1 The graph of $y=\dfrac{x+1}{ax^2+bx+c}$, where a, b and c are non–zero constants, has an asymptote at $x=-\dfrac{1}{2}$. The graph also has a turning point at $\left(-2,\ -\dfrac{1}{9}\right)$.
Find the values of a, b and c. [4]

[Solution]

Asymptote at $x = -\dfrac{1}{2}$ $\Rightarrow$ $a\left(-\dfrac{1}{2}\right)^2+b\left(-\dfrac{1}{2}\right)+c=0$

$$a-2b+4c=0 \qquad \ldots\text{Eq(1)}$$

When $x = -2$ and $y = -\dfrac{1}{9}$, $\quad -\dfrac{1}{9}=\dfrac{-2+1}{4a-2b+c}$

$$4a-2b+c=9 \qquad \ldots\text{Eq(2)}$$

$y=\dfrac{x+1}{ax^2+bx+c}$ $\quad\Rightarrow\quad$ $\left(ax^2+bx+c\right)y=x+1$

Differentiating wrt x: $\quad \left(ax^2+bx+c\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+\left(2ax+b\right)y=1$

When $x = -2$, $y = -\dfrac{1}{9}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad 0+\left(-4a+b\right)\left(-\dfrac{1}{9}\right)=1$

$$-4a+b=-9 \qquad \ldots\text{Eq(3)}$$

From GC, $a = 2$, $b = -1$ and $c = -1$

2 Two vectors **a** and **b** are such that |**a** × **b**| = 3. It is given that **a** is a unit vector and **b.b** = 9. Show that **a** and **b** are perpendicular. [3]

[Solution]

$|\mathbf{a}\times\mathbf{b}|=3 \;\Rightarrow\; |\mathbf{a}||\mathbf{b}||\sin\theta|=3$ where θ is the angle between vectors **a** and **b**.

$(1)(3)|\sin\theta|=3$ since $|\mathbf{a}| = 1$ and $|\mathbf{b}| = \sqrt{9} = 3$

$|\sin\theta| = 1$

$\Rightarrow$ $\sin\theta = 1$ or -1
$\Rightarrow$ $\theta = 90°$ or $270°$
$\Rightarrow$ **a** and **b** are perpendicular. (Shown)

3 The equation $z^3 + az^2 + bz + c = 0$, where a, b and c are constants, has roots 3 and w is a complex number.

(a) State a condition on a, b and c for the third root to be w^*. [1]

(b) Given that the condition in part **(a)** holds, and that $w = -1 + 2\text{i}$, find the values of a, b and c. [3]

[Solution]

(a) All three numbers, i.e. a, b, and c, are real numbers.

(b) Let $\text{f}(z) = z^3 + az^2 + bz + c$

$\text{f}(3) = 27 + 9a + 3b + c = 0 \quad \Rightarrow \quad 9a + 3b + c = -27$ …Eq(1)

$\text{f}(-1 + 2\text{i}) = (-1+2\text{i})^3 + a(-1+2\text{i})^2 + b(-1+2\text{i}) + c = 0$

From GC, $(11 - 2\text{i}) + a(-3 - 4\text{i}) - b + 2b\text{i} + c = 0$

$(11 - 3a - b + c) + \text{i}(-2 - 4a + 2b) = 0$

By comparing real and imaginary parts, $3a + b - c = 11$ …Eq(2)

$4a - 2b = -2$ …Eq(3)

From GC, $a = -1$, $b = -1$, $c = -15$

4 **(a)** Without using a calculator, solve the inequality $\dfrac{4}{x+2} \geq \dfrac{x-3}{x}$. [4]

(b) Hence, solve the inequality $\dfrac{4}{|x|+2} \geq \dfrac{|x|-3}{|x|}$. [2]

[Solution]

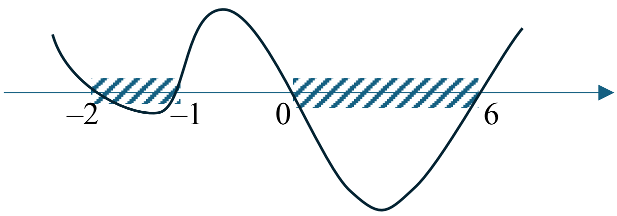
(a) $$\frac{4}{x+2} \geq \frac{x-3}{x}$$

$$\frac{4}{x+2} - \frac{x-3}{x} \geq 0$$

$$\frac{4x-(x+2)(x-3)}{x(x+2)} \geq 0$$

$$\frac{4x - x^2 + x + 6}{x(x+2)} \geq 0$$

$\frac{x^2-5x-6}{x(x+2)} \le 0$



$x(x+2)(x-6)(x+1) \le 0$

$\Rightarrow -2 < x \le -1$ or $0 < x \le 6$

(b) Replacing x with $|x|$, we have $-2 < |x| \le -1$ (reject since $|x| \ge 0$) or $0 < |x| \le 6$

$\Rightarrow -6 < x < 6$

Since $x \ne 0$, the solution is $-6 < x < 0$ or $0 < x < 6$

5 A geometric series has first term a and common ratio r, where $r < 0$. The sum to infinity of the series is 18 and its third term is $\frac{8}{3}$.

(a) Show that $27r^3 - 27r^2 + 4 = 0$. [3]

(b) Find the values of r and a. [1]

The nth term of the series is u_n.

(c) It is given that

$$80 \sum_{r=k+1}^{\infty} u_r = \sum_{r=1}^{k} u_r .$$

Find the value of k. [3]

[Solution]

(a) $S_\infty = \frac{a}{1-r} = 18 \Rightarrow a = 18 - 18r$ …Eq(1)

$ar^2 = \frac{8}{3} \Rightarrow 3ar^2 = 8$ …Eq(2)

Subst Eq(1) into Eq(2): $3(18 - 18r)r^2 = 8$

$54r^2 - 54r^3 = 8$

$27r^3 - 27r^2 + 4 = 0$ (Shown)

(b) From GC, $r = -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ (reject $\frac{2}{3}$ since $r < 0$)

From Eq(1): $a = 18 - 18\left(-\frac{1}{3}\right) = 24$

(c) $$80\sum_{r=k+1}^{\infty} u_r = \sum_{r=1}^{k} u_r \Rightarrow 80\sum_{r=k+1}^{\infty} u_r + 80\sum_{r=1}^{k} u_r = \sum_{r=1}^{k} u_r + 80\sum_{r=1}^{k} u_r$$

$$\Rightarrow 80\sum_{r=1}^{\infty} u_r = 81\sum_{r=1}^{k} u_r$$

$$\Rightarrow 80(18) = 81\frac{24\left[1-\left(-\frac{1}{3}\right)^k\right]}{1-\left(-\frac{1}{3}\right)}$$

$$\Rightarrow \left(-\frac{1}{3}\right)^k = \frac{1}{81} = \left(-\frac{1}{3}\right)^4$$

$$\Rightarrow k = 4$$

6 A curve C has equation $y = 2\mathrm{e}^{x^3} + a$, where a is a positive constant. The point T lies on C and has an x – coordinate of 1.

(a) Use calculus to find the equation of tangent to C at T. Give the equation in the form $y = \mathrm{e}(px+q)+ra$, where p, q and r are exact constants to be found. [5]

(b) It is given that the tangent to C at T passes through the origin.

(i) Find the exact value of a. [1]

(ii) Find the area of the region bounded by C, the line $x = 0$ and the tangent at T. Give your answer correct to 1 decimal place. [2]

[Solution]

(a) $y = 2\mathrm{e}^{x^3} + a$

Differentiating wrt x: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(3x^2\right)\mathrm{e}^{x^3} = 6x^2\mathrm{e}^{x^3}$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}$ and $y = 2\mathrm{e} + a$

Hence the equation of the tangent at T is $y - 2\mathrm{e} - a = 6\mathrm{e}(x - 1)$

$y = \mathrm{e}(6x - 6 + 2) + a$

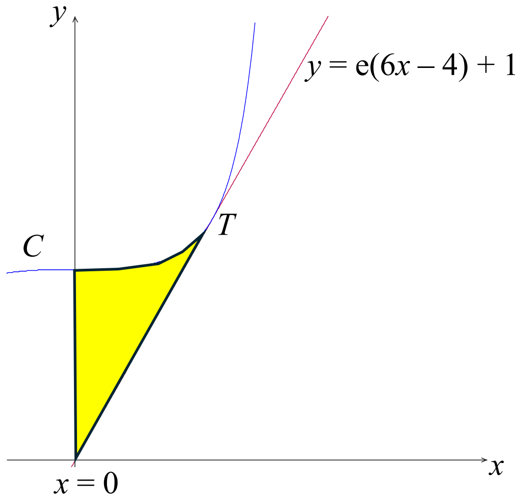
$y = \mathrm{e}(6x - 4) + a$

$\therefore p = 6$, $q = -4$ and $r = 1$

(b) (i) Since the y-intercept is 0, $-4\mathrm{e} + a = 0$

$a = 4\mathrm{e}$

(ii) Area required



$= \int_0^1 \left(2\mathrm{e}^{x^3} + 4\mathrm{e}\right) - \left(6\mathrm{e}x - 4\mathrm{e} + 4\mathrm{e}\right)\,\mathrm{d}x$

$= 5.402$

$= 5.4$ units^2 (1 dp)

7 It is given that $\mathrm{f}(r) = \cos(r\theta)$.

(a) Show that $\mathrm{f}(2r-1) - \mathrm{f}(2r+1) = 2\sin\theta\sin(2r\theta)$. [2]

(b) Hence, given that $\sin\theta \neq 0$, show that [3]

$$\sum_{r=1}^{n} \sin(2r\theta) = \frac{\cos\theta - \cos\left((2n+1)\theta\right)}{2\sin\theta}.$$

(c) Hence find the three possible values of [3]

$$\sum_{r=1}^{n} \sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right).$$

[Solution]

(a) $\mathrm{f}(2r-1) - \mathrm{f}(2r+1) = \cos\left[(2r-1)\theta\right] - \cos\left[(2r+1)\theta\right]$

$$= -2\sin\frac{(2r-1)\theta - (2r+1)\theta}{2}\sin\frac{(2r-1)\theta + (2r+1)\theta}{2}$$

$$= -2\sin(-\theta)\sin 2r\theta$$

$$= 2\sin\theta\sin 2r\theta \qquad \text{(Shown)}$$

(b) $$\sum_{r=1}^{n}\sin(2r\theta) = \sum_{r=1}^{n}\frac{\mathrm{f}(2r-1) - \mathrm{f}(2r+1)}{2\sin\theta}$$

$$= \frac{1}{2\sin\theta}\sum_{r=1}^{n}\left[\mathrm{f}(2r-1) - \mathrm{f}(2r+1)\right]$$

$$= \frac{1}{2\sin\theta}\begin{bmatrix} f(1)-f(3) \\ +f(3)-f(5) \\ +f(5)-f(7) \\ \cdots \\ +f(2n-5)-f(2n-3) \\ +f(2n-3)-f(2n-1) \\ +f(2n-1)-f(2n+1) \end{bmatrix}$$

$$= \frac{f(1)-f(2n+1)}{2\sin\theta}$$

$$= \frac{\cos\theta-\cos(2n+1)\theta}{2\sin\theta} \qquad \text{(Shown)}$$

(c) $\displaystyle\sum_{r=1}^{n}\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right) = \frac{1}{2}\sum_{r=1}^{n}2\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right)$

$$= \frac{1}{2}\sum_{r=1}^{n}\sin\left[2r\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\cos\frac{\pi}{6}-\cos(2n+1)\frac{\pi}{6}}{2\sin\frac{\pi}{6}}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}-\cos(2n+1)\frac{\pi}{6}}{2\left(\frac{1}{2}\right)}\right]$$

$$= \frac{\sqrt{3}}{4}-\frac{1}{2}\cos(2n+1)\frac{\pi}{6}$$

When $n = 1$, $\frac{\sqrt{3}}{4}-\frac{1}{2}\cos(2n+1)\frac{\pi}{6} = \frac{\sqrt{3}}{4}-\frac{1}{2}\cos\frac{\pi}{2} = \frac{\sqrt{3}}{4}$

When $n = 2$, $\frac{\sqrt{3}}{4}-\frac{1}{2}\cos(2n+1)\frac{\pi}{6} = \frac{\sqrt{3}}{4}-\frac{1}{2}\cos\frac{5\pi}{6} = 0$

When $n = 3$, $\frac{\sqrt{3}}{4}-\frac{1}{2}\cos(2n+1)\frac{\pi}{6} = \frac{\sqrt{3}}{4}-\frac{1}{2}\cos\frac{7\pi}{6} = \frac{\sqrt{3}}{2}$

Hence the 3 possible values required are 0, $\frac{\sqrt{3}}{4}$ and $\frac{\sqrt{3}}{2}$.

8 It is given that $y=\cos\left(1-\mathrm{e}^{2x}\right)$.

(a) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=k\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}-2y\mathrm{e}^{4x}\right)$, where k is a constant to be found. [3]

(b) By differentiation of the result in part **(a)**, find the first three non-zero terms of the Maclaurin expansion of $\cos\left(1-\mathrm{e}^{2x}\right)$. [4]

(c) The first two non-zero terms of the Maclaurin expansion of $\cos\left(1-\mathrm{e}^{2x}\right)$ are equal to the first two non-zero terms of the series expansion of $\dfrac{1}{\sqrt{a+bx^2}}$, where a and b are constants.
Using standard series from the List of Formulae (MF26), find the values of a and b. [2]

[Solution]

(a) $y=\cos\left(1-\mathrm{e}^{2x}\right)$

Differentiating wrt x: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(-2\mathrm{e}^{2x}\right)\left[-\sin\left(1-\mathrm{e}^{2x}\right)\right]=2\mathrm{e}^{2x}\sin\left(1-\mathrm{e}^{2x}\right)$

Differentiating wrt x: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4\mathrm{e}^{2x}\sin\left(1-\mathrm{e}^{2x}\right)+2\mathrm{e}^{2x}\left(-2\mathrm{e}^{2x}\right)\cos\left(1-\mathrm{e}^{2x}\right)$

$$=4\mathrm{e}^{2x}\sin\left(1-\mathrm{e}^{2x}\right)-4\mathrm{e}^{4x}\cos\left(1-\mathrm{e}^{2x}\right)$$

$$=2\frac{\mathrm{d}y}{\mathrm{d}x}-4\mathrm{e}^{4x}y$$

$$=2\left(\frac{\mathrm{d}y}{\mathrm{d}x}-2y\mathrm{e}^{4x}\right);\ k=2 \qquad \text{(Shown)}$$

(b) Differentiating wrt x: $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-4\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{4x}+4y\mathrm{e}^{4x}\right)=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-4\dfrac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{4x}-16y\mathrm{e}^{4x}$

When $x=0$, $y=1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(0-2)=-4,$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2(-4)-0-16=-24$$

Hence the Maclaurin's expansion required is

$$y=1+0+\frac{-4}{2!}x^2+\frac{-24}{3!}x^3+\ldots=1-2x^2-4x^3+\ldots$$

(c) $$\frac{1}{\sqrt{a+bx^2}}=\frac{1}{\sqrt{a}}\left(1+\frac{bx^2}{a}\right)^{\frac{-1}{2}}$$

$$=\frac{1}{\sqrt{a}}\left[1-\frac{1}{2}\left(\frac{bx^2}{a}\right)+...\right]$$ (From $(1+x)^n$ expansion in MF26)

$$\therefore\ 1-2x^2=\frac{1}{\sqrt{a}}\left[1-\frac{1}{2}\left(\frac{bx^2}{a}\right)\right]=\frac{1}{\sqrt{a}}-\frac{bx^2}{2a^{\frac{3}{2}}}$$

$$\Rightarrow\quad 1=\frac{1}{\sqrt{a}}\quad\text{and}\quad -2=-\frac{b}{2a^{\frac{3}{2}}}$$

$$\Rightarrow\quad a=1\quad\text{and}\quad b=4(1)^{\frac{3}{2}}=4$$

9 A curve C has parametric equations

$$x=3t^2+2t,\ \ y=t^2+2t^3\ \text{for}\ t\geq 0.$$

(a) Show that $\frac{dy}{dx}=kt$, where k is a constant to be found. [2]

(b) The tangent to C at a point P makes an angle of 60° with the x-axis. Find the exact coordinates of P. [3]

The point Q with coordinates (16, 20) lies on C.

(c) Using calculus, find the exact area of the region bounded by the curve C, the x-axis and the line parallel to the y-axis through Q. [5]

[Solution]

(a) $$\frac{dx}{dt}=6t+2\quad\text{and}\quad\frac{dy}{dt}=2t+6t^2$$

$$\therefore\ \frac{dy}{dx}=\frac{2t+6t^2}{6t+2}=\frac{t(2+6t)}{6t+2}=t\ ;\ k=1$$ (Shown)

(b) Let $\frac{dy}{dx}=t=\tan 60^\circ$

$\Rightarrow t=\sqrt{3}$

When $t=\sqrt{3}$, $x=3\left(\sqrt{3}\right)^2+2\sqrt{3}=9+2\sqrt{3}$

$$y=\left(\sqrt{3}\right)^2+2\left(\sqrt{3}\right)^3=3+6\sqrt{3}$$

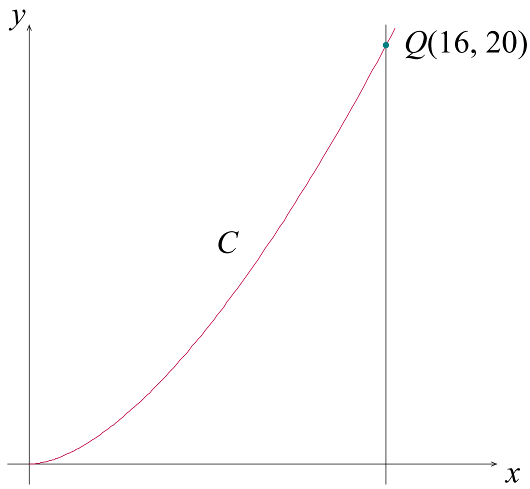
$\therefore$ Coordinates of P are $\left(9+2\sqrt{3},3+6\sqrt{3}\right)$

(c) When $x = 0$, $t = 0$

When $x = 16$, $3t^2 + 2t = 16$

$$t = \frac{-2 \pm \sqrt{4+192}}{6}$$

$$= 2,\ \frac{-8}{3} \text{ (rejected since } t \geq 0\text{)}$$



Area required

$$= \int_0^{16} y \,\mathrm{d}x$$

$$= \int_0^2 \left(t^2 + 2t^3\right) \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t$$

$$= \int_0^2 \left(t^2 + 2t^3\right)\left(6t + 2\right) \mathrm{d}t$$

$$= \int_0^2 6t^3 + 2t^2 + 12t^4 + 4t^3 \,\mathrm{d}t$$

$$= \left[\frac{10}{4}t^4 + \frac{2}{3}t^3 + \frac{12}{5}t^5\right]_0^2$$

$$= 40 + \frac{16}{3} + \frac{384}{5} - 0$$

$$= \frac{1832}{15} \text{ units}^2$$

10 **(a)** Using the substitution $u=\sqrt{4x+1}$, show that [3]

$$\int \frac{\sqrt{4x+1}}{x-2}\,\mathrm{d}x = \int \frac{2u^2}{u^2-9}\,\mathrm{d}u.$$

(b) The region R lies in the first quadrant and is bounded by the curve $y=\dfrac{\sqrt{4x+1}}{x-2}$ and the lines $x = 6$, $x = 12$ and $y = 0.5$.

(i) Sketch the graph of $y=\dfrac{\sqrt{4x+1}}{x-2}$ for $x \geq 0$, giving the coordinates of any intercepts with the axes and the equations of any asymptotes. Shade the region R on your sketch. [3]

(ii) Find the exact area of R, giving your answer in the form $a + b \ln c$. [5]

(i) Find the volume of the solid generated when R is rotated through 2π radians about the x-axis. Give your answer correct to 2 decimal places. [3]

[Solution]

(a) $u=\sqrt{4x+1} \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{4}{2\sqrt{4x+1}}=\dfrac{2}{\sqrt{4x+1}}=\dfrac{2}{u}$

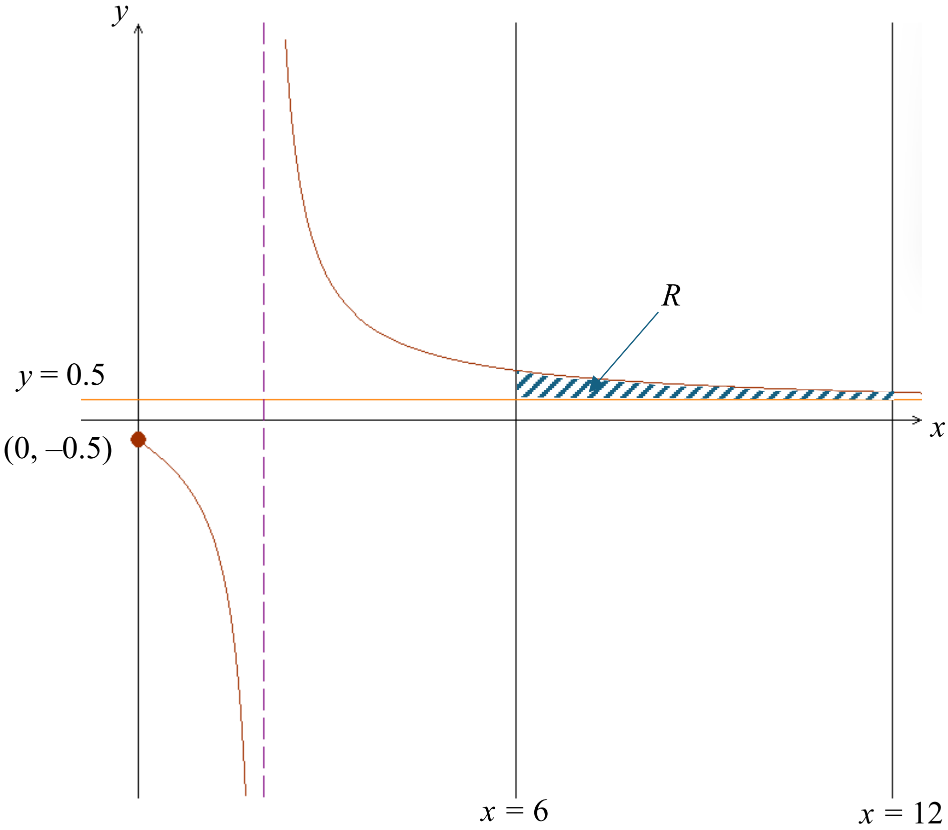
$$\int \frac{\sqrt{4x+1}}{x-2}\,\mathrm{d}x = \int \frac{u}{x-2}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= \int \frac{u}{\frac{1}{4}\left(u^2-1\right)-2}\left(\frac{u}{2}\right)\mathrm{d}u$$

$$= \int \frac{2u^2}{\left(u^2-1\right)-8}\,\mathrm{d}u$$

$$= \int \frac{2u^2}{u^2-9}\,\mathrm{d}u \qquad \text{(Shown)}$$

(b) (i)



(ii) Area of R

$$= \int_6^{12} \frac{\sqrt{4x+1}}{x-2}\,dx - (0.5)(12-6)$$

$$= \int_5^7 \frac{2u^2}{u^2-9}\,du - 3$$

$$= 2\int_5^7 \frac{u^2-9+9}{u^2-9}\,du - 3$$

$$= 2\int_5^7 1\,du + 18\int_5^7 \frac{1}{u^2-3^2}\,du - 3$$

$$= 2[u]_5^7 + 18\left[\frac{1}{2(3)}\ln\frac{u-3}{u+3}\right]_5^7 - 3$$

$$= 2[7-5] + 3\left[\ln\frac{4}{10} - \ln\frac{2}{8}\right] - 3$$

$$= 1 + 3\ln\frac{8}{5} \text{ units}^2 \text{ ; } a = 1,\ b = 3 \text{ and } c = \frac{8}{5}$$

(iii) Volume required

$$= \pi\int_6^{12}\left(\frac{\sqrt{4x+1}}{x-2}\right)^2 \mathrm{d}x - \pi\left(\frac{1}{2}\right)^2(12-6)$$

$= 11.0432$

$= 11.04$ units^3

11 A helicopter is hovering at a great height above the ground. A parachutist drops from the helicopter and his parachute opens. The parachutist falls vertically. At time t seconds after leaving the helicopter he has fallen a distance of x metres. At this time he has velocity v ms^{-1}. The motion of the parachutist is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}x}{\mathrm{d}t} = 9.8,$$

where α is a constant.

(a) **(i)** Use the equation $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ to write down a differential equation in v and t. [1]

(ii) Given that $\frac{\mathrm{d}v}{\mathrm{d}t} = 5$ when $v = 2.4$, find the value of α. [1]

(b) It is given that the initial velocity of the parachutist is zero. Show that $v = A\left(1-\mathrm{e}^{Bt}\right)$, where A and B are constants to be found. [5]

(c) The helicopter is at a height of 2000 m above the ground. Find the time taken for the parachutist to reach the ground. [5]

[Solution]

(a) (i) $v = \frac{\mathrm{d}x}{\mathrm{d}t} \quad\Rightarrow\quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

$$\therefore \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}x}{\mathrm{d}t} = 9.8 \quad\Rightarrow\quad \frac{\mathrm{d}v}{\mathrm{d}t} + \alpha v = 9.8$$

(ii) When $v = 2.4$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = 5$, $\quad 5 + 2.4\alpha = 9.8$

$$\therefore \alpha = 2 \text{ s}^{-1}$$

(b) $\dfrac{dv}{dt}+2v=9.8 \quad\Rightarrow\quad \dfrac{dv}{dt}=9.8-2v$

$$\int \frac{1}{4.9-v}\,dv=\int 2\,dt$$

$-\ln|4.9-v| = 2t+c$ where c is an arbitrary constant

$|4.9-v| = e^{-2t-c}$

$4.9 - v = ke^{-2t}$ where $k = \pm e^{-c}$

$v = 4.9 - ke^{-2t}$

When $t = 0$ and $v = 0$, $k = 4.9$

$\therefore\ v = 4.9 - 4.9e^{-2t}$

$= 4.9(1 - e^{-2t})$; $A = 4.9$ and $B = -2$ (Shown)

(c) $\dfrac{dx}{dt} = 4.9(1 - e^{-2t}) \quad\Rightarrow\quad x = 4.9\int 1-e^{-2t}\,dt$

$= 4.9\left[t+\dfrac{1}{2}e^{-2t}\right] + d$ where d is a constant

When $t = 0$ and $x = 0$, $0 = 4.9\left[0+\dfrac{1}{2}\right] + d$

$\Rightarrow d = -2.45$

$\therefore\ x = 4.9\left[t+\dfrac{1}{2}e^{-2t}\right] - 2.45$

When $x = 2000$, $2000 = 4.9\left[t+\dfrac{1}{2}e^{-2t}\right] - 2.45$

From GC, $t = 408.663$

$\therefore$ The time taken for the parachutist to reach the ground = 409 s (to nearest second)

12 Air traffic controllers need to know the exact locations of aircraft. They use coordinates (x, y, z), with units in kilometres, to locate individual aircraft relative to the base of the control tower which is at the position $(0, 0, 0)$. The ground is horizontal and is modelled as the x-y plane.

An aircraft flies along a path with equation $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. The air traffic controllers are situated in the control tower 0.1 km above the base of the control tower.

(a) Due to the atmospheric conditions at the time, the air traffic controllers can only see aircraft that come within 4 km of their position.

(i) Find the coordinates of this aircraft when it is at its closest point to the air traffic controllers. [4]

(ii) Hence determine whether the air traffic controllers will see this aircraft. [2]

A drone flies along a path with equation

$$\frac{2x-3}{3} = y+1 = \frac{z-k}{-1},$$

where k is a positive constant.

(b) Determine the possible values of k if the paths of the aircraft and the drone do not intersect. [3]

(c) Find the acute angle between the paths of the aircraft and the drone. [2]

(d) It is now given that $k = 2$.
The aircraft and the drone are observed on radar to be closest to each other when they are at the points $(4, 3, 1)$ and $(4.2, 0.8, 0.2)$ on their respective paths.

(i) Find the distance between the aircraft and the drone at this instant. [2]

(ii) By considering the vector between the aircraft and the drone at this instant, determine whether the distance found in part **(d)(i)** is the shortest distance between the two paths. [2]

[Solution]

(i) Let O, T and A denote the positions of the base of the tower, the aircraft controller and the aircraft respectively.

$$\therefore \left|\overrightarrow{TA}\right| = \left|\begin{pmatrix} 4+\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix}\right| = \sqrt{(4+\lambda)^2 + (3-\lambda)^2 + (0.9+2\lambda)^2}$$

$$= \sqrt{16+8\lambda+\lambda^2+9-6\lambda+\lambda^2+0.81+3.6\lambda+4\lambda^2}$$

$$= \sqrt{6\lambda^2+5.6\lambda+25.81}$$

$$= \sqrt{6\left(\lambda+\frac{7}{15}\right)^2 + 25.81 - \frac{49}{225}}$$

$$= \sqrt{6\left(\lambda + \frac{7}{15}\right)^2 + 25\frac{533}{900}}$$

$\Rightarrow \left|\overrightarrow{TA}\right|$ is minimum when $\lambda = -\dfrac{7}{15}$

When $\lambda = -\dfrac{7}{15}$, $\overrightarrow{OA} = \begin{pmatrix} 4 - \frac{7}{15} \\ 3 + \frac{7}{15} \\ 1 - \frac{14}{15} \end{pmatrix} = \begin{pmatrix} \frac{53}{15} \\ \frac{52}{15} \\ \frac{1}{15} \end{pmatrix}$

Hence coordinates required are $\left(\dfrac{53}{15}, \dfrac{52}{15}, \dfrac{1}{15}\right)$.

(ii) Since $\sqrt{25\frac{533}{900}} = 5.6$ (3 sf) > 4, then the aircraft controller will not be able to see the aircraft.

(iii) Let D denote the position of the drone.

$\therefore$ Line OD: $\mathbf{r} = \begin{pmatrix} 1.5 \\ -1 \\ k \end{pmatrix} + \mu \begin{pmatrix} 1.5 \\ 1 \\ -1 \end{pmatrix}$ for $\mu \in \mathbb{R}$.

Let $\begin{pmatrix} 1.5 + 1.5\mu \\ -1 + \mu \\ k - \mu \end{pmatrix} = \begin{pmatrix} 4 + \lambda \\ 3 - \lambda \\ 1 + 2\lambda \end{pmatrix}$

$\lambda - 1.5\mu = -2.5$ …Eq(1)

$\lambda + \mu = 4$ …Eq(2)

$2\lambda + \mu = k - 1$ …Eq(3)

From GC, $\lambda = 1.4$ and $\mu = 2.6$ using Eq(1) and Eq(2)

If the 2 paths do not meet, $k \neq 2(1.4) + 2.6 + 1 = 6.4$

Hence $0 < k < 6.4$ or $k > 6.4$

(c) Let θ denote the acute angle between the 2 paths.

$$\cos\theta = \frac{\left|\begin{pmatrix}1.5\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}{\sqrt{1.5^2+1^2+1^2}\sqrt{1^2+1^2+2^2}}$$

$$\therefore \theta = \cos^{-1}\frac{|1.5-1-2|}{\sqrt{4.25}\sqrt{6}} = 1.2692 = 72.7° \text{ (nearest degree)}$$

(d) (i) Distance required $= \left|\begin{pmatrix}4\\3\\1\end{pmatrix}-\begin{pmatrix}4.2\\0.8\\0.2\end{pmatrix}\right| = \left|\begin{pmatrix}-0.2\\2.2\\0.8\end{pmatrix}\right|$

$= \sqrt{(-0.2)^2+2.2^2+0.8^2}$

$= 2.3494 = 2.35$ km (3 sf)

(ii) At that instant, $\overrightarrow{DA} = \begin{pmatrix}-0.2\\2.2\\0.8\end{pmatrix}$.

Since $\begin{pmatrix}-0.2\\2.2\\0.8\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix} = -0.2-2.2+1.6 = -0.8 \neq 0$, $\overrightarrow{DA}$ is not perpendicular to the path of the aircraft. Hence the distance found in part (d)(i) is not the shortest distance between the 2 paths.