^n$

$27 = 9^n$

$3^3 = 3^{2n}$

$n = 1.5$

(c) Using the values of k and n found in **part (b)**, find the total distance travelled by the particle in the first 8 seconds. [4]

$v = (16 - 2t)^{1.5}$

$s = \dfrac{(16 - 2t)^{2.5}}{(2.5)(-2)} + c$

When $t = 8,\ s = 4.8$

$4.8 = c$

$s = -\dfrac{1}{5}(16 - 2t)^{2.5} + 4.8$

When $t = 0,\ s = -200$

Total distance = 200 + 4.8 = 204.8 m

(d) Explain whether the given model can continue to describe the motion of the particle after 8 seconds. [1]

No since when $t > 8$, v and s are undefined.

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