**RAFFLES INSTITUTION**
**H2 Mathematics 9758**
**2025 Year 6 Term 3 Revision 2 (Zeta)**
**Topic(s): Complex Numbers and Sequences and Series (Solutions)**

**Source of Question: 9740/2012/I/Q3(modified)**

**1** A sequence $u_1$, $u_2$, $u_3$, … is given by

$$u_1 = 2 \text{ and } u_{n+1} = \frac{3u_n - 1}{6} \text{ for } n \geq 1.$$

**(i)** Find the exact values of $u_2$ and $u_3$, and the value of $u_{10}$ to 5 decimal places. [3]

**(ii)** It is given that $u_n \to l$ as $n \to \infty$. Showing your working, find the exact value of $l$. [2]

**(iii)** Describe the behaviour of the sequence. [1]

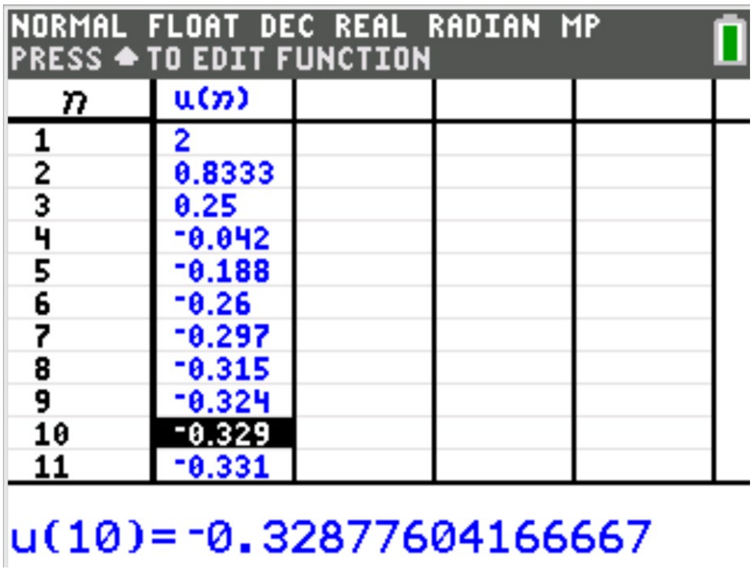
**Solution:**

**(i)**

$$u_2 = \frac{3(2)-1}{6} = \frac{5}{6}$$

$$u_3 = \frac{3\left(\frac{5}{6}\right)-1}{6} = \frac{1}{4}$$

From the GC, $u_{10} = -0.32878$ (5 dp)



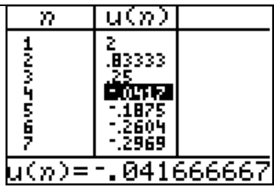

**(ii)**

As $n \to \infty$, $u_n \to l$ and $u_{n+1} \to l$

We have $l = \frac{3(l)-1}{6} \Rightarrow 6l - 3l = -1 \Rightarrow l = -\frac{1}{3}$

**(iii)**

The sequence decreases and converges to $-\frac{1}{3}$.



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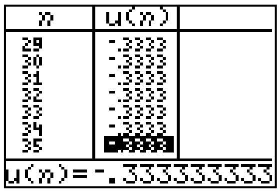
| |  |

**Source of Question: CJC Prelim 9758/2022/II/Q2**

**2** A sequence $a_1, a_2, a_3, \ldots$ is such that $a_n = 2a_{n-1} - 3n + K$, where $K$ is a constant, and $n \ge 2$.

**(i)** Given that $a_1 = 2$ and $a_2 = 4$, find $K$ and $a_3$. [2]

It is known that the $n$th term of this sequence is given by $a_n = p(2^n) + qn + r$, where $p$, $q$ and $r$ are constants.

**(ii)** Find $p$, $q$ and $r$. [3]

**(iii)** Find $\sum_{n=1}^{N} a_n$ in terms of $N$. [2]

**Solution:**

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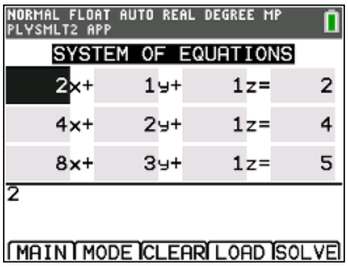
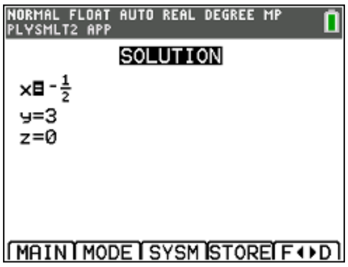
|---|---|
| **(i)** | Given that $a_1 = 2$ and $a_2 = 4$,<br>$4 = 2(2) - 3(2) + K \quad \Rightarrow K = 6$<br>$\therefore a_3 = 2a_2 - 3(3) + K$<br>$= 2(4) - 3(3) + 6 \quad = 5$ |
| **(ii)** | Since $a_n = p(2^n) + qn + r$,<br>$\left.\begin{array}{lll} a_1 = 2 & \Rightarrow & p(2^1) + q(1) + r = 2 \\ a_2 = 4 & \Rightarrow & p(2^2) + q(2) + r = 4 \\ a_3 = 5 & \Rightarrow & p(2^3) + q(3) + r = 5 \end{array}\right\}$<br>Solving this system of linear equations :<br><br>$\therefore \; p = -\frac{1}{2}$, $q = 3$, and $r = 0$.<br>$\therefore \; a_n = -\dfrac{1}{2}(2^n) + 3n$ |
| **(iii)** | $a_n = -\dfrac{1}{2}(2^n) + 3n$ |

| | |
|---|---|
| | $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N}\left(-\frac{1}{2}(2^n)+3n\right)$<br>$=-\frac{1}{2}\sum_{n=1}^{N}2^n+3\sum_{n=1}^{N}n$<br>$=-\frac{1}{2}\left(2^1+2^2+2^3+...+2^N\right)+3\left(1+2+3+...+N\right)$<br>$=-\frac{1}{2}\left(\frac{2^1(2^N-1)}{2-1}\right)+3\left(\frac{N}{2}(1+N)\right)$<br>$=(1-2^N)+\frac{3}{2}N(1+N)$ |

**Source of Question: HCI Prelim 9740/2012/I/Q6(b)**

**3** Given that all terms in a geometric progression $\{u_n\}$, $n=1, 2, 3, ...$ are positive with first term $a$ and common ratio $r$, where $r \neq 1$, the sums $H$ and $C$ are defined as follows:

$$H = u_1 + u_2 + ... + u_n, \qquad C = \frac{1}{u_1}+\frac{1}{u_2}+...+\frac{1}{u_n}.$$

**(i)** By expressing $H$ and $C$ in terms of $a$ and $r$, prove that $\frac{H}{C}=u_1u_n$. [3]

**(ii)** Express the product $u_1u_2u_3 ... u_n$ in the form of $\left(\frac{H}{C}\right)^{f(n)}$, where f($n$) is a function of $n$. [2]

**Solution:**

| | |
|---|---|
| **(i)** | Since $\{u_n\}$ is a GP with common ratio $r$, $H=\frac{a(r^n-1)}{r-1}$.<br>$\left\{\frac{1}{u_n}\right\}$ is a GP with common ratio $\frac{1}{r}$ and thus we have $C=\frac{\frac{1}{a}(1-\frac{1}{r^n})}{1-\frac{1}{r}}=\frac{1}{ar^{n-1}}\frac{r^n-1}{r-1}$.<br>$\frac{H}{C}=\frac{a(r^n-1)}{r-1}\times\frac{ar^{n-1}(r-1)}{r^n-1}$<br>$=a\times ar^{n-1}$<br>$=u_1u_n$ |
| **(ii)** | $u_1 \cdot u_2 \cdot ... \cdot u_n = a(ar)(ar^2)(ar^3)...(ar^{n-1})$<br>$=a^n(r^{1+2+3+...+(n-1)})$<br>$=a^n r^{\frac{(n-1)n}{2}}$<br>Since $\frac{H}{C}=u_1u_n=a^2r^{n-1}$, |

| | |
|---|---|
| | $u_1 \cdot u_2 \cdot \ldots \cdot u_n = a^n r^{\frac{(n-1)n}{2}}$ <br> $= (a^2 r^{n-1})^{\frac{n}{2}} = \left(\frac{H}{C}\right)^{\frac{n}{2}}$ |

**Source of Question: DHS Prelim 9740/2012/II/Q4**

**4** A finite sequence $\{a_n\}$ has 50 terms and is such that $a_{n+1} = a_n + 0.15$ for $n = 1, 2, 3, \ldots, 49$.

**(i)** Given that $a_{50} = 99a_1$, show that $a_1 = 0.075$. [2]

**(ii)** Find, without using a calculator, the value of $\sum_{n=1}^{50} a_n$. [2]

Another infinite sequence $\{b_m\}$ is such that $b_1 = a_{50}$ and $\frac{b_m}{b_{m-1}} = 0.98$ for $m \geq 2$.

**(iii)** Determine the smallest value of *k* such that $b_k < a_{25}$. [2]

**(iv)** Find the least value of *h* such that the sum of the first *h* terms of $\{b_m\}$ is more than 99% of its sum to infinity. [2]

**(v)** If $\frac{b_m}{b_{m-1}} = -0.98$ instead, find $\sum_{m=0}^{\infty} b_{1+3m}$. [2]

**Solution:**

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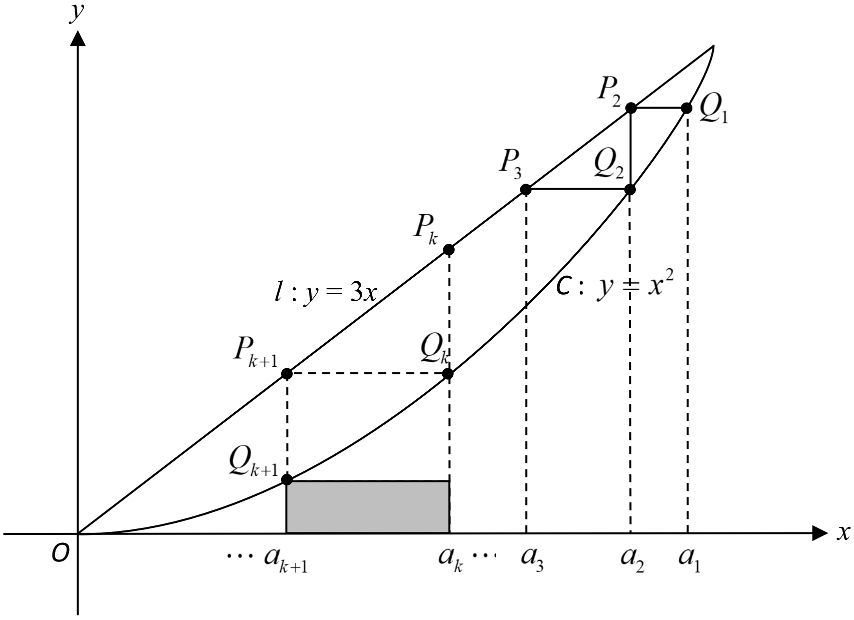
|---|---|
| **(i)** | $a_{50} = 99a_1$ <br> $a_1 + 49d = 99a_1$ <br> $a_1 = \frac{1}{2}(0.15) = 0.075$ (Shown) |
| **(ii)** | $\sum_{n=1}^{50} a_n = \frac{50}{2}(0.075 + 99 \times 0.075) = 187.5$ |
| **(iii)** | $b_k < a_{25}$ <br> $(99 \times 0.075)(0.98)^{k-1} < (0.075) + 24(0.15)$ <br> $k > 35.8$ <br> least $k = 36$ |
| **(iv)** | Consider <br> $\frac{b_1(1-0.98^h)}{1-0.98} > 0.99\left(\frac{b_1}{1-0.98}\right)$ <br> $0.98^h < 0.01$ <br> $h > 227.9$ <br> least $h = 228$ |

| | |
|---|---|
| **(v)** | $\sum_{m=0}^{\infty} b_{1+3m}$<br>$= b_1 + b_4 + b_7 + \ldots$<br>$= \dfrac{99 \times 0.075}{1-(-0.98)^3}$<br>$= 3.82$ (3s.f.) |

**Source of Question: RI Prelim 9740/2011/I/Q6(modified)**

**5**




The line $l$ with equation $y = 3x$ and the curve $C$ with equation $y = x^2$ are shown in the diagram above. The $x$-coordinates of the point $Q_1$ on $C$ is $a_1$, where $0 < a_1 < 3$.

A line parallel to the $x$-axis passing through the point $Q_k$, ($k \geq 1$) on the curve $C$ intersects the line $l$ at $P_{k+1}$, and another line parallel to the $y$-axis passing through $P_{k+1}$ intersects $C$ at $Q_{k+1}$.

The sequence $a_1$, $a_2$, …, is defined by the $x$-coordinates of the points $Q_1$, $Q_2$, …, respectively.

**(i)** Show that the sequence satisfies $a_{k+1} = \dfrac{a_k^2}{3}$ for all $k \geq 1$. [2]

It is given that the $k$-th term of the sequence is given by $a_k = 3\left(\dfrac{a_1}{3}\right)^{2^{k-1}}$ for all $k \geq 1$.

For the rest of the question, assume $a_1 = 1$.

It is given that $a_1 = 1$.

**(ii)** Find $a_3$. [1]

**(iii)** Find the area of the shaded rectangle in terms of $a_k$ and $a_{k+1}$. By considering the area of the region bounded by $C$, the lines $x = 0$, $x = 1$ and the $x$-axis, give a geometrical reason why $\sum_{k=1}^{n}(a_k - a_{k+1})a_{k+2} < \frac{1}{9}$ for all $n \geq 1$. [3]

**(iv)** Given that $\sum_{k=1}^{n}(a_k - a_{k+1}) = 1 - \left(\frac{1}{3}\right)^{2^n - 1}$, show that $\sum_{k=1}^{n}(a_k - a_{k+1})a_{k+2} < \frac{1}{27}$ for all $n \geq 1$. [2]

**Solution:**

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|---|---|
| **(i)** | The coordinates of the point $Q_k$ is given by $\left(a_k, a_k^2\right)$.<br>Hence the coordinates of the point $P_{k+1}$ is given by $\left(a_{k+1}, a_k^2\right)$.<br>Since this point lies on the line $y = 3x$, we have $a_k^2 = 3a_{k+1}$ and thus $a_{k+1} = \frac{a_k^2}{3}$. |
| **(ii)** | Using $a_k = 3\left(\frac{a_1}{3}\right)^{2^{k-1}}$, we have $a_3 = 3\left(\frac{1}{3}\right)^{2^{3-1}} = \frac{1}{27}$. |
| **(iii)** | Area of rectangle with base $(a_k - a_{k+1})$ and height $a_{k+1}^2$ is $(a_k - a_{k+1})(a_{k+1}^2)$.<br>Using the recurrence $a_{k+1} = \frac{a_k^2}{3}$ we have<br>$(a_k - a_{k+1})(a_{k+1}^2) = (a_k - a_{k+1})(3a_{k+2})$. So $\sum_{k=1}^{n}(a_k - a_{k+1})(3a_{k+2})$<br>represents the sum of the areas of rectangles under the curve $y = x^2$ from $a_{n+1}$ to 1. This is clearly less than the exact area under the curve $y = x^2$ from 0 to 1 which is given by $\int_0^1 x^2 \, dx = \frac{1}{3}$.<br>Hence we have $\sum_{k=1}^{n}(a_k - a_{k+1})(3a_{k+2}) < \frac{1}{3} \Rightarrow \sum_{k=1}^{n}(a_k - a_{k+1})(a_{k+2}) < \frac{1}{9}$. |
| **(iv)** | Since $a_n = 3\left(\frac{1}{3}\right)^{2^{n-1}}$, the sequence is decreasing and thus, $a_n \leq a_3 = \frac{1}{27}$ for $n \geq 3$.<br>Therefore,<br>$\sum_{k=1}^{n}(a_k - a_{k+1})a_{k+2} < \sum_{k=1}^{n}(a_k - a_{k+1})a_3 = a_3\sum_{k=1}^{n}(a_k - a_{k+1})$<br>$= \frac{1}{27}\sum_{k=1}^{n}(a_k - a_{k+1})$<br>$= \frac{1}{27}\left(1 - \left(\frac{1}{3}\right)^{2^n - 1}\right) < \frac{1}{27}$. |

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**Source of Question: ASRJC Prelim 9758/2019/I/Q11**

**6** **(a)** Find the sum of all integers between 200 and 1000 (both inclusive) that are not divisible by 7. [4]

**(b)**

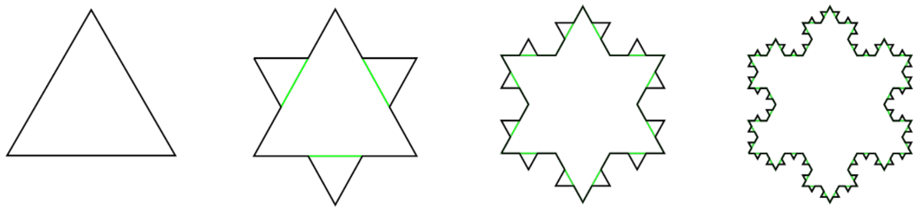

Fig. 1 Fig. 2 Fig. 3 Fig. 4

Snowflakes can be constructed by starting with an equilateral triangle (Fig. 1), then repeatedly altering each line segment of the resulting polygon as follows:

1. Divide each outer line segments into three segments of equal length.
2. Add an equilateral triangle that has the middle segment from step 1 as its base.
3. Remove the line segment that is the base of the triangle from step 2.
4. Repeat the above steps for a number of iterations, $n$.

The 1$^{\text{st}}$, 2$^{\text{nd}}$ and 3$^{\text{rd}}$ iteration produces the snowflakes in Fig. 2, Fig. 3 and Fig. 4 respectively.

**(i)** If $a_0$ denotes the area of the original triangle and the area of each new triangle added in the $n^{\text{th}}$ iteration is denoted by $a_n$, show that $a_n = \frac{1}{9}a_{n-1}$ for all positive integers $n$. [2]

**(ii)** Write down the number of sides in the polygons in Fig.1, Fig. 2 and Fig. 3, and deduce, with clear explanations, that the number of new triangles added in the $n^{\text{th}}$ iteration is $T_n = k\left(4^n\right)$ where $k$ is a constant to be determined. [2]

**(iii)** Find the total area of triangles added in the $n^{\text{th}}$ iteration, $A_n$ in terms of $a_0$ and $n$. [2]

**(iv)** Show that the total area of the snowflake produced after the $n^{\text{th}}$ iteration is $a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]$ units$^2$. [3]

The snowflake formed when the above steps are followed indefinitely is called the Koch snowflake.

**(v)** Determine the least number of iterations needed for the area of the snowflake to exceed 99% of the area of a Koch snowflake. [3]

**Solution:**

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| **(a)** | Sum required $= (200 + 201 + 202 + \ldots + 1000) - (203 + 210 + 217 + \ldots + 994)$ <br> $= \dfrac{1000-200+1}{2}(200+1000) - 7(29 + 30 + 31 + \ldots + 142)$ <br> $= 480600 - 7\left(\dfrac{142-29+1}{2}\right)(29+142)$ <br> $= 412371$ |
| **(b)(i)** | Each new triangle added in the $n^{\text{th}}$ iteration is similar to the triangle added in the previous iteration (AAA). <br> Since the length of a side of the new triangle is $\frac{1}{3}$ of the length of a side of the triangle in the previous iteration, $a_n = \left(\frac{1}{3}\right)^2 a_{n-1} = \frac{1}{9}a_{n-1}$ (Shown) |
| **(ii)** | No of sides in the polygon (Fig 1) = 3, <br> No of sides in the polygon (Fig 2) = 12 <br> No of sides in the polygon (Fig 3) = 48 <br> GP with first term 3 and common ratio 4 <br> Since there is 1 new triangle for every side in the next iteration, the number of new triangles added in the $n^{\text{th}}$ iteration = the number of sides in the polygon after the $(n-1)^{\text{th}}$ iteration <br> i.e. $T_1 = 3$, $T_2 = 12$, $T_3 = 48$, … <br> $\therefore\ T_n = 3(4)^{n-1} = \frac{3}{4}(4)^n$ and $k = \frac{3}{4}$ |
| **(iii)** | $A_n = T_n \times a_n = \frac{3}{4}(4^n)\times\left(\frac{1}{9}a_{n-1}\right)$ <br> $= \frac{3}{4}(4^n)\times\left(\underbrace{\frac{1}{9}\times\frac{1}{9}\times\frac{1}{9}\times\ldots\times\frac{1}{9}}_{n\text{ terms}}a_0\right)$ |

| | |
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| | $= \frac{3}{4}\left(\frac{4}{9}\right)^n a_0$ units$^2$ |
| **(iv)** | Total area required $= a_0 + \sum_{r=1}^{n} \frac{3}{4}\left(\frac{4}{9}\right)^r a_0$<br>$= a_0 + \frac{3}{4}a_0 \frac{\frac{4}{9}\left(1-\frac{4}{9}^n\right)}{1-\frac{4}{9}}$<br>$= a_0\left[1+\frac{3}{5}\left(1-\frac{4}{9}^n\right)\right]$<br>$= a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]$ (Shown) |
| **(v)** | Area of a Koch snowflake $= \lim_{n\to\infty} a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right] = \frac{8}{5}a_0$<br>Consider $a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right] > 0.99\times\frac{8}{5}a_0$<br>$0.016-\frac{3}{5}\left(\frac{4}{9}\right)^n > 0$<br>From GC,<br><table><tr><td>$n$</td><td>$0.016-\frac{3}{5}\left(\frac{4}{9}\right)^n$</td></tr><tr><td>4</td><td>–0.007 < 0</td></tr><tr><td>5</td><td>0.0056 > 0</td></tr><tr><td>6</td><td>0.0114 > 0</td></tr></table>Hence the least value of $n$ is 5. |

**Source of Question: SAJC Prelim 9758/2019/I/Q8**

**7** **(a)** Meredith owns a set of screwdrivers numbered 1 to 17 in decreasing lengths. The lengths of the screwdrivers form a geometric progression. It is given that the total length of the longest 3 screwdrivers is equal to three times the total length of the 5 shortest screwdrivers. It is also given that the total length of all the odd-numbered screwdrivers is 120 cm. Find the total length of all the screwdrivers, giving your answer correct to 2 decimal places. **[4]**

**(b)** Meredith is building a DIY workbench, and she needs to secure several screws by twisting them with a screwdriver drill. Each time Meredith presses the button on the drill, the screw is rotated clockwise by $u_n$ radians, where $n$ is the number of times the button is pressed. Each press rotates the screw more than the previous twist, and on the first press, the screw is rotated by $\frac{2\pi}{3}$ radians. It is given that $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ and $\sin u_{n+1} = \frac{1}{2}\sin u_n + \frac{\sqrt{3}}{2}\cos u_n$ for all $n \geq 1$.

**(i)** By considering $\cos\left(u_{n+1} - u_n\right)$ or otherwise, and assuming that the increase in rotation in successive twists is less than $\pi$ radians, prove that $\{u_n\}$ is an arithmetic progression with common difference $\frac{\pi}{3}$ radians. **[3]**

**(ii)** Each screw requires at least 25 complete revolutions to ensure that it does not fall out. Find the minimum number of times Meredith has to press the drill button to ensure the screw is fixed in place. **[3]**

**(iii)** The distance the screw is driven into the workbench on the $n$th press of the drill, $d_n$, is proportional to the angle of rotation $u_n$. If the total distance the screw is driven into the workbench after 21 presses is 144mm, find the distance the screw is driven into the workbench on the first press. **[2]**

**Solution:**

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|---|---|
| **(a)** | Let the length of the first screwdriver be $b$, and the common ratio be $r$.<br>$b + br + br^2 = 3\left(br^{12} + br^{13} + br^{14} + br^{15} + br^{16}\right)$<br>$1 + r + r^2 - 3r^{12} - 3r^{13} - 3r^{14} - 3r^{15} - 3r^{16} = 0$<br>Using the GC, $r$ = 0.882854 (6 s.f.) since $0 < r < 1$ .<br>There are 9 odd numbered screwdrivers, with common ratio of lengths being $r^2$.<br>Total length = $\frac{b\left(1-\left(r^2\right)^9\right)}{1-r^2} = 120$<br>$\frac{b\left(1-r^{18}\right)}{1-r^2} = 120$<br>Using the GC, $b$ = 29.6122 (6 s.f.)<br>The total length is $\frac{b\left(1-r^{17}\right)}{1-r} = 222.38 \text{ cm} \quad (\text{to 2 d.p.})$ |

| | |
|---|---|
| **(b)(i)** | $\cos(u_{n+1}-u_n)=\cos u_{n+1}\cos u_n+\sin u_{n+1}\sin u_n$<br>$=\frac{1}{2}\cos^2 u_n-\frac{\sqrt{3}}{2}\sin u_n\cos u_n$<br>$+\frac{1}{2}\sin^2 u_n+\frac{\sqrt{3}}{2}\sin u_n\cos u_n$<br>$=\frac{1}{2}$<br><br>$\therefore u_{n+1}-u_n=\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$ (since the difference is less than $\pi$)<br>is a constant independent of $n$.<br><br>Hence $\{u_n\}$ is an arithmetic progression.<br><br>The common difference is $\frac{\pi}{3}$.<br><br>**<u>Alternatively</u>**<br><br>$\cos u_{n+1}=\frac{1}{2}\cos u_n-\frac{\sqrt{3}}{2}\sin u_n$<br>$=\cos u_n\cos\frac{\pi}{3}-\sin u_n\sin\frac{\pi}{3}$<br>$=\cos\left(u_n+\frac{\pi}{3}\right)$<br><br>$\therefore u_{n+1}=u_n+\frac{\pi}{3}$ since the difference is less than $\pi$.<br><br>$u_{n+1}-u_n$ is a constant, therefore $\{u_n\}$ is an arithmetic progression.<br><br>The common difference is $\frac{\pi}{3}$. |
| **(ii)** | The total angle rotated over $n$ twists is $\frac{n}{2}\left(2\left(\frac{2\pi}{3}\right)+(n-1)\frac{\pi}{3}\right)$.<br><br>$\frac{n}{2}\left(\frac{4\pi}{3}+(n-1)\frac{\pi}{3}\right)\geq 25\times(2\pi)$<br>$\frac{2n\pi}{3}+\frac{\pi}{6}n(n-1)-50\pi\geq 0$<br>$4n+n(n-1)-300\geq 0$<br>$n^2+3n-300\geq 0$<br><br>Using the GC, |

| $n$ | $n^2+3n-300$ |
|---|---|
| 15 | $-30$ |
| 16 | 4 |
| 17 | 40 |

The minimum number of buttons presses required is 16.

**(iii)** Total angle rotated after 21 button presses is

$$\frac{21}{2}\left(2\left(\frac{2\pi}{3}\right)+(20)\frac{\pi}{3}\right)$$
$$=14\pi+70\pi$$
$$=84\pi$$

$84\pi k=144,$ where $k$ is a positive real constant.

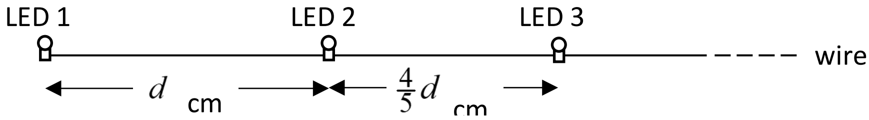
The first button press rotates the screw by $\frac{2\pi}{3}$.

$$d_1=ku_1=\frac{144}{84\pi}\left(\frac{2\pi}{3}\right)=\frac{8}{7}$$

The first button press drills the screw in by $\frac{8}{7}$ mm $(1.14\text{mm})$.

**Source of Question: HCI Prelim 9758/2019/I/Q9**

**8** A company produces festive decorative Light Emitting Diode (LED) string lights, where micro LEDs are placed at intervals along a thin wire. In a particular design, the first LED (LED 1) is placed on one end of a wire with the second LED (LED 2) placed at a distance of $d$ cm from LED 1, and each subsequent LED is placed at a distance $\frac{4}{5}$ times the preceding distance as shown.




**(i)** If the distance between LED 8 and LED 9 is 56.2 cm, find the value of $d$ correct to 1 decimal place. [2]

**(ii)** Find the theoretical maximum length of the wire, giving your answer in centimetres correct to 1 decimal place. [2]

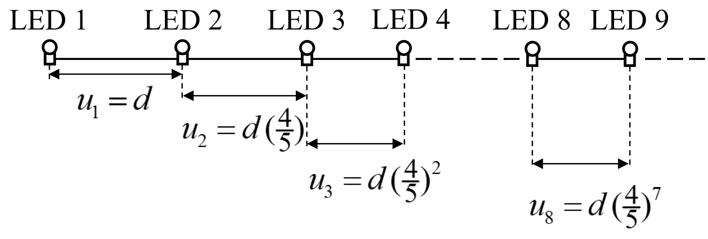
The LEDs consisting of three colours red, orange and yellow, are arranged in that order in a repeated manner, that is, LED 1 is red, LED 2 is orange, LED 3 is yellow, LED 4 is red, LED 5 is orange, LED 6 is yellow, and so on.

**(iii)** Find the colour of the LED nearest to a point on the wire 12.9 m from LED 1. [3]

**(iv)** If the minimum distance between any two consecutive LEDs is 1 cm so that they can be mounted on the wire, find the colour of the last LED. [3]

**Solution:**

**(i)**




$d(\frac{4}{5})^7 = 56.2$

$\therefore\ d = 267.9824829 = 268.0$ (1 d.p.)

**(ii)**

Since $|r| = \left|\frac{4}{5}\right| < 1$, sum to infinity exists.

Hence maximum theoretical length of wire

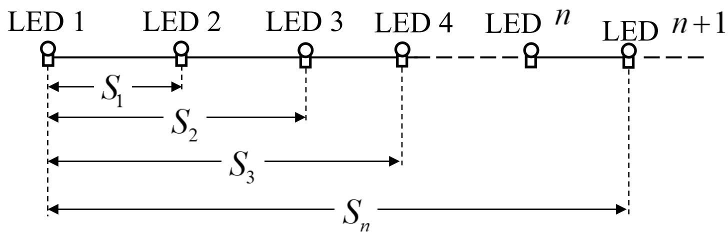
$= \dfrac{d}{1-r}$

$= \dfrac{267.9824829}{1-\frac{4}{5}}$

$= 1339.912415$

$= 1339.9$ cm (1 d.p.)

**(iii)**




Let $S_n$ be distance from LED 1 to LED $n+1$.

<u>Method 1</u>: (using GC table)

Using sum of GP, $S_n = \dfrac{267.9824829[1-(\frac{4}{5})^n]}{1-\frac{4}{5}}$

Using GC,

| $n$ | $S_n$ | $\lvert S_n - 1290\rvert$ |
|---|---|---|
| 14 | 1281.0 | 9.0 |

| | | |
|---|---|---|
| 15 | 1292.8 | 2.8 |

$\therefore$ closest LED to 1290 is when $n = 15$.

$\therefore$ required LED is LED $(15+1)$ = LED 16.

Hence colour of LED 16 is red.

Method 2: (using algebraic manipulation)

Let $S_n = \dfrac{267.9824829[1-(\frac{4}{5})^n]}{1-\frac{4}{5}} = 1290$

$1-(\frac{4}{5})^n = 0.9627494943$

$n\ln(\frac{4}{5}) = \ln 0.0372505057$

$\therefore\ n = 14.74427443$

Using GC,

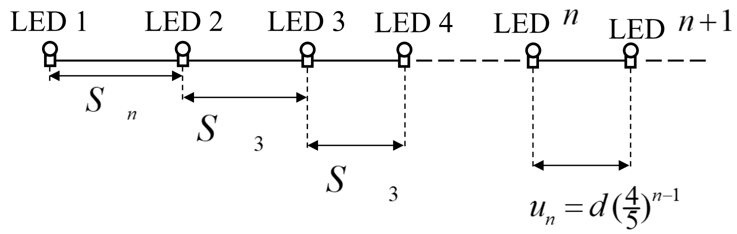
| $n$ | $S_n$ | $\lvert S_n - 1290\rvert$ |
|---|---|---|
| 14 | 1281.0 | 9.0 |
| 15 | 1292.8 | 2.8 |

$\therefore$ closest LED to 1290 is when $n = 15$.

$\therefore$ required LED is LED $(15+1)$ = LED 16.

Hence colour of LED 16 is red.

**(iv)**




Let $u_n = d\,(\frac{4}{5})^{n-1} = 267.9824829(\frac{4}{5})^{n-1} \geq 1$

$(\frac{4}{5})^{n-1} \geq \frac{1}{267.9824829}$

$\ln(\frac{4}{5})^{n-1} \geq \ln\frac{1}{267.9824829}$

$(n-1)\ln(\frac{4}{5}) \geq \ln\frac{1}{267.9824829}$

$n-1 \leq 25.05526859$

$\therefore\ n \leq 26.05526859$

Since largest integer $n = 26$, last LED is LED $(26+1) =$ LED 27

| | |
|---|---|
| | Hence colour of last LED 27 is yellow. |

**Source of Question: MI Prelim 9758/2021/I/Q11**

**9** Bank *A* offers a study loan to students enrolled in an undergraduate course of study. The key features of the loan are:

- interest-free during the course of study,
- fixed monthly interest of $0.3\%$ upon graduation,
- minimum monthly repayment of $100.

Ali decides to take a study loan of $50000 from Bank *A* on 1 January 2021 at the start of his 3-year undergraduate course.

**(a)** During his course of study, Ali pays $200 at the end of January 2021 and on the last day of each subsequent month, he pays $10 more than in the previous month. Thus on 28 February 2021, he pays $210 and on 31 March 2021, he pays $220, and so on. How much does Ali owe the bank at the end of his course of study? [2]

**(b)** Bank *A* charges interest on any outstanding amount of the loan on the first day of each month, starting on the month right after a student graduates.

**(i)** Upon graduation, Ali immediately found a job and decides to pay $900 to bank *A* on the last day of each month, starting on the month right after he graduated. Show that Ali owes the bank $34121 (to the nearest dollar) on the last day of the 3rd month after he graduated. [2]

**(ii)** Use the formula for the sum of a geometric progression to find an expression for the amount owed by Ali on the last day of the *n*th month after he graduated. Hence find in which month Ali pays off his study loan. [5]

**(iii)** Find the total amount of interest that Ali paid. [2]

**(iv)** If Ali decides to pay off his study loan within 3 years upon graduation, i.e. at the end of December 2026, what is the minimum amount, to the nearest dollar, that he needs to pay per month after graduation? [3]

**Solution:**

| | |
|---|---|
| **(a)** | Amount of money Ali paid at the end of 3 years<br>$=\frac{36}{2}\left[2(200)+(36-1)(10)\right]$<br>$=13500$<br>Amount Ali owes the bank at the end of 3 years<br>$=50000-13500$<br>$=36500$ |

**(b)(i)**

At the end of 1 month, amount owed

$=1.003(36500)-900$

$=35709.50$

At the end of 2 months, amount owed

$=1.003(35709.50)-900$

$=34916.6285$

At the end of 3 months, amount owed

$=1.003(34916.6285)-900$

$=34121.38$

$=34121$ (nearest dollar)

**(ii)**

| Month | Amount owed at the start of month | Amount owed at the end of month |
|---|---|---|
| 1 | $1.003(36500)$ | $1.003(36500)-900$ |
| 2 | $1.003\left[1.003(36500)-900\right]$ <br> $=1.003^2(36500)-1.003(900)$ | $1.003^2(36500)-1.003(900)$ <br> $-900$ |
| 3 | $1.003\left[1.003^2(36500)-1.003(900)-900\right]$ <br> $=1.003^3(36500)-1.003^2(900)-1.003(900)$ | $1.003^3(36500)-1.003^2(900)-1.003(900)$ <br> $-900$ |
| … | … | … |
| $n$ | | $1.003^n(36500)-1.003^{n-1}(900)$ <br> $-1.003^{n-2}(900)-\ldots-900$ |

On the last day of the $n$th month, Ali owed

$$1.003^n(36500)-1.003^{n-1}(900)-1.003^{n-2}(900)-\ldots-900$$

$$=1.003^n(36500)-\left[900+1.003(900)\ldots+1.003^{n-2}(900)+1.003^{n-1}(900)\right]$$

$$=1.003^n(36500)-900\left(1+1.003+\ldots+1.003^{n-2}+1.003^{n-1}\right)$$

$$=1.003^n(36500)-900\left[\frac{1.003^n-1}{1.003-1}\right]$$

$$=1.003^n(36500)-300000\left(1.003^n-1\right)$$

When Ali pays off his study loan,

$$1.003^n(36500)-300000\left(1.003^n-1\right)\le 0$$

Using GC,

<table>
<tr><td></td><td>

| $n$ | $1.003^n(36500)-300000(1.003^n-1)$ |
|---|---|
| 43 | 276.54 |
| 44 | $-622.6$ |
| 45 | $-1524$ |

Ali takes 44 months to pay off his study loan, i.e. he pays off his study loan in August 2027.
</td></tr>
<tr><td>(iii)</td><td>

**Method 1:**
Since Ali pays $\$(1.003\times276.54)$ in the last month, total interest Ali paid
$=43\times900+(1.003\times276.54)-36500$
$=2477.37$ (nearest cent)

**Method 2:**
Since Ali pays \$276.54 in the last month, total amount he paid
$=13500+43\times900+(1.003\times276.54)$
$=52477.37$
Total amount of interest Ali paid
$=52477.37-50000$
$=2477.37$ (nearest cent)
</td></tr>
<tr><td>(iv)</td><td>

Let the amount that Ali pays per month upon graduation be \$$x$.
$1.003^{36}(36500)-x\left[\frac{1.003^{36}-1}{1.003-1}\right]\le 0$
$40656.16902-37.956x\le 0$
$x\ge 1071.14$
Ali needs to pay \$1072 per month.
</td></tr>
</table>

**Source of Question: HCI Prelim 9758/2022/01/Q8**

**10 Do not use a graphing calculator for this question.**

**(i)** The complex number $a+\mathrm{i}$ is a root of the equation $z^2-2\sqrt{2}z+b=0$ where $a$ and $b$ are positive real constants. Find the exact values of $a$ and $b$. [3]

It is given that $\mathrm{f}(z)=z^6-3z^4+11z^2-9$.

**(ii)** Show that $\mathrm{f}(z)=\mathrm{f}(-z)$. [1]

**(iii)** It is given that $a+\mathrm{i}$ is a root of the equation $\mathrm{f}(z)=0$. Using the results found in parts **(i)** and **(ii)**, show that $\mathrm{f}(z)$ can be factorised into 3 quadratic factors with real coefficients, leaving your answer in exact form. [4]

**Solution:**

| | |
|---|---|
| **(i)** | Since the coefficients of the polynomial are all real and $a+\mathrm{i}$ is a root, then $a-\mathrm{i}$ is also a root.<br><br>$z^2-2\sqrt{2}z+b=\left[z-(a+\mathrm{i})\right]\left[z-(a-\mathrm{i})\right]$<br>$=z^2-2az+\left(a^2+1\right)$<br><br>By comparing $z$ term, $-2\sqrt{2}=-2a$<br>$a=\sqrt{2}$.<br><br>By comparing constant term, $b=a^2+1=\left(\sqrt{2}\right)^2+1=3$.<br><br>**Alternative Method**<br>$(a+\mathrm{i})^2-2\sqrt{2}(a+\mathrm{i})+b=0$<br>$\left(a^2-1-2\sqrt{2}a+b\right)+\left(2a-2\sqrt{2}\right)\mathrm{i}=0$<br>By comparing real and imaginary parts,<br>$\left(2a-2\sqrt{2}\right)=0$<br>$a=\sqrt{2}$<br><br>$a^2-1-2\sqrt{2}a+b=0$<br>$2-1-4+b=0$<br>$b=3$ |
| **(ii)** | $\mathrm{f}(-z)=(-z)^6-3(-z)^4+11(-z)^2-9$<br>$=z^6-3z^4+11z^2-9$<br>$=\mathrm{f}(z)$ |
| **(iii)** | Since the coefficients of $\mathrm{f}(z)$ are all real and $\sqrt{2}+\mathrm{i}$ is a root, then $\sqrt{2}-\mathrm{i}$ is also a root.<br>Thus $z^2-2\sqrt{2}z+3$ is a quadratic factor of $\mathrm{f}(z)$.<br><br>Since $\mathrm{f}(z)=\mathrm{f}(-z)$, then $-\sqrt{2}+\mathrm{i}$ and $-\sqrt{2}-\mathrm{i}$ are also roots of $\mathrm{f}(z)=0$.<br>$\left[z-\left(-\sqrt{2}-\mathrm{i}\right)\right]\left[z-\left(-\sqrt{2}+\mathrm{i}\right)\right]=\left(z+\sqrt{2}\right)^2-\mathrm{i}^2$<br>$=z^2+2\sqrt{2}+3$<br><br>Thus $z^2+2\sqrt{2}z+3$ is a quadratic factor of $\mathrm{f}(z)$.<br>$\mathrm{f}(z)=z^6-3z^4+11z^2-9$<br>$=\left(z^2-2\sqrt{2}z+3\right)\left(z^2+2\sqrt{2}z+3\right)\left(z^2+Cz+D\right)$<br><br>By comparing constant term, $-9=9D\Rightarrow D=-1$ |

| | |
|---|---|
| | $\mathrm{f}(z)=\left(z^2-2\sqrt{2}z+3\right)\left(z^2+2\sqrt{2}z+3\right)\left(z^2+Cz-1\right)$<br>$=\left[\left(z^2+3\right)^2-\left(2\sqrt{2}z\right)^2\right]\left(z^2+Cz-1\right)$<br>$=\left(z^4-2z^2+9\right)\left(z^2+Cz-1\right)$<br><br>By comparing coefficient of $z$ term, then $0=9Cz\Rightarrow C=0$.<br><br>Therefore,<br>$\mathrm{f}(z)=\left(z^2-2\sqrt{2}z+3\right)\left(z^2+2\sqrt{2}z+3\right)\left(z^2-1\right)$. |

**Source of Question: HCI JC2 CT1 9758/2017/P1/Q5**

**11** **(i)** Show that $\tan\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{\sin\alpha}$. [2]

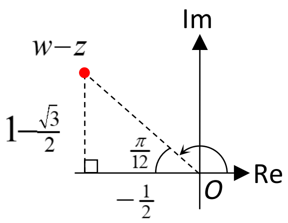
It is given that $w=-\dfrac{1}{2}+\mathrm{i}$ and $z=\dfrac{\sqrt{3}}{2}\mathrm{i}$ .

**(ii)** Show that $\left|w-z\right|=\sqrt{2-\sqrt{3}}$ . [2]

**(iii)** Using part **(i)**, find the exact value of $\arg\left(w-z\right)$. [3]

**Solution:**

| | |
|---|---|
| **(i)** | $\text{RHS}=\dfrac{1-\cos\alpha}{\sin\alpha}$<br>$=\dfrac{1-\left(1-2\sin^2\dfrac{\alpha}{2}\right)}{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}$<br>$=\dfrac{2\sin^2\dfrac{\alpha}{2}}{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}$<br>$=\dfrac{\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}}$<br>$=\tan\dfrac{\alpha}{2}=\text{LHS}$ (shown) |
| **(ii)** | $w-z=-\dfrac{1}{2}+\mathrm{i}-\dfrac{\sqrt{3}}{2}\mathrm{i}$ |

| | |
|---|---|
| | $= -\frac{1}{2} + \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{i}$<br><br>$\lvert w - z \rvert = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(1 - \frac{\sqrt{3}}{2}\right)^2}$<br>$= \sqrt{\frac{1}{4} + \left(1 - 2(1)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)^2\right)}$<br>$= \sqrt{\frac{1}{4} + 1 - \sqrt{3} + \frac{3}{4}}$<br>$= \sqrt{2 - \sqrt{3}}$ |
| **(iii)** | $w - z = -\frac{1}{2} + \mathrm{i}\left(1 - \frac{\sqrt{3}}{2}\right)$ is in 2nd quadrant.<br><br>Basic angle $= \tan^{-1}\left(\dfrac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$<br>$= \tan^{-1}\left(\dfrac{1 - \cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}\right)$<br>$= \tan^{-1}\left(\tan\dfrac{\left(\frac{\pi}{6}\right)}{2}\right)$<br>$= \frac{\pi}{12}$<br><br><br><br><br>Since $w - z$ lies in 2nd quadrant,<br><br>$\arg(w - z) = \pi - \frac{\pi}{12}$<br>$= \frac{11\pi}{12}$ |

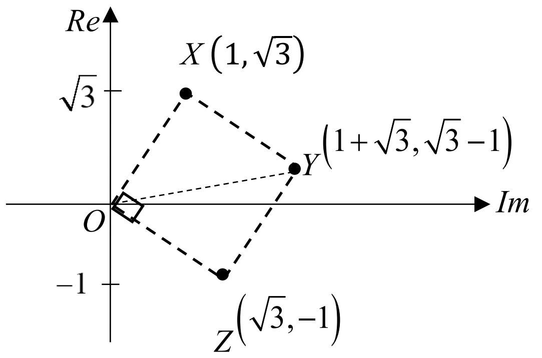
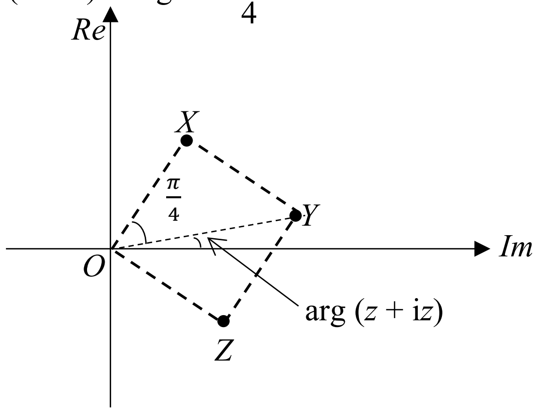
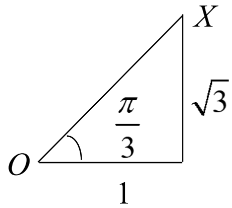
**Source of Question: HCI JC2 CT2 9758/2017/P1/Q4**

**12** The complex number $z$ is given by $z=\sqrt{3}-\mathrm{i}$. In an Argand diagram, mark and label clearly the points *Z, X* and *Y* representing the complex numbers $z$ , $\mathrm{i}z$ and $z+\mathrm{i}z$ respectively. [2]

**(i)** State, with a reason, the geometrical representation of *OXYZ*. [1]

**(ii)** Using your Argand diagram, determine the argument of $z+\mathrm{i}z$. Hence deduce that $\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$. [5]

**Solution:**

| | |
|---|---|
| **(i)** | <br>angle $XOZ=90^{\circ}$ …… (1)<br>$OZ = \lvert z\rvert = 2$ and $OX = \lvert \mathrm{i}z\rvert = 2$ ….. (2)<br>*OXYZ* is a square. |
| **(ii)** | arg $(z + \mathrm{i}z)$ = arg $\mathrm{i}z - \frac{\pi}{4}$<br> |
| | arg $(z + \mathrm{i}z) = \frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$<br><br>**Alternatively,**<br>arg $(z + \mathrm{i}z) = \angle ZOY - \angle\lvert\arg(z)\rvert = \frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$<br>Also,<br>$z+\mathrm{i}z=\left(\sqrt{3}-\mathrm{i}\right)+\left(\mathrm{i}\sqrt{3}+1\right)=\left(\sqrt{3}+1\right)+\mathrm{i}\left(\sqrt{3}-1\right)$<br>Hence arg $(z + \mathrm{i}z) = \tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)=\frac{\pi}{12}$<br>$\therefore \tan\left(\frac{\pi}{12}\right)=\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}.$ |

**13 VJC Promo 9758/2024/Q7**

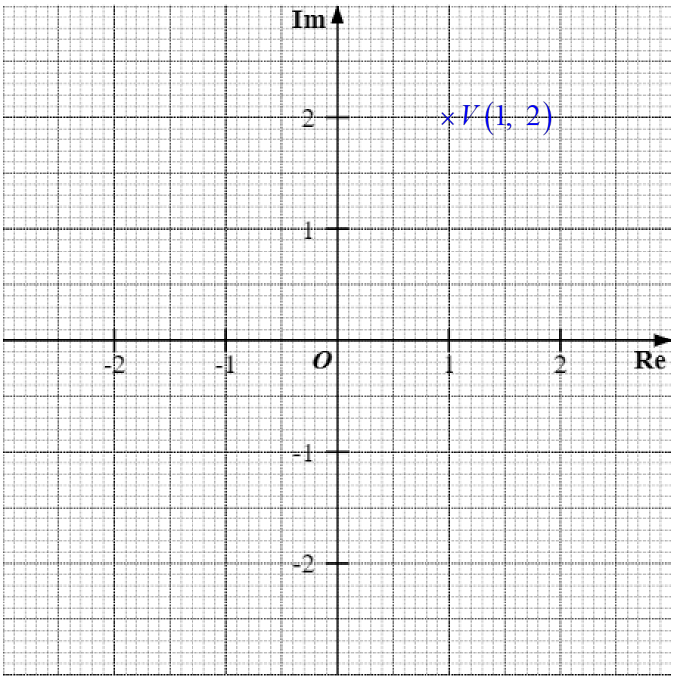
**(a) (i)** Showing your working, find the complex numbers $v$ and $w$ which satisfy the following simultaneous equations.

$$v + 2w = 5$$
$$3v - w^* = 1 + 5\mathrm{i}$$ [4]

**(ii)** Points $O$, $V$ and $W$ on an Argand diagram represent the complex numbers 0, $v$ and $w$ respectively.
Plot these points on an Argand diagram and state the transformation that maps the point $W$ onto $V$. [2]

**(b)** The roots of the equation $z^4 - 2z^3 - 3z^2 + az + b = 0$, where $a$ and $b$ are real numbers, are $z_1$, $z_2$, $z_3$ and $z_4$. It is given that $z_1^2 + z_2^2 + z_3^2 + z_4^2 < 0$. Explain why at most two of $z_1$, $z_2$, $z_3$ and $z_4$ are real. [2]

| No. | Solution |
|---|---|
| 3(a)(i) | $v + 2w = 5 \Rightarrow 3v + 6w = 15$ ---------- (1)<br>$3v - w^* = 1 + 5\mathrm{i}$ ------ (2)<br>(1) – (2): $6w + w^* = 14 - 5\mathrm{i}$<br>Let $w = x + \mathrm{i}y$<br>$6(x + \mathrm{i}y) + (x - \mathrm{i}y) = 14 - 5\mathrm{i}$<br>$7x + 5y\mathrm{i} = 14 - 5\mathrm{i}$<br>Comparing real and imaginary parts:<br>$7x = 14 \Rightarrow x = 2$<br>$5y = -5 \Rightarrow y = -1$<br>Hence, $w = 2 - \mathrm{i}$<br>Sub $w = 2 - \mathrm{i}$ into $v + 2w = 5$: $v = 5 - 2(2 - \mathrm{i}) = 1 + 2\mathrm{i}$<br>**<u>Alternatively</u>**<br>$v + 2w = 5 \Rightarrow v = 5 - 2w$ and sub into eqn (2):<br>$3(5 - 2w) - w^* = 1 + 5\mathrm{i}$<br>$15 - 6w - w^* = 1 + 5\mathrm{i}$<br>Let $w = x + y\mathrm{i}$:<br>$15 - 6(x + y\mathrm{i}) - (x - y\mathrm{i}) = 1 + 5\mathrm{i}$<br>$7x + 5y\mathrm{i} = 14 - 5\mathrm{i}$ and compare real and imaginary parts |

| | |
|---|---|
| | |
| (a)(ii) | <br>Transformation: $W$ is rotated **anti-clockwise** through $\frac{\pi}{2}$ about $O$ to obtain $V$ **OR** $W$ is rotated **clockwise** through $\frac{3\pi}{2}$ about $O$ to obtain $V$ |
| (b) | Since $z_1^2+z_2^2+z_3^2+z_4^2<0$, the equation has at least one non-real root. In addition, since the coefficients of the polynomial equation are all real, by conjugate root theorem, there is at least 1 pair of complex conjugate roots. Hence, at most two of $z_1, z_2, z_3$ and $z_4$ are real. |