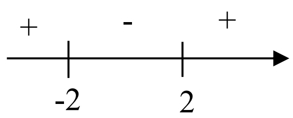
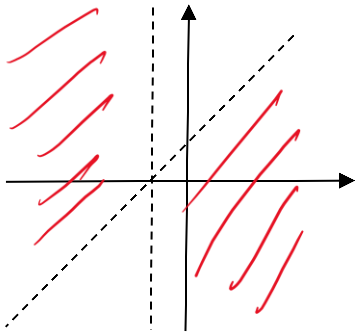
2025 ACJC H2 Math Preliminary Examination Paper 1 Markers' Report

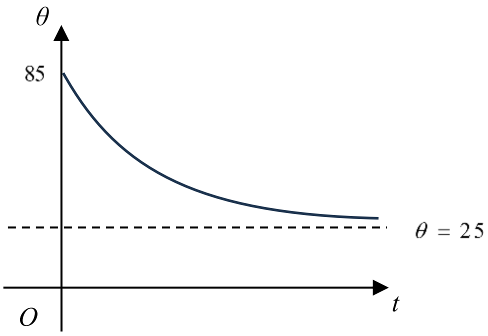
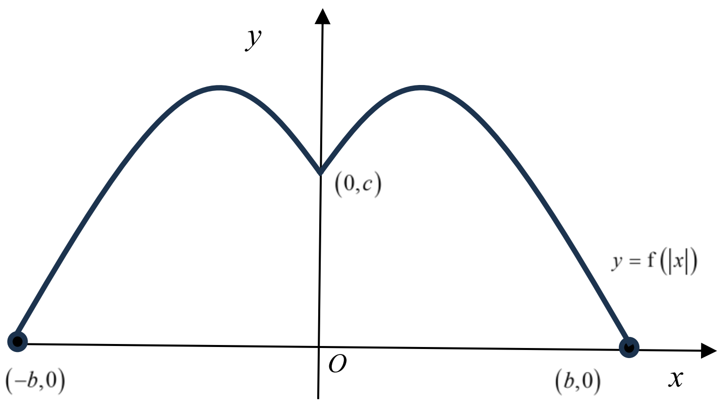
Qn	Solution	Remarks
1(a)	$\frac{x}{x+2} \geq \frac{2}{2-x}$ $\frac{x}{x+2} - \frac{2}{2-x} \geq 0$ $\frac{x(2-x)-2(x+2)}{(x+2)(2-x)} \geq 0$ $\frac{-x^2-4}{(x+2)(2-x)} \geq 0$ $\frac{x^2+4}{(x+2)(x-2)} \geq 0$ As $x^2+4>0$ for all real x, consider: $(x-2)(x+2) \geq 0$ $x<-2$ or $x>2$  	This is a standard inequality question and generally it was quite well done. Method: Move expressions to one side, common denominator, sign test Some common issues: 1. Multiplying -1 to both sides of inequality should result in an inequality sign flip. 2. When considering x^2+4, it is not sufficient to just look at whether it has roots (it matters whether it is always positive/negative). 3. $(x+2)(2-x)$ is a n-shaped quadratic curve. 4. In the final solution, $x \neq \pm 2$ is often forgotten.
1(b)	$\frac{\lvert x\rvert}{\lvert x\rvert+2} \geq \frac{2}{2-\lvert x\rvert}$ Replace x with $\lvert x\rvert$, $\lvert x\rvert<-2$ (no solutions) or $\lvert x\rvert>2$ $x<-2$ or $x>2$	The modulus function is still quite poorly understood by a significant minority. $\lvert x\rvert<-2$ has NO real solutions.
2(a)	Since vertical asymptote is $x=-1$, $d=1$. Since the oblique asymptote is $y=2x+2$, consider $y=2x+2+\frac{k}{x+1}$ $=\frac{(2x+2)(x+1)+k}{x+1}$ $=\frac{2x^2+4x+2+k}{x+1}$ Comparing with $y=\frac{ax^2+bx+c}{x+d}$, $a=2$ and $b=4$. **Alternatively,** Since vertical asymptote is $x=-1$, $d=1$. $y=\frac{ax^2+bx+c}{x+1}=ax+(b-a)+\frac{c-(b-a)}{x+1}$ (by long division). Comparing coefficients of $y=ax+(b-a)$ with $y=2x+2$, $a=2$ and $b=4$.	Graphing question, generally well-done. For students doing long division, there were still a minority who made careless slips in the procedure, resulting (thankfully for this part) in wrong remainders. Those who got the quotient ($ax+(b-a)$) would not have been able to show the given result. Also, as this is a 'show' question, the presentation of their method should be clear and substantial.

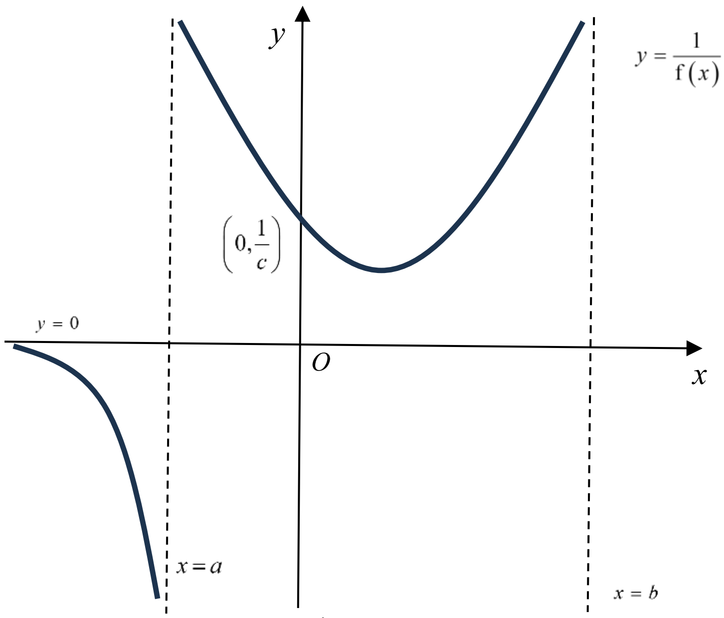
2(b)	$y=\dfrac{2x^2+4x+c}{x+1}=2x+2+\dfrac{c-2}{x+1}$, $c\neq 2$. For the graph to have no stationary points, the graphs will have to be in these 2 segments.  As $x\to\infty$, $y\to(2x+2)^-$ Hence $c-2<0 \quad\Rightarrow\quad c<2$. **Alternatively,** $\dfrac{dy}{dx}=2-\dfrac{c-2}{(x+1)^2}$ Considering $\dfrac{dy}{dx}=0$, $\dfrac{c-2}{(x+1)^2}=2$ $c-2=2(x+1)^2$ $2x^2+4x+4-c=0$ For the graph to have no stationary points, the $\dfrac{dy}{dx}=0$ equation should have no solutions, $16-4(2)(4-c)<0$ $-16+8c<0$ $c<2$	This part proved more challenging than **(a)**. Those who got the remainder ($c-2$) wrong would not have been able to get the correct answer. Most students considered the discriminant < 0, but a significant percentage of these solutions considered the discriminant of $2x^2+4x+c$ (which is the numerator of y). This is conceptually wrong. Students should be considering the discriminant of $\dfrac{dy}{dx}=0$. There were also many careless mistakes differentiating $\dfrac{c-2}{x+1}$, with some even arriving at a ln function.
3(a)	$y=x^{xy}$ $\ln y=xy\ln x$ $\dfrac{\ln y}{y}=x\ln x$ $\left(\dfrac{1-\ln y}{y^2}\right)\dfrac{dy}{dx}=x\left(\dfrac{1}{x}\right)+(1)\ln x$ $\left(\dfrac{1-\ln y}{y^2}\right)\dfrac{dy}{dx}=1+\ln x$ $\dfrac{dy}{dx}=\dfrac{y^2(1+\ln x)}{1-\ln y}$	Well done. Most realised the need to take ln on both sides before differentiating implicitly. Those who rearranged to $\dfrac{\ln y}{y}=x\ln x$ were most successful, while some who differentiated $\ln y=xy\ln x$ missed out the third term in applying the product rule on the RHS. Some mistakes observed: 1. $x^{xy}\neq x^x x^y$ 2. $\dfrac{\ln x}{\ln y}\neq\ln(x-y)$ 3. Assuming the base is a constant, e.g. $\dfrac{d}{dx}\left(x^{xy}\right)\neq\ln x\times x^{xy}\times\dfrac{d}{dx}(xy)$ 4. Assuming the index is a constant, e.g. $\dfrac{d}{dx}\left(x^{xy}\right)\neq\dfrac{d}{dx}(xy)\times x^{xy-1}$

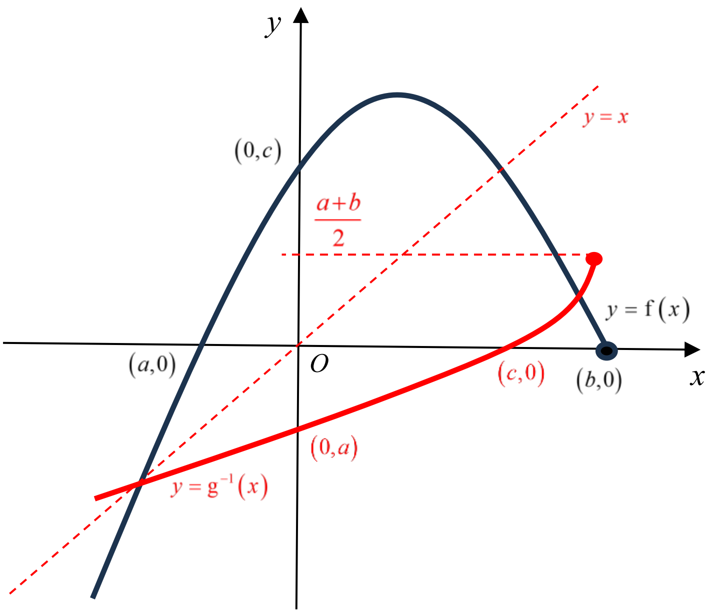
	Alternatively, $y = x^{xy}$ $\ln y = xy \ln x$ $\frac{1}{y}\frac{dy}{dx} = x\ln x\frac{dy}{dx} + y(1+\ln x)$ $\frac{dy}{dx} = \frac{y(1+\ln x)}{\frac{1}{y} - x\ln x}$ $= \frac{y^2(1+\ln x)}{1-xy\ln x}$ $= \frac{y^2(1+\ln x)}{1-\ln y}$	
3(b)	Tangent // to y-axis, $\frac{dy}{dx}$ is undefined. $1-\ln y = 0$ $\ln y = 1$ $\therefore y = e$ $\ln e = xe\ln x \Rightarrow x\ln x = \frac{1}{e}$ $\therefore x = 1.32$ (by GC) The coordinates of the point is $(1.32, e)$.	Most equated the denominator of $\frac{dy}{dx}$ to zero correctly. Many got stuck solving $x\ln x = \frac{1}{e}$, and they should be reminded to use the GC. A handful did not give the final answer in coordinates, swopped x- and y-coordinates, or gave $\left(1.32, \frac{1}{e}\right)$ as the final answer.
4	Volume $= \pi\int_e^{e^2} y^2 \, dx = \pi\int_e^{e^2} x^2(\ln x)^2 \, dx$ $\int x^2(\ln x)^2 \, dx = \frac{x^3}{3}(\ln x)^2 - \int \frac{x^3}{3}(2\ln x)\left(\frac{1}{x}\right) dx$ $= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx$ $= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{1}{x}\right) dx\right)$ $= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 \, dx\right)$ $= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right) + c$ $= \frac{1}{27}x^3\left(9(\ln x)^2 - 6\ln x + 2\right) + c$ $\pi\int_e^{e^2} x^2(\ln x)^2 \, dx = \frac{\pi}{27}\left[x^3\left(9(\ln x)^2 - 6\ln x + 2\right)\right]_e^{e^2}$ $= \frac{\pi}{27}\left(e^6(36-12+2) - e^3(9-6+2)\right)$ $= \frac{\pi}{27}(26e^6 - 5e^3) = \frac{\pi e^3}{27}(26e^3 - 5)$ $a = 26, b = -5$	Most students knew the procedure to be carried out, i.e. obtain the correct limits, apply the formula, apply integration by parts, and evaluate the limits. Almost all cited the correct volume formula, with common mistakes (i) using $\pi\int y \, dx$ and $2\pi\int y^2 \, dx$; and (ii) finding the wrong region or limits of integration. Most could apply integration by parts correctly once, but many made sign errors applying it twice. Some made algebraic slips at the outset that made them think that only one integration by parts was necessary. These mistakes include: 1. $(\ln x)^2 \neq \ln(x^2) = 2\ln x$ 2. $\int (x\ln x)^2 \, dx \neq \left(\int x\ln x \, dx\right)^2$

5(a)	$\mathbf{a}\cdot\mathbf{n}=\mathbf{b}\cdot\mathbf{n}$ $(\mathbf{b}-\mathbf{a})\cdot\mathbf{n}=0$ $\overrightarrow{AB}\cdot\mathbf{n}=0$ $\overrightarrow{AB}\perp\mathbf{n}$ $\therefore\overrightarrow{AB}$ is <u>parallel</u> to the plane p.	Not well done. It is insufficient to say that **a** and **b** are coplanar with normal vector **n**, as the property that $\mathbf{r}\cdot\mathbf{n}$ is a constant for planes is derived from this proof. In writing proofs, students should ensure logical consistency, i.e. 1. It should not insinuate that $\mathbf{a}\cdot\mathbf{n}=0$ or $\mathbf{b}\cdot\mathbf{n}=0$ (both are given to be non-zero quantities) 2. $\mathbf{a}\cdot\mathbf{n}=\mathbf{b}\cdot\mathbf{n}$ does <u>not</u> imply that $\mathbf{a}=\mathbf{b}$ State *geometrical relationships* precisely (e.g. "parallel" or "perpendicular"), as opposed to: 1. More colloquial expressions like "lies in" and "contains", which is also vague here as neither A nor B are on the plane p; 2. "length of projection" when neither vector is unit and is a geometrical *interpretation*; and 3. "direction vector of the plane" which applies to *lines*.
5(b)	$\mathbf{r}=\lambda(\mathbf{b}-\mathbf{a})$ (Since origin lies on p.)	Not well done. Common errors are to think that A or B lie on P (and hence on the line), or that the line is parallel to **n**. Some did not realise that $\mathbf{r}\cdot\mathbf{n}=0$ meant that the origin is a known point on p. Abuse of notation, e.g. $l=\mathbf{r}+\lambda(\mathbf{a}-\mathbf{b})$, was observed.
5(c)	Substituting $\mathbf{r}=\mathbf{c}+\mu\mathbf{n}$ into $\mathbf{r}.\mathbf{n}=0$, $(\mathbf{c}+\mu\mathbf{n}).\mathbf{n}=0$ $\mathbf{c}.\mathbf{n}+\mu\lvert\mathbf{n}\rvert^2=0$ $\mu=-\dfrac{\mathbf{c}.\mathbf{n}}{\lvert\mathbf{n}\rvert^2}$ $\therefore\overrightarrow{OF}=\mathbf{c}-\left(\dfrac{\mathbf{c}.\mathbf{n}}{\lvert\mathbf{n}\rvert^2}\right)\mathbf{n}$ **Alternatively,** $\overrightarrow{FC}=\left(\overrightarrow{OC}.\hat{\mathbf{n}}\right)\hat{\mathbf{n}}=\dfrac{(\mathbf{c}.\mathbf{n})}{\lvert\mathbf{n}\rvert^2}\mathbf{n}$ $\therefore\overrightarrow{OF}=\overrightarrow{OC}+\overrightarrow{CF}=\mathbf{c}-\dfrac{(\mathbf{c}.\mathbf{n})}{\lvert\mathbf{n}\rvert^2}\mathbf{n}$	Not well done. Many could not use the line equation or vector projection formula, probably due to the abstraction of this question. Many arrived at $\mathbf{c}-\left(\dfrac{\mathbf{c}.\mathbf{n}}{\lvert\mathbf{n}\rvert^2}\right)\mathbf{n}$, but went ahead to "cancel" by **n** to get $\mathbf{c}-\dfrac{\mathbf{c}}{\mathbf{n}}\mathbf{n}=\mathbf{0}$. The final answer mark was withheld for this conceptual error, and students are reminded that this is not a valid vector operation. In a similar vein, $\mathbf{n}\cdot\mathbf{n}=\mathbf{n}^2$ was also seen. Some who attempted the line equation approach only considered $\overrightarrow{CF}\cdot\mathbf{n}$ instead of the equation of line CF. More mistakes were seen by students using the vector projection approach, including:

		1. Stating $\left(\overrightarrow{OC}.\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$ is $\overrightarrow{CF}$ or $\overrightarrow{OF}$ 2. Using $\overrightarrow{FC}=\left\lvert\overrightarrow{OC}.\hat{\mathbf{n}}\right\rvert\hat{\mathbf{n}}$ 3. Using $\overrightarrow{OF}=\left\lvert\overrightarrow{OC}\times\hat{\mathbf{n}}\right\rvert\hat{\mathbf{n}}$
6(a)	$\pi(r-a)^2(h-a)=1000\pi$ $h-a=\dfrac{1000}{(r-a)^2}\Rightarrow h=\dfrac{1000}{(r-a)^2}+a$ $V=\pi r^2h-\pi(r-a)^2(h-a)$ $=\pi r^2\left(\dfrac{1000}{(r-a)^2}+a\right)-1000\pi$ $=\dfrac{1000\pi r^2}{(r-a)^2}+\pi r^2a-1000\pi$ $=1000\pi\left[\dfrac{r^2}{(r-a)^2}-1\right]+\pi r^2a$ $k=1000$	Most students were able to get at least 2/3 marks. Common mistakes were 1. unnecessary expansion of $(r-a)^2(h-a)$. Students should look at the form to be shown as a guide 2. a lot of unnecessary working from not seeing the internal cylinder volume as , resulting in loss of time 3. misread 1000π to be 100π Note also that since *h* is defined in the question as the external height, some students still use *h* for the height of the internal cylinder, which resulted in presentation error even though they manage to show the expression.
6(b)	$\dfrac{\mathrm{d}V}{\mathrm{d}r}=1000\pi\left[\dfrac{2r(r-a)^2-2r^2(r-a)}{(r-a)^4}\right]+\pi r^2a$ $=1000\pi\left[\dfrac{2r(r-a)\left[(r-a)-r\right]}{(r-a)^4}\right]+2\pi ar$ $=-\dfrac{2000\pi ar(r-a)}{(r-a)^4}+2\pi ar$ $=-\dfrac{2000\pi ar}{(r-a)^3}+2\pi ar$ $\dfrac{\mathrm{d}V}{\mathrm{d}r}=-\dfrac{2000\pi ar}{(r-a)^3}+2\pi ar=0$ $\dfrac{2000\pi ar}{(r-a)^3}=2\pi ar$ $(r-a)^3=1000\Rightarrow r=10+a$	Most can differentiate using quotient/product rule correctly, although some reversed the order of the numerator when using quotient rule. However many were not able to solve $\dfrac{\mathrm{d}V}{\mathrm{d}r}=0$ due to inadequacies in algebraic manipulation, such as 1. unnecessary expansion, instead of factorisation and cancellation of common terms 2. not simplifying the expression, e.g. $r-a$ can be cancelled in $\dfrac{2000\pi ar(r-a)}{(r-a)^4}$

7(a)	$\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=k(\theta-25)$ $\displaystyle\int \frac{1}{\theta-25}\,\mathrm{d}\theta=kt+c$ $\Rightarrow \ln\|\theta-25\|=kt+c$ $\Rightarrow \theta-25=A\mathrm{e}^{kt}$ When $t=0,\theta=85\Rightarrow A=60\Rightarrow \theta-25=60\mathrm{e}^{kt}$ When $t=20,\theta=55\Rightarrow 30=60\mathrm{e}^{20k}\Rightarrow k=\dfrac{1}{20}\ln\dfrac{1}{2}(=)-0.0347$ $\therefore \theta=25+60\mathrm{e}^{\left(\frac{1}{20}\ln\frac{1}{2}\right)t}=25+60\left(\dfrac{1}{2}\right)^{\frac{t}{20}}=25+60\mathrm{e}^{-0.0347t}$	There are many who got full marks for this question even though it is in some sense unguided, showing that most students know the process of solving differential equations and find particular solutions clearly. There were a few who used the conditions right at the start. students need to know that if the conditions involve time, the conditions can only be used after obtaining the general solution.
7(b)		Most students were able to get the graph using the form given. Comon errors were 1. sketching for $t<0$ 2. the curve did not cut the y-axis 3. not indicating the horizontal asymptote or y-intercept
8(a) (i)		Most got this correct. Some who didn't study of course got it wrong. They reflected $x<0$ about the y-axis.

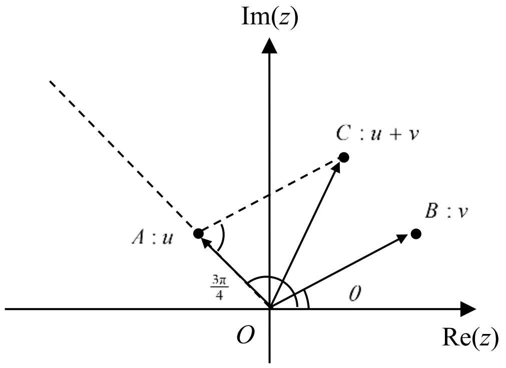
8(a) (ii)	 Fig. 1	Quite well done too. Comon errors 1. no y-intercept indicated 2. extra curve on $x>b$
8(a) (iii)	Scale the graph of $y=f(2x+1)$ by factor 2 parallel to the x-axis. (To obtain $y=f\left(2\left(\frac{x}{2}\right)+1\right)=f(x+1)$.) Translate the graph of $y=f(x+1)$ 1 unit in the positive x-direction. (To obtain $y=f((x-1)+1)=f(x)$.) **Alternatively,** Translate the graph of $y=f(2x+1)$ 0.5 units in the positive x-direction. (To obtain $y=f(2(x-0.5)+1)=f(2x)$.) Scale the graph of $y=f(2x)$ by factor 2 parallel to the x-axis. (To obtain $y=f\left(2\left(\frac{x}{2}\right)\right)=f(x)$.)	There were a lot of reading errors in this question. Many read as obtaining $y=f(2x+1)$ from $y=f(x)$. Those who did translation first often translated by 1 unit in the positive direction. This will result in a replacement of x by $x-1$, hence will give $y=f(2(x-1)+1)=f(2x-1)$ and not $y=f(2x)$. There were many who did scaling parallel to the y-axis. Marks were also deducted for students who use the terms "shift" or "scale by 2 units"
8(b) (i)	Since $y=f(x)$ is a quadratic curve, the maximum point occurs at $x=\frac{a+b}{2}$. Hence the maximum value of k is $\frac{a+b}{2}$ for $y=g^{-1}(x)$ to exist.	About one third got this right. Many put $x=\frac{b-a}{2}$.

8(b) (ii)	 Fig. 1 $R_{\mathrm{g}^{-1}}=D_g=\left(-\infty,\dfrac{a+b}{2}\right]$	This was poorly done. There were many answers that gave the reflection of the whole curve, instead of the part $x\le\dfrac{a+b}{2}$. There were many who gave me curves that are not one-one. (The curve "turn" too much and became concave upwards) There were also many answers that gave the y-intercept as $(0,-a)$ because it is below the x-axis. Marks were not deducted but students should realise that a need not always be a positive number.
8(b) (iii)	The solution to $\mathrm{g}^{-1}(x)=x$ is the x-coordinate of the intersection point between the graph of $y=\mathrm{g}^{-1}(x)$ and the line $y=x$. As the graphs of $y=\mathrm{g}^{-1}(x)$ and $y=\mathrm{g}(x)$ are the reflections of each other in the line $y=x$, if the graph of $y=\mathrm{g}^{-1}(x)$ intersects the line $y=x$, then the graph of $y=\mathrm{g}(x)$ will also intersect the line $y=x$ at the same point. Hence, the solution to $\mathrm{g}^{-1}(x)=x$ will also satisfy the equation $\mathrm{g}^{-1}(x)=\mathrm{g}(x)$.	The explanation "$y=\mathrm{g}^{-1}(x)$ and $y=\mathrm{g}(x)$ are the reflections of each other in the line $y=x$" is not sufficient, they have to mention that the two curves intersect at $y=x$ or suggest that is the case. The language used is also particularly bad as students do not how to describe curves. They will say "$\mathrm{g}(x)$ and $\mathrm{g}^{-1}(x)$ intersect at x", instead of "the graphs of $\mathrm{g}(x)$ and $\mathrm{g}^{-1}(x)$ intersect at the line $y=x$".
9(a)	Substitute $(1,3,-2)$ into $3x+c(y+z)-2=0$, $3(1)+c(3-2)-2=0\Rightarrow c=-1$	

	Alternatively: $\pi_1 : \mathbf{r}.\begin{pmatrix}3\\c\\c\end{pmatrix}=2$ $\begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}3\\c\\c\end{pmatrix}=3+3c-2c=2 \Rightarrow c=-1$	
9(b)	Sub $\mathbf{r}=\begin{pmatrix}1+2s+t\\3-s\\-2+3s-t\end{pmatrix}$ into $\mathbf{r}.\begin{pmatrix}3\\-1\\-1\end{pmatrix}=2$ $\begin{pmatrix}1+2s+t\\3-s\\-2+3s-t\end{pmatrix}.\begin{pmatrix}3\\-1\\-1\end{pmatrix}=2$ $3+6s+3t-3+s+2-3s+t=2$ $4s+4t=0$ $s=-t$ $\therefore$ line of intersection is $\mathbf{r}=\begin{pmatrix}1-2t+t\\3+t\\-2-3t-t\end{pmatrix}=\begin{pmatrix}1-t\\3+t\\-2-4t\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}-t\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+\alpha\begin{pmatrix}1\\-1\\4\end{pmatrix}$ **Alternatively:** $\mathbf{n}_2=\begin{pmatrix}2\\-1\\3\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\1\end{pmatrix}$ Since $A(1,3,-2)$ lies in both planes and $\begin{pmatrix}3\\-1\\-1\end{pmatrix}\times\begin{pmatrix}1\\5\\1\end{pmatrix}=\begin{pmatrix}4\\-4\\16\end{pmatrix} // \begin{pmatrix}1\\-1\\4\end{pmatrix}$ $\therefore$ line of intersection is $\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+\alpha\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1+\alpha\\3-\alpha\\-2+4\alpha\end{pmatrix}$ Cartesian form: $\pi_1 : 3x-y-z=2$ $\pi_2 : x+5y+z=14$ From GC, line of intersection is **not** accepted.	To show the line of intersection between 2 planes, GC method is not accepted. Line of intersection is given by $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{n}_1\times\mathbf{n}_2)$ where $\mathbf{a}$ is the p.v. of the point $(1,3,-2)$ that lies in both planes.
9(c)	Since P lies on the line, $\overrightarrow{OP}=\begin{pmatrix}1+\alpha\\3-\alpha\\-2+4\alpha\end{pmatrix}$ $\overrightarrow{BP}=\begin{pmatrix}1+\alpha\\3-\alpha\\-2+4\alpha\end{pmatrix}-\begin{pmatrix}2\\-3\\7\end{pmatrix}=\begin{pmatrix}-1+\alpha\\6-\alpha\\-9+4\alpha\end{pmatrix}$ $\left\lvert\overrightarrow{BP}\right\rvert=\left\lvert\begin{pmatrix}-1+\alpha\\6-\alpha\\-9+4\alpha\end{pmatrix}\right\rvert=3\sqrt{2}$	Students who had difficulty did not use "any point that lies on the line l" is given by $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+\alpha\\3-\alpha\\-2+4\alpha\end{pmatrix}$ for some α.

	$\sqrt{(-1+\alpha)^2+(6-\alpha)^2+(-9+4\alpha)^2}=3\sqrt{2}$ $1-2\alpha+\alpha^2+36-12\alpha+\alpha^2+81-72\alpha+16\alpha^2=18$ $18\alpha^2-86\alpha+118=18$ $9\alpha^2-43\alpha+50=0$ $\alpha=2 \quad \text{or} \quad \dfrac{25}{9}$ The points are $\begin{pmatrix}1+2\\3-2\\-2+4(2)\end{pmatrix}=\begin{pmatrix}3\\1\\6\end{pmatrix} \qquad \begin{pmatrix}1+\frac{25}{9}\\3-\frac{25}{9}\\-2+4\left(\frac{25}{9}\right)\end{pmatrix}=\begin{pmatrix}\frac{34}{9}\\\frac{2}{9}\\\frac{82}{9}\end{pmatrix}=\frac{1}{9}\begin{pmatrix}34\\2\\82\end{pmatrix}$	
9(d)	$\pi_2:\mathbf{r}.\begin{pmatrix}1\\5\\1\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}1\\5\\1\end{pmatrix}=1+15-2=14$ $\pi_3:\mathbf{r}.\begin{pmatrix}1\\5\\1\end{pmatrix}=\begin{pmatrix}2\\-3\\7\end{pmatrix}.\begin{pmatrix}1\\5\\1\end{pmatrix}=2-15+7=-6$ Distance = $\dfrac{\lvert 14-(-6)\rvert}{\sqrt{1^2+5^2+1^2}}=\dfrac{20}{\sqrt{27}}=\dfrac{20}{3\sqrt{3}}$ **Alternatively (for distance),** $\overrightarrow{AB}=\begin{pmatrix}2\\-3\\7\end{pmatrix}-\begin{pmatrix}1\\3\\-2\end{pmatrix}=\begin{pmatrix}1\\-6\\9\end{pmatrix}$ $\left\lvert\overrightarrow{AB}.\hat{\mathbf{n}}\right\rvert=\left\lvert\begin{pmatrix}1\\-6\\9\end{pmatrix}.\frac{1}{\sqrt{27}}\begin{pmatrix}1\\5\\1\end{pmatrix}\right\rvert=\frac{1}{3\sqrt{3}}\left\lvert\begin{pmatrix}1\\-6\\9\end{pmatrix}.\begin{pmatrix}1\\5\\1\end{pmatrix}\right\rvert=\frac{20}{3\sqrt{3}}$ Hence the perpendicular from π_3 to π_2 is $\dfrac{20}{3\sqrt{3}}$.	Using the equations of planes: $\pi_1:\mathbf{r}.\mathbf{n}=p_1$ $\pi_2:\mathbf{r}.\mathbf{n}=p_2$ to find the distance between two planes = $\dfrac{\lvert p_1-p_2\rvert}{\lvert\mathbf{n}\rvert}$. If $p_1>0,\ p_2<0$ then the planes lies on opposite sides of the origin. Adding distances from orign is the distance between two planes. Otherwise students should avoid using $\dfrac{\lvert p_1\rvert+\lvert p_2\rvert}{\lvert\mathbf{n}\rvert}$ as a formula, it <u>will not</u> work if both $p_1>0,\ p_2>0$ or both $p_1<0,\ p_2<0$.

10(a)	$\omega^4-2\omega^3+10\omega^2+p\omega+q=0$ Since the coefficients are all real, complex roots occur in conjugate pairs. Therefore $2-3\mathrm{i}$ is also a root. $\therefore \omega^4-2\omega^3+10\omega^2+p\omega+q$ $=\left(\omega-(2+3\mathrm{i})\right)\left(\omega-(2-3\mathrm{i})\right)\left(\omega^2+a\omega+b\right)$ $=\left(\omega^2-4\omega+13\right)\left(\omega^2+a\omega+b\right)$ Comparing coefficients: $\omega^3:-2=-4+a\Rightarrow a=2$ $\omega^2:10=-4a+13+b\Rightarrow b=5$ $\therefore \omega^4-2\omega^3+10\omega^2+p\omega+q\left(\omega^2-4\omega+13\right)\left(\omega^2+2\omega+5\right)$ $\Rightarrow p=-20+26=6$ $\Rightarrow q=13\times 5=65$ The other roots are $\omega=\dfrac{-2\pm\sqrt{4-4(5)}}{2}=\dfrac{-2\pm\sqrt{-16}}{2}=-1\pm 2\mathrm{i}$ **Alternatively,** $(2+3\mathrm{i})^2=-5+12\mathrm{i}$ $(2+3\mathrm{i})^3=(-5+12\mathrm{i})(2+3\mathrm{i})=-46+9\mathrm{i}$ $(2+3\mathrm{i})^4=(-5+12\mathrm{i})^2=-119-120\mathrm{i}$ Substitute $\omega=2+3\mathrm{i}$ into $\omega^4-2\omega^3+10\omega^2+p\omega+q=0$ $(-119-120\mathrm{i})-2(-46+9\mathrm{i})$ $+10(-5+12\mathrm{i})+p(2+3\mathrm{i})+q=0$ Compare real and imaginary parts, $\mathrm{Re}:-119+92-50+2p+q=0\Rightarrow 2p+q=77$ $\mathrm{Im}:-120-18+120+3p=0\Rightarrow p=6\Rightarrow q=65$ Since the coefficients are all real, complex roots occur in conjugate pairs. Therefore $2-3\mathrm{i}$ is also a root. $\therefore \omega^4-2\omega^3+10\omega^2+6\omega+65$ $=\left(\omega-(2+3\mathrm{i})\right)\left(\omega-(2-3\mathrm{i})\right)\left(\omega^2+a\omega+b\right)$ $=\left(\omega^2-4\omega+13\right)\left(\omega^2+a\omega+b\right)$ Comparing coefficients: $\omega^0:65=13b\Rightarrow b=5$ $\omega^3:-2=-4+a\Rightarrow a=2$ Therefore the other factor is $\omega^2+2\omega+5$. Hence the other roots are $\omega=\dfrac{-2\pm\sqrt{4-4(5)}}{2}=\dfrac{-2\pm\sqrt{-16}}{2}=-1\pm 2\mathrm{i}$	Lots of errors when expanding $2+3\mathrm{i}$ up to power $(2+3\mathrm{i})^4$. Lots of errors when expanding factors of conjugate roots. Regards to comparing coefficients, some students needed more effort than others. And for other students, long division works well. Some students expressed the quadratic factor as $\left(\omega-(a+b\mathrm{i})\right)\left(\omega-(a-b\mathrm{i})\right)$ or $(\omega-a)(\omega-b)$ where the latter clearly does not suffice.

<table>
<tr><td>10(b)
(i)</td><td>$|u| = \sqrt{\left(-\sqrt{2}\right)^2 + \sqrt{2}^2} = 2$
Basic angle $= \tan^{-1}\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\pi}{4}$
$\arg u = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$</td><td>This is wrong:
$|u| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\mathbf{i}\right)^2} = 0$
The <u>modulus</u> of u is the length of OA.
The <u>argument</u> of u is the angle (in radians) that OA made with the positive Re(z).
Students can use GC to get angle/arg.</td></tr>
<tr><td>10(b)
(ii)</td><td>
</td><td>Argand diagram mostly lack details.
Given $0 < \arg \mathbf{v} < \dfrac{\pi}{4}$ but many students drew OB at 45°, resulting in a rectangle or square OACB, instead of a parallelogram.
Since $|\mathbf{v}| = 3 > 2$, the length of OB should be drawn longer in comparison to OA.</td></tr>
<tr><td>10(b)
(iii)</td><td>$\angle AOB = \dfrac{3\pi}{4} - \theta$
$\angle OAC = \pi - \left(\dfrac{3\pi}{4} - \theta\right) = \theta + \dfrac{\pi}{4}$
Using cosine rule on ΔOAC
$OC^2 = AO^2 + AC^2 - 2(AO)(AC)\cos\angle OAC$
$|u+v|^2 = 2^2 + 3^2 - 2(2)(3)\cos\left(\theta + \dfrac{\pi}{4}\right)$
$= 13 - 12\cos\left(\theta + \dfrac{\pi}{4}\right)$
$a = 13, b = -12, K = \dfrac{\pi}{4}$

Alternative Method 1
$|u+v|^2 = (u+v)\cdot(u+v)$
$= |u|^2 + 2u.v + |v|^2$ $u.v = |u||v|\cos\angle AOB$
$= 2^2 + 3^2 + 2(2)(3)\cos\left(\dfrac{3\pi}{4} - \theta\right)$
$= 13 + 12\cos\left(\dfrac{3\pi}{4} - \theta\right)$
$= 13 + 12\cos\left(\theta - \dfrac{3\pi}{4}\right)$ $a = 13, b = 12, K = -\dfrac{3\pi}{4}$</td><td>O level geometry & cosine rule.

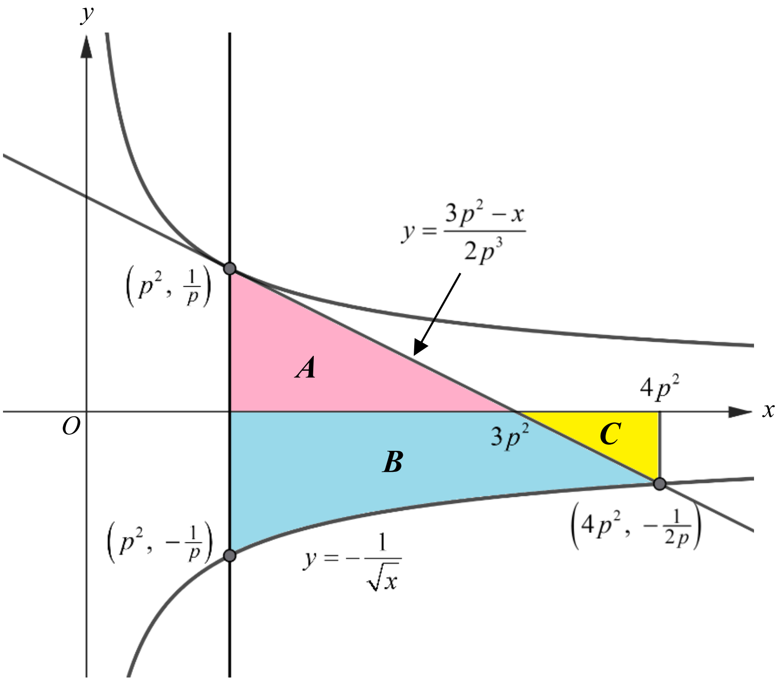
Alternative method 1 uses $|z|^2 = z.z$.

Alternative method 2 is for students who applied trigo to rewrite complex no. v.
$x = \mathrm{Re}(v) = 3\cos\theta$
$y = \mathrm{Im}(v) = 3\sin\theta$
$v = x + \mathrm{i}y = 3(\cos\theta + \mathrm{i}\sin\theta)$</td></tr>
</table>

	Alternative Method 2 $\lvert u+v\rvert^2=\lvert -\sqrt{2}+\mathrm{i}\sqrt{2}+3(\cos\theta+\mathrm{i}\sin\theta)\rvert^2$ $=\left(-\sqrt{2}+3\cos\theta\right)^2+\left(\sqrt{2}+3\sin\theta\right)^2$ $=\left(2-6\sqrt{2}\cos\theta+9\cos^2\theta\right)+\left(2+6\sqrt{2}\sin\theta+9\sin^2\theta\right)$ $=4+9\left(\sin^2\theta+\cos^2\theta\right)-6\sqrt{2}\left(\cos\theta-\sin\theta\right)$ $=13-6\sqrt{2}\left(\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)\right)$ $=13-12\cos\left(\theta+\frac{\pi}{4}\right)$	
11(a)	$\frac{\mathrm{d}x}{\mathrm{d}t}=2at \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{a}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{a}{t^2}\div 2at=-\frac{1}{2t^3}$ At $P\left(ap^2,\frac{a}{p}\right)$, $t=p \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2p^3}$ $y-\frac{a}{p}=-\frac{1}{2p^3}\left(x-ap^2\right)$ $y=-\frac{1}{2p^3}x+\frac{a}{2p}+\frac{a}{p}$ $y=-\frac{1}{2p^3}x+\frac{3a}{2p}$ $2p^3y+x=3ap^2$	Most students were able to apply chain rule correctly to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and use the coordinates of the point to arrive at the correct equation. The value of $t=p$ should be substituted into the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ first to obtain the value of the gradient <u>before</u> forming the equation of the tangent. An equation with 3 variables (x, y, t) is not an equation for a line. As this is a "show" question and the final answer is provided, students must be reminded to show <u>sufficient</u> algebraic working to demonstrate how their formula for equation of tangent leads to the required solution. The examiner cannot fill in the blanks for the steps that are not written down explicitly.
11(b)	$x=a\Rightarrow p^2=1 \therefore p=\pm 1$ Substitute $p=\pm 1$ into the equations of tangents in **(a)**: $2(1)^3y+x=3a(1)^2 \Rightarrow 2y+x=3a$ or $y=-\frac{1}{2}x+\frac{3a}{2}$ $2(-1)^3y+x=3a(-1)^2 \Rightarrow -2y+x=3a$ or $y=\frac{1}{2}x-\frac{3a}{2}$ Angle between the tangents is $\tan^{-1}\frac{1}{2}-\tan^{-1}\left(-\frac{1}{2}\right)=53.1^\circ$ or 0.927 radians	Most students were able to find $t=\pm 1$, but some did not realise that these were the values of p to substitute into the equation of the tangent from part **(a)**. Many of them end up with algebraic errors as they tried to find the equations of tangents from scratch, for example in finding the values of the gradients and the coordinates of the points. Another common error was carelessness in matching the gradient to the correct coordinates. To find the angle between the tangents, the concept to use is that the gradient of a line that makes an angle of θ with the positive x-axis is $\tan\theta$. Students who didn't know this often tried to find the

		angle by geometry, which wasn't always successful as students used incorrect values from the equations of the tangent for the calculation. A common misconception was assuming the tangents were at right angles as their gradients differ by a factor of -1 – the correct concept is that two lines are perpendicular if the *product* of their gradients is -1.
11(c) (i)	$(q-p)^2(q+2p)=(q^2-2qp+p^2)(q+2p)$ $=q^3\cancel{-2q^2p}+p^2q\cancel{+2pq^2}-4p^2q+2p^3$ $=q^3-3p^2q+2p^3$	This is a straightforward question that is meant to help students for **(c)(ii)**. Again, since this is a "show" question, students must write down sufficient algebraic working to reach the final answer.
11(c) (ii)	Sub $Q\left(aq^2,\dfrac{a}{q}\right)$ into $2p^3y+x=3ap^2$, $2p^3\left(\dfrac{a}{q}\right)+aq^2=3ap^2$ $2ap^3+aq^3=3ap^2q$ $q^3-3p^2q+2p^3=0$ $(q-p)^2(q+2p)=0$ Since $q\neq p\ \therefore q=-2p$	A significant number of students were stuck after substituting the coordinates of Q into the equation of the tangent. Some tried to solve for q using some form of the "zero product rule", but end up with an answer that also contains q. These students usually did not realise that the result in **(c)(i)** was meant to be a hint to help students factorise the cubic expression for q. For those who arrived at the solutions $q=p$ or $-2p$, many of them did not go on to reject the first answer, not realising that Q must be a different point from P and therefore $q\neq p$. Some students did not understand the task required and attempted to find the equation of the tangent at point Q instead.
11(d)	**Method 1: Area between 2 graphs** Area $=\displaystyle\int_{p^2}^{4p^2}\frac{1}{2p^3}\left(-x+3p^2\right)-\left(-\frac{1}{\sqrt{x}}\right)\mathrm{d}x$ $=\dfrac{1}{2p^3}\left[-\dfrac{x^2}{2}+3p^2x\right]_{p^2}^{4p^2}+\left[2x^{\frac{1}{2}}\right]_{p^2}^{4p^2}$ $=\dfrac{1}{2p^3}\left(\left(-\dfrac{16p^4}{2}+12p^4\right)-\left(-\dfrac{p^4}{2}+3p^4\right)\right)+(4p-2p)$ $=\dfrac{1}{2p^3}\left(4p^4-\dfrac{5p^4}{2}\right)+2p$ $=\dfrac{3p^4}{4p^3}+2p$ $=\dfrac{11p}{4}$	There was a roughly equal proportion of students who tried to find the area with respect to the x-axis and the y-axis. The easiest approach was to observe that the region is simply bounded by the tangent $y=\dfrac{3p^2-x}{2p^3}$ from above and the curve $y=-\dfrac{1}{\sqrt{x}}$ from below, so the formula for area between 2 curves would suffice. Students were more likely to make algebraic errors if they tried to divide the region into more parts. Some of the common algebraic errors include:

Method 2: Area with respect to x-axis



$$\text{Area of } \boldsymbol{A} = \frac{1}{2}\times\left(3p^2 - p^2\right)\times\frac{1}{p} = p$$

$$\begin{aligned}\text{Area of } \boldsymbol{B} \text{ and } \boldsymbol{C} &= -\int_{p^2}^{4p^2} -\frac{1}{\sqrt{x}}\ \mathrm{d}x \\ &= \int_{p^2}^{4p^2} x^{-\frac{1}{2}}\ \mathrm{d}x \\ &= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{p^2}^{4p^2} \\ &= 2\sqrt{4p^2} - 2\sqrt{p^2} \\ &= 2(2p) - 2p \\ &= 2p\end{aligned}$$

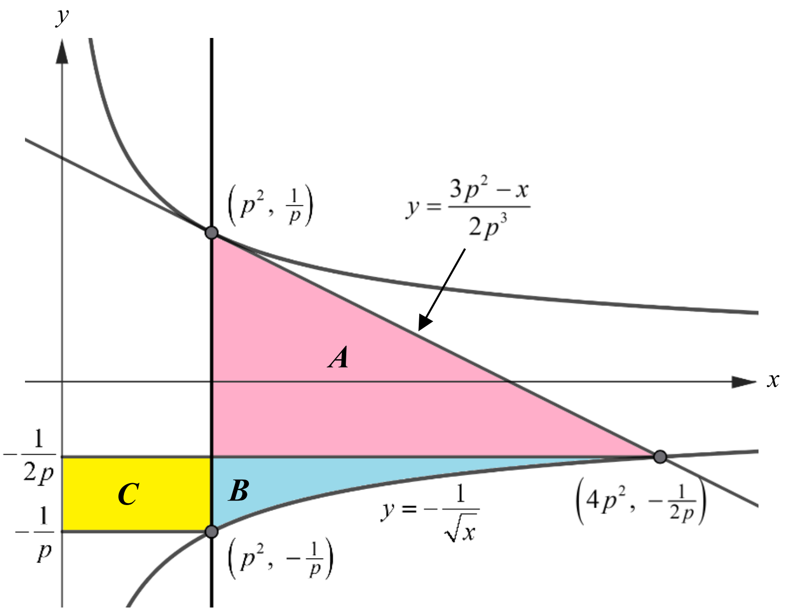
$$\text{Area of } \boldsymbol{C} = \frac{1}{2}\times\left(4p^2 - 3p^2\right)\times\frac{1}{2p} = \frac{1}{4p}$$

$$\therefore \text{ Required area } = p + 2p - \frac{1}{4p} = \frac{11p}{4}$$

- Writing the equation of the curve as $y = -\frac{1}{x}$ or $y = \sqrt{1-x}$
- Using the wrong curve, or not accounting for the negative area bounded by the curve and x-axis
- Using the wrong limits, or mixing up the x- and y-coordinates
- Forgetting to subtract the areas of certain triangles and rectangles
- Carelessness in the actual integration

Students should **sketch a diagram in the solution space** in the Printed Answer Booklet to complement their workings. Any sketches or indications done on the question booklet will not be seen by the marker, so their descriptions of “rectangles” and “triangles” would not make sense to the marker.

Method 3: Area with respect to y-axis



$$\text{Area of } \boldsymbol{A} = \frac{1}{2}\times\left(4p^2 - p^2\right)\times\left[\frac{1}{p}-\left(-\frac{1}{2p}\right)\right]$$
$$= \frac{9p}{4}$$

$$\text{Area of } \boldsymbol{B} \text{ and } \boldsymbol{C} = \int_{-\frac{1}{p}}^{-\frac{1}{2p}} \frac{1}{y^2}\, \mathrm{d}y$$
$$= \left[-\frac{1}{y}\right]_{-\frac{1}{p}}^{-\frac{1}{2p}}$$
$$= 2p - p$$
$$= p$$

$$\text{Area of } \boldsymbol{C} = p^2 \times \left[-\frac{1}{2p}-\left(-\frac{1}{p}\right)\right]$$
$$= \frac{p}{2}$$

$$\therefore \text{ Required area } = \frac{9p}{4} + p - \frac{p}{2}$$
$$= \frac{11p}{4}$$

12(a)	Amount Sarah plans to repay (before graduation): $\frac{1}{5}\times\frac{90}{100}\times 8250\times 4 = \$5940$ Number of repayments made from August 2021 to July 2025 is 48 (i.e. $n = 48$). The repayments follow an arithmetic progression (AP) with first term $a = x$ and common difference $d = 2$. The sum of an arithmetic progression is $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$ $5940 = \frac{48}{2}\left[2x+(48-1)2\right]$ $x = 76.75$	Many students were unable to handle the many details involved in the calculation of the loan amount and repayment, often missing out at least one piece of information. The most common error in calculating the amount repaid in the first phase was omitting to multiply by 4, as \$8250 was the **annual** tuition fee for the **4-year course**. Students often applied the wrong formula, using $a+(n-1)d$ instead of the formula for the sum of an AP. The number of repayments was also a common error – 47 was commonly seen, perhaps due to careless counting, but other numbers were also seen, which indicated inadequacy in understanding the problem. Students should also not round off exact answers unnecessarily.
12(b)	Amount of outstanding loan after Sarah's graduation: $\frac{4}{5}\times\frac{90}{100}\times 8250\times 4 = \$23760$ On 31st August 2025, apply interest of 0.4% to outstanding loan to get $\$\left(23760(1.004)\right)$. On 1st September 2025, make 1st repayment of \$$y$ to the bank. $\therefore$ Outstanding loan after 1st repayment $= \$\left(23760(1.004)-y\right)$ or \$(23855.04 − y)	To answer parts **(b)** and **(c)** correctly, students needed to understand the interest and repayment scheme and be very meticulous with the timeline of both elements when tracking how the outstanding loan changes. The first important point to note is that **interest** is the **first** element to be applied to the post-graduation loan amount. This is then followed by **repayment** on the next day. Students who misinterpreted and did repayment before interest would have been unsuccessful in both parts. Next, students must multiply the correct number reflecting the interest rate of 0.4%, which is 100.4% or 1.004. Common errors involving the wrong number of zeroes in this common ratio were often seen. Part **(c)** was poorly done because students did not "show" sufficiently all the information needed to lead to the given result. Most students realised the formula for the sum of a geometric progression (GP) was needed, but the application of this formula was slipshod. Students must write down the full series before the formula can be applied, and this full series must be sufficiently
12(c)	On 30th September 2025, apply interest of 0.4%. On 1st October 2025, make 2nd repayment of \$$y$. Outstanding loan after 2nd repayment $= \left(23760(1.004)-y\right)1.004-y$ $= 23760(1.004)^2-1.004y-y$ Outstanding loan after 3rd repayment (on 1st Nov 2025) $= \left(23760(1.004)^2-1.004y-y\right)1.004-y$ $= 23760(1.004)^3-1.004^2y-1.004y-y$ Outstanding loan after nth repayment $= 23760(1.004)^n-1.004^{n-1}y-\ldots-1.004^2y-1.004y-y$ $= 23760(1.004)^n-y\left(1+1.004+1.004^2+\ldots+1.004^{n-1}\right)$ $= 23760(1.004)^n-y\left(\frac{1.004^n-1}{1.004-1}\right)$ $= 23760(1.004)^n-250y\left(1.004^n-1\right)$ (shown)	

		written to demonstrate the pattern – simply writing down "$1.004^{n-1}+...+1$" does not mean that this series is a GP with common ratio 1.004. Another technique students must grasp is to identify and track the **correct amount** for the sequence. The required sequence was the outstanding loan after repayment, but there were students who tried to arrive at the pattern by tracking the amount after interest was applied. It would have been impossible to get the required result in this way. Also, students must understand that patterns can only be established with at least 3 terms in the sequence. Since the amount after first repayment has already been done in part **(b)**, students must write down the outstanding loan after at least 2 more repayments in order to generate a pattern. In short, when a question requires students to "show" a result, they must be careful to show all necessary steps that demonstrate application of concept and not simply summarise workings and take shortcuts that occur in their minds.
12(d)	To repay remaining loan by the end of 3 years (i.e. 36 monthly repayments of \$$y$), let $n = 36$. To fully repay the loan, we want outstanding loan amount after 36 repayments to be zero (or less). $23760(1.004)^{36} - 250y(1.004^{36} - 1) \le 0$ $y \ge 709.9769$ $\therefore\ y = 709.98$ (nearest cent)	This was generally well done by students who correctly substituted n to be 36 in part **(c)**'s result. Some students seemed to overthink the problem and used 37 instead. Some forgot to convert years to months, while a small portion of students did not know how many months there were in a year. Another common problem was equating the expression for the outstanding loan from part **(c)** to a number other than zero (e.g. 5940). This perhaps comes from not understanding the significance of the expression in part **(c)** – to fully repay the loan is to bring the outstanding loan amount down to zero.

12(e)	After 24 monthly repayments of \$500 (i.e. $n=24$, $y=500$), outstanding loan amount $=23760(1.004)^{24}-250(500)\left((1.004)^{24}-1\right)$ $=13580.49$ Outstanding loan amount as of 1st August 2027 $=13580.49$ In a similar method to **(c)**, the outstanding loan amount after m monthly repayments of \$1000 (i.e. y = 1000) is $13580.49(1.004)^{m}-250(1000)\left((1.004)^{m}-1\right)$ To fully repay the loan, we need $13580.49(1.004)^{m}-250(1000)\left((1.004)^{m}-1\right)\le 0$ $m\ge 13.991185$ Least m = 14 Sarah would require another 14 monthly repayments to fully repay her loan, with her first repayment of \$1000 on 1st September 2027. Hence, she would clear her loan on <u>(1st) October 2028</u>. The outstanding loan amount after her 13th monthly repayment of \$1000 (on 1st Sep 2028) is $13580.49(1.004)^{13}-250(1000)\left((1.004)^{13}-1\right)$ $=987.2529$ Interest is applied on 30th Sep 2028. Hence her final repayment amount is $987.2529\times 1.004=\underline{\$991.20}$ (nearest cent) (Alternatively, outstanding loan amount after 14th monthly repayment is $13580.49(1.004)^{14}-250(1000)\left((1.004)^{14}-1\right)$ $=-8.7981$ Hence the final repayment is $1000-8.7981=\$991.20$ (nearest cent).	This question was poorly done. Many students disregarded the interest rate completely and simply considered the total outstanding loan amount after graduation minus 24 repayments of \$500. Some students repeated the working from part **(c)** all over again without realising that the required outstanding loan amount after 24 monthly repayments of \$500 can be calculated using the same result from part **(c).** For the repayments of \$100, students need to realise that the first term in the result in part **(c)** should change to reflect the new outstanding loan amount at the start of this new repayment phase. For those who made it to this part, most of them obtained n = 14, but then went on to incorrectly calculate the month and year as November 2028. Also, only a handful correctly spotted that to obtain the amount of the final repayment, if they had substituted n = 13 into the expression for the outstanding loan amount, they need to multiply 1.004 to apply interest on the loan amount before the final repayment takes place. (On a related note, students also misread or misunderstood "final repayment" to mean "total repayment".)