RAFFLES INSTITUTION
H2 Mathematics 9758
2025 Year 6 Term 3 Revision 5 (Zeta)
Topic(s): Vectors
(Vectors II and III (Solutions))

Source of Question: JPJC Prelim 9758/2022/02/Q2

1 **(a)** With reference to the origin O, the points A, B and X are $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OX}=\frac{1}{8}\mathbf{a}+\frac{3}{8}\mathbf{b}$. The point Y lies on AB such that O, X and Y are collinear. Express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and find the ratio of $AY:YB$. [5]

(b) The points P, Q and R have non zero position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively. P and Q are fixed, and R varies. Describe geometrically the set of possible positions of the point R such that

(i) $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=\mathbf{0}$, [2]

(ii) $(\mathbf{r}-\mathbf{p}).\mathbf{q}=0$. [2]

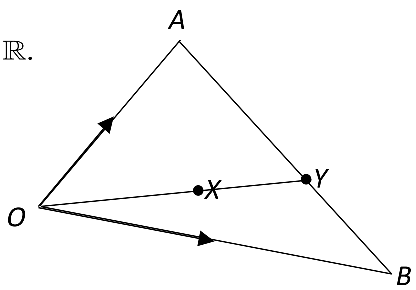
Solution:

1(a)

Y lies on line $AB \Rightarrow \overrightarrow{OY}=\mathbf{a}+\beta(\mathbf{b}-\mathbf{a})$, for some $\beta\in\mathbb{R}$.

Since O, X and Y are collinear,

$\mathbf{a}+\beta(\mathbf{b}-\mathbf{a})=\alpha\left(\frac{1}{8}\mathbf{a}+\frac{3}{8}\mathbf{b}\right)$ for some $\alpha\in\mathbb{R}$.



Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel,

Comparing,

$1-\beta=\frac{\alpha}{8}$ ---(1)

$\beta=\frac{3\alpha}{8}$ ---(2)

Solving (1) and (2), $\alpha=2,\quad \beta=\frac{3}{4}$

$\overrightarrow{OY}=2\left(\frac{1}{8}\mathbf{a}+\frac{3}{8}\mathbf{b}\right)=\frac{1}{4}\mathbf{a}+\frac{3}{4}\mathbf{b}$

$\overrightarrow{AY}=\frac{3}{4}\overrightarrow{AB}$

$AY:YB=3:1$

(b)(i)	$(\mathbf{r}-\mathbf{p})\times\mathbf{q}=\mathbf{0}$ We see that either $(\mathbf{r}-\mathbf{p})\,//\,\mathbf{q}$ or $(\mathbf{r}-\mathbf{p})=\mathbf{0}$ $\mathbf{r}=\mathbf{p}+k\mathbf{q},\ k\neq 0$ or $\mathbf{r}=\mathbf{p}$ Thus $\mathbf{r}=\mathbf{p}+k\mathbf{q},\ k\in\mathbb{R}$ The set of all possible positions of the point R is the line that passes through the point P and parallel to the vector $\mathbf{q}$.
(b)(ii)	$(\mathbf{r}-\mathbf{p}).\mathbf{q}=0$ $\mathbf{r}.\mathbf{q}-\mathbf{p}.\mathbf{q}=0$ $\mathbf{r}.\mathbf{q}=\mathbf{p}.\mathbf{q}$ The set of all possible positions of the point R is the plane that is perpendicular to $\mathbf{q}$ and containing the point P.

Source of Question: HCI Prelim 9758/2023/02/Q5

2 The equations of two planes p_1 and p_2 are

$$2x + sy + z = -3 \ ,$$
$$x - 2y + 3z = 1 \ ,$$

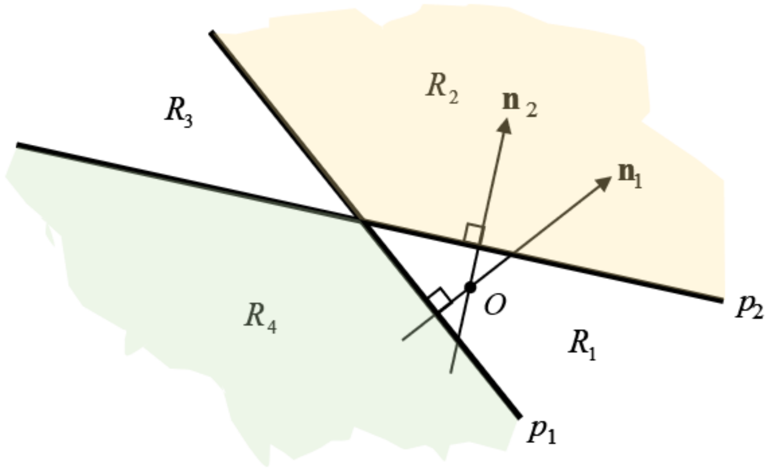
respectively, where s is a negative real constant.

(a) Find a vector parallel to both p_1 and p_2, giving your answer in terms of s. [2]

(b) It is given that p_1 and p_2 intersect in a line l, and l meets the xz-plane at a point A. Find the coordinates of A. [3]

(c) The shortest distance between the origin O and p_1 is $\frac{3}{\sqrt{6}}$. Find the value of s. [2]

It is now given that $s = -2$, with the normal vectors to p_1 and p_2 denoted as $\mathbf{n_1} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{n_2} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ respectively. It is given that p_1 and p_2 divide the real space into four regions R_1, R_2, R_3 and R_4 as shown in the following two-dimensional diagram (not drawn to scale).



(d) Relative to the origin O, the point B has position vector $\mathbf{k}$. Determine whether B lies in R_1, R_2, R_3 or R_4, justifying your answer. [2]

Solution:

2(a)	A vector parallel to both p_1 and p_2 is $\begin{pmatrix} 2 \\ s \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3s+2 \\ -5 \\ -4-s \end{pmatrix}$
(b)	xz-plane $\Rightarrow$ $y = 0$ $2x + z = -3$ ...(1) $x + 3z = 1$ ...(2) $2\times(2) - (1)$: $5z = 5 \Rightarrow z = 1$ $\therefore\ x = -2$

Hence coordinates of A is $(-2,0,1)$.

(c)

$$p_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ s \\ 1 \end{pmatrix} = -3$$

$$\underset{\sim}{r} \cdot \frac{1}{\sqrt{2^2+s^2+1^2}} \begin{pmatrix} 2 \\ s \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2^2+s^2+1^2}}(-3) = \frac{-3}{\sqrt{s^2+5}}$$

$\therefore$ shortest distance between O and p_1 is

$$\left|\frac{-3}{\sqrt{s^2+5}}\right| = \frac{3}{\sqrt{6}}$$

$$\frac{3}{\sqrt{s^2+5}} = \frac{3}{\sqrt{6}}$$

$$\sqrt{\frac{6}{s^2+5}} = 1$$

$$\frac{6}{s^2+5} = 1$$

$$6 = s^2 + 5$$

$$s^2 = 1$$

$$s = \pm 1 \quad (\text{reject } s = 1 \text{ since } s < 0)$$

$\therefore\ s = -1$

(d)

Given $\overrightarrow{OB} = \underset{\sim}{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $p_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -3$, $p_2 : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 1$

<u>Method 1:</u>

Consider the origin O:

$p_1 : 2(0) - 2(0) + (0) = 0 > -3$

$p_2 : (0) - 2(0) + 3(0) = 0 < 1$

Consider the point B:

$p_1 : 2(0) - 2(0) + (1) = 1 > -3$

$p_2 : (0) - 2(0) + 3(1) = 3 > 1$

Hence, O and B are on the same side of p_1 but O and B are on opposite side of p_2 .

$\therefore\ B$ is in R_2 .

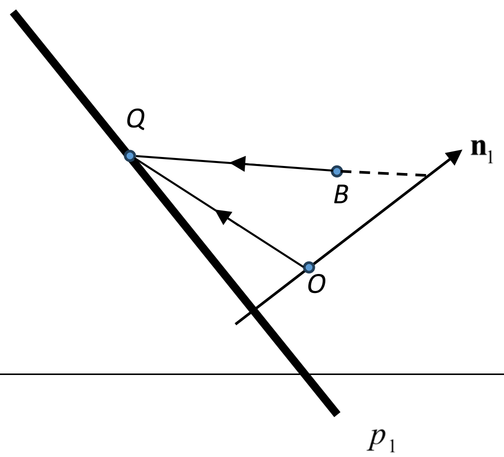
<u>Method 2: Using angles</u>

$$p_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -3$$

$Q(0,0,-3)$ is a point that lies in p_1 .

Since $\overrightarrow{OQ} \cdot \mathbf{n}_1 = -3 < 0$,

then the angle between OQ and $\mathbf{n}_1$ is obtuse.

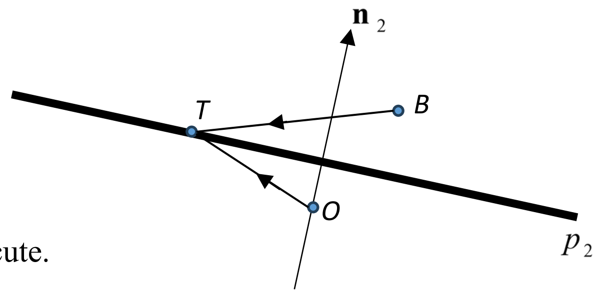


$\overrightarrow{BQ} \cdot \mathbf{n}_1 = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -4 < 0$,

then the angle between BQ and $\mathbf{n}_1$ is obtuse.

$\therefore$ O and B are on same side of p_1. $---(1)$

$p_2 : \underset{\sim}{r} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 1$



$T(1,0,0)$ is a point that lies in p_2.

Since $\overrightarrow{OT} \cdot \mathbf{n}_2 = 1 > 0$,

then the angle between OT and $\mathbf{n}_2$ is acute.

$\overrightarrow{BT} \cdot \mathbf{n}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = -2 < 0$,

then the angle between BT and $\mathbf{n}_2$ is obtuse.

$\therefore$ O and B are on opposite side of p_2. $---(2)$

Combining (1) and (2) with reference to given diagram, $\therefore$ B lies in R_2.

<u>Method 3</u>: (using shortest distance of planes to origin O)

Let p_3 be plane containing B and parallel to p_1.

Equation of p_3 is $\underset{\sim}{r}. \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$

$$\underset{\sim}{r}. \frac{1}{\sqrt{2^2+(-2)^2+1^2}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2^2+(-2)^2+1^2}}$$

i.e. $\underset{\sim}{r}. \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3} > 0$

Since equation of p_1 $\underset{\sim}{r}. \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{-3}{3} = -1 < 0$,

$\therefore$ p_1 and p_3 are on opposite sides of origin O. …(1)

Let p_4 be plane containing B and parallel to p_2.

Equation of p_4 is $\underset{\sim}{r}. \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 3$

	$r.\frac{1}{\sqrt{1^2+(-2)^2+3^2}}\begin{pmatrix}1\\-2\\3\end{pmatrix}=\frac{3}{\sqrt{1^2+(-2)^2+3^2}}=\frac{3}{\sqrt{14}}>0$ i.e. $r.\frac{1}{\sqrt{14}}\begin{pmatrix}1\\-2\\3\end{pmatrix}=\frac{3}{\sqrt{14}}>0$ Since equation of p_2 $\underset{\sim}{r}.\frac{1}{\sqrt{14}}\begin{pmatrix}1\\-2\\3\end{pmatrix}=\frac{1}{\sqrt{14}}>0$, $\therefore$ p_2 and p_4 are on same side of origin O. …(2) Combining (1) and (2) with reference to given diagram, $\therefore$ B lies in R_2.

Source of Question: NYJC Prelim 9758/2022/01/Q10

3 Two charged particles, U and V, are confined to the planes F_1 and F_2 with position vectors given by

$$(3+6p)\mathbf{i}+(1+4p+q)\mathbf{j}+(6+2p-4q)\mathbf{k} \text{ and } (-9+3p)\mathbf{i}+(1+p-2q)\mathbf{j}+(3-p+8q)\mathbf{k}$$

respectively, where $p,q\in\mathbb{R}$.

(i) Obtain the equation of F_1 in scalar product form. [2]

(ii) Find the acute angle between F_1 and F_2. [3]

The forces of the two particles U and V allow another charged particle W to remain suspended between them, such that $2UW = WV$.

(iii) As the positions of U and V vary, show that the set of points described by the path of W is a line l, whose vector equation is to be determined. [3]

(iv) An uncharged particle A is fired along a path described by a line $\mathbf{r} = s\mathbf{k}$, where $s\in\mathbb{R}$ and crosses F_1 at some instant in time. Find the shortest distance of A from l at this instant. [4]

Solution:

3(i)	$F_1:\ \mathbf{r}=\begin{pmatrix}3+6p\\1+4p+q\\6+2p-4q\end{pmatrix}=\begin{pmatrix}3\\1\\6\end{pmatrix}+p\begin{pmatrix}6\\4\\2\end{pmatrix}+q\begin{pmatrix}0\\1\\-4\end{pmatrix}$ $\mathbf{n}_1=\begin{pmatrix}6\\4\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\-4\end{pmatrix}=\begin{pmatrix}-18\\24\\6\end{pmatrix}=6\begin{pmatrix}-3\\4\\1\end{pmatrix}$

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$F_1 : \mathbf{r} \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = 1$

$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = 1$

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$F_2 :\ \mathbf{r} = \begin{pmatrix} -9+3p \\ 1+p-2q \\ 3-p+8q \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \\ 3 \end{pmatrix} + p \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ -2 \\ 8 \end{pmatrix}$

$\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ -24 \\ -6 \end{pmatrix} = -6 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$\theta = \cos^{-1} \dfrac{\left| \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \right|}{\sqrt{9+16+1}\sqrt{1+16+1}} = \cos^{-1} \left| \dfrac{20}{\sqrt{26}\sqrt{18}} \right| \approx 22.4^0$

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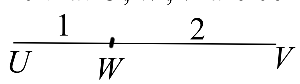
$\overrightarrow{OU} = \begin{pmatrix} 3+6p \\ 1+4p+q \\ 6+2p-4q \end{pmatrix}$ $\qquad$ $\overrightarrow{OV} = \begin{pmatrix} -9+3p \\ 1+p-2q \\ 3-p+8q \end{pmatrix}$

Given $2UW = WV$ (or equivalently $\dfrac{UW}{WV} = \dfrac{1}{2}$). We assume that U, W, V are collinear.



$\overrightarrow{OW} = \dfrac{1}{3}\left(\overrightarrow{OV} + 2\overrightarrow{OU}\right)$

$= \dfrac{1}{3}\left[\begin{pmatrix} -9+3p \\ 1+p-2q \\ 3-p+8q \end{pmatrix} + 2 \begin{pmatrix} 3+6p \\ 1+4p+q \\ 6+2p-4q \end{pmatrix} \right]$

$= \dfrac{1}{3} \begin{pmatrix} -3+15p \\ 3+9p \\ 15+3p \end{pmatrix} = \begin{pmatrix} -1+5p \\ 1+3p \\ 5+1p \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} + p \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$

$\therefore$ Path of W is a line l with equation $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} + p \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$, $p \in \mathbb{R}$

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<u>Method 1: Use the scalar product form of F₁</u>
Since point A is on F_1 and on the line $\mathbf{r} = s\mathbf{k} = s\begin{pmatrix}0\\0\\1\end{pmatrix}$,
we have $s\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=1$.
Therefore, $s = 1$ and $\overrightarrow{OA}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

<u>Method 2: Use the parametric form of F₁</u>
Since point A is on F_1 and on the line $\mathbf{r} = s\mathbf{k} = s\begin{pmatrix}0\\0\\1\end{pmatrix}$,we have
$\mathbf{s}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}3\\1\\6\end{pmatrix}+p\begin{pmatrix}6\\4\\2\end{pmatrix}+q\begin{pmatrix}0\\1\\-4\end{pmatrix}---(*)$
$3+6p=0$ and $1+4p+q=0$
$p=-\frac{1}{2}$ and $1+4(-\frac{1}{2})+q=0$
$q=1$
Sub back (*), $\overrightarrow{OA}=\begin{pmatrix}3\\1\\6\end{pmatrix}-\frac{1}{2}\begin{pmatrix}6\\4\\2\end{pmatrix}+\begin{pmatrix}0\\1\\-4\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$
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<u>For the perpendicular distance between A and line l</u>
<u>Method 1: Use length of projection formula</u>
Distance of A to $l=\dfrac{\left|\left(\begin{pmatrix}-1\\1\\5\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\times\begin{pmatrix}5\\3\\1\end{pmatrix}\right|}{\sqrt{25+9+1}}=\dfrac{\left|\begin{pmatrix}-1\\1\\4\end{pmatrix}\times\begin{pmatrix}5\\3\\1\end{pmatrix}\right|}{\sqrt{35}}$
$=\dfrac{\left|\begin{pmatrix}-11\\21\\-8\end{pmatrix}\right|}{\sqrt{35}}=\dfrac{\sqrt{626}}{\sqrt{35}}=4.23\text{ (to 3 s.f.)}$
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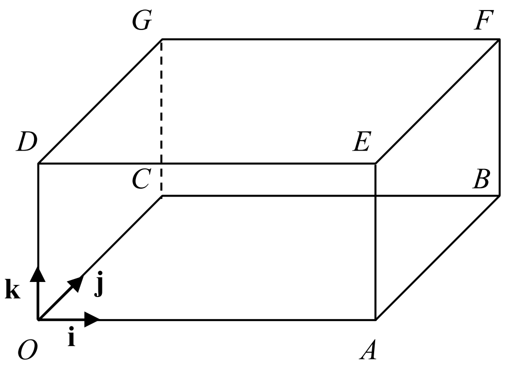
	Method 2: Find the distance between *A* and its foot of perpendicular to *l* (*not recommended*) Let P be the foot of perpendicular from A to l $\overrightarrow{OP}=\begin{pmatrix}-1\\1\\5\end{pmatrix}+p\begin{pmatrix}5\\3\\1\end{pmatrix}$, for some $p\in\mathbb{R}$, and $\overrightarrow{AP}=\begin{pmatrix}-1\\1\\5\end{pmatrix}+p\begin{pmatrix}5\\3\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\1\\4\end{pmatrix}+p\begin{pmatrix}5\\3\\1\end{pmatrix}$ Since $\overrightarrow{AP}$ is perpendicular to $\begin{pmatrix}5\\3\\1\end{pmatrix}$, $\left[\begin{pmatrix}-1\\1\\4\end{pmatrix}+p\begin{pmatrix}5\\3\\1\end{pmatrix}\right]\bullet\begin{pmatrix}5\\3\\1\end{pmatrix}=0$ $2+(25+9+1)p=0$ $p=-\frac{2}{35}$ $\overrightarrow{AP}=\begin{pmatrix}-1\\1\\4\end{pmatrix}-\frac{2}{35}\begin{pmatrix}5\\3\\1\end{pmatrix}=\frac{1}{35}\begin{pmatrix}-45\\29\\138\end{pmatrix}$ Distance of A to $l=\left\|\overrightarrow{AP}\right\|=\frac{\sqrt{(-45)^2+(29)^2+(138)^2}}{35}\approx 4.23$

Source of Question: NYJC CT2 9758/2023/02/Q3

4 A refinery installs pipelines to transport natural gas from one part of the refinery to another. In one section of the refinery, there is a rectangular frame structure as shown in the diagram below. The point O is at one corner of the structure and is taken as the origin for the position vectors. Perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OD respectively. The lengths of OA, OC and OD are 15 units, 10 units and 4 units respectively.

The first pipe enters the lower corner of the structure at O and exits at the diametrically opposed corner F. The second pipe enters at the lower corner A and exits at the opposite lower corner C. The pipes are laid in straight lines and the widths of the pipes can be neglected.



(a) Find the acute angle between the two pipes. [3]

Engineers at the refinery have determined a need to install a support strut between the two pipes to reduce damaging vibrations. To minimise cost, the plan is to install this strut by connecting the closest points between these two skew pipes.

(b) Find a vector perpendicular to both the pipes. It is also known that point P is on the first pipe and point Q is on the second pipe such that PQ is perpendicular to both pipes. Hence, by finding the position vectors of points P and Q, determine the length of the strut in order to minimise cost. [4]

(c) A solar panel $-2x+3y+\beta z=k\beta$, where k and β are constants, is to be installed along DF. For the solar panel to get their best performance in producing the maximum energy, the tilt of the solar panel should be at an angle of 60° with respect to the roof $DEFG$ of the structure. Find the exact values of k and β. [3]

Solution:

4(a)	$\overrightarrow{OA}=\begin{pmatrix}15\\0\\0\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}0\\10\\0\end{pmatrix}$, $\overrightarrow{OF}=\begin{pmatrix}15\\10\\4\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}0\\10\\0\end{pmatrix}-\begin{pmatrix}15\\0\\0\end{pmatrix}=\begin{pmatrix}-15\\10\\0\end{pmatrix}=5\begin{pmatrix}-3\\2\\0\end{pmatrix}$ Line OF : $\mathbf{r}=\lambda\begin{pmatrix}15\\10\\4\end{pmatrix},\lambda\in\mathbb{R}$ Line AC : $\mathbf{r}=\begin{pmatrix}15\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-3\\2\\0\end{pmatrix},\lambda\in\mathbb{R}$ Let the acute angle between line *OF* and line *AC* be θ . $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}15\\10\\4\end{pmatrix}\bullet\begin{pmatrix}-3\\2\\0\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}15\\10\\4\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}-3\\2\\0\end{pmatrix}\right\rvert}=\dfrac{25}{\sqrt{341}\sqrt{13}}$ $\theta=67.9^\circ$ (1 dp)
(b)	$\begin{pmatrix}15\\10\\4\end{pmatrix}\times\begin{pmatrix}-3\\2\\0\end{pmatrix}=\begin{pmatrix}-8\\-12\\60\end{pmatrix}=-4\begin{pmatrix}2\\3\\-15\end{pmatrix}$ A vector perpendicular to both pipes is $\begin{pmatrix}2\\3\\-15\end{pmatrix}$. $\overrightarrow{OP}=\lambda\begin{pmatrix}15\\10\\4\end{pmatrix}$, $\overrightarrow{OQ}=\begin{pmatrix}15\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-3\\2\\0\end{pmatrix}$ for some $\lambda,\mu\in\mathbb{R}$ $\overrightarrow{PQ}=\begin{pmatrix}15\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-3\\2\\0\end{pmatrix}-\lambda\begin{pmatrix}15\\10\\4\end{pmatrix}$ Since $\overrightarrow{PQ}$ is parallel to $\begin{pmatrix}2\\3\\-15\end{pmatrix}$, $m\begin{pmatrix}2\\3\\-15\end{pmatrix}=\begin{pmatrix}15\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-3\\2\\0\end{pmatrix}-\lambda\begin{pmatrix}15\\10\\4\end{pmatrix}$, for some $m\in\mathbb{R}$ $2m=15-3\mu-15\lambda\Rightarrow 15\lambda+3\mu+2m=15$ $3m=2\mu-10\lambda\Rightarrow 10\lambda-2\mu+3m=0$ $-15m=-4\lambda\Rightarrow 4\lambda-15m=0$ From GC,

	$\lambda=\dfrac{225}{476},\mu=\dfrac{1215}{476},m=\dfrac{15}{119}$ Length of struts $=\left\|\dfrac{15}{119}\begin{pmatrix}2\\3\\-15\end{pmatrix}\right\|=\dfrac{15}{119}\sqrt{238}=1.9446=1.94$ (to 3 s.f.)
(c)	Since $D(0,0,4)$ lies on the solar panel, $-2(0)+3(0)+\beta(4)=k\beta$ Hence, $k=4$. $\cos 60^\circ=\left\|\dfrac{\begin{pmatrix}0\\0\\1\end{pmatrix}\bullet\begin{pmatrix}-2\\3\\\beta\end{pmatrix}}{\sqrt{1}\sqrt{13+\beta^2}}\right\|$ $\dfrac{1}{2}=\left\|\dfrac{\beta}{\sqrt{13+\beta^2}}\right\|$ $\beta^2+13=4\beta^2$ $3\beta^2=13$ $\beta=\pm\sqrt{\dfrac{13}{3}}$

Source of Question: HCI CT2 9758/2023/01/Q6

5 The line l has vector equation

$$\mathbf{r}=\begin{pmatrix} b \\ -1 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ a \\ -5 \end{pmatrix},$$

where λ is a real parameter and a , b are real constants.

The equation of plane π_1 is given as $3x-y+2z=7$.

(a) Given that the distance between l and π_1 is non-zero, find the value of a, and the set of values of b. [4]

For the rest of the question, let a be the value found in part **(a)**, and $b=2$.

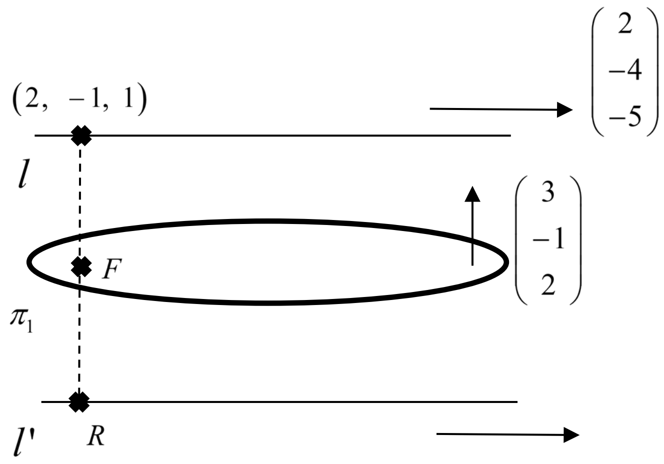
(b) Find the vector equation of the line l', where l' is the reflection of l in π_1. [4]

The equation of plane π_2 is given as $-3x+y-2z=2$.

(c) Given that the ratio of the distance between π_1 and π_2 to the distance between π_2 and plane π_3 is $2:3$, find the possible cartesian equations of π_3. [4]

Solution:

5(a)	Distance between l and π_1 is non-zero $\Rightarrow$ l and π_1 are parallel to each other, and l does not lie on π_1. $\pi_1:\mathbf{r}\cdot\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}=7$ l and π_1 are parallel to each other $\Rightarrow$ l is perpendicular to the normal of π_1 $\begin{pmatrix} 2 \\ a \\ -5 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}=0$ $6-a-10=0$ $a=-4$ l does not lie on π_1 $\Rightarrow$ the point $(b,\ -1,\ 1)$ on l does not lie on π_1

	$\begin{pmatrix} b \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \neq 7$ $3b+1+2 \neq 7$ $b \neq \frac{4}{3}$ The set of values of b is $\left\{ b \in \mathbb{R} \middle| b \neq \frac{4}{3} \right\}$
(b)	$a=-4$ implies that l and π_1 are parallel to each other  The direction vector of l' is parallel to l, i.e. a direction vector of l' is $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$. Let F be the foot of perpendicular of point $(2,\ -1,\ 1)$ to π_1 $\overrightarrow{OF} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ for some value of μ Since F is on π_1, $\overrightarrow{OF} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 7 \Rightarrow \begin{pmatrix} 2+3\mu \\ -1-\mu \\ 1+2\mu \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 7$ $\Rightarrow 6+9\mu+1+\mu+2+4\mu=7$ $\Rightarrow 14\mu=-2$ $\Rightarrow \mu=-\frac{1}{7}$ Therefore $\overrightarrow{OF} = \begin{pmatrix} 2+3\left(-\frac{1}{7}\right) \\ -1-\left(-\frac{1}{7}\right) \\ 1+2\left(-\frac{1}{7}\right) \end{pmatrix} = \begin{pmatrix} \frac{11}{7} \\ -\frac{6}{7} \\ \frac{5}{7} \end{pmatrix}$

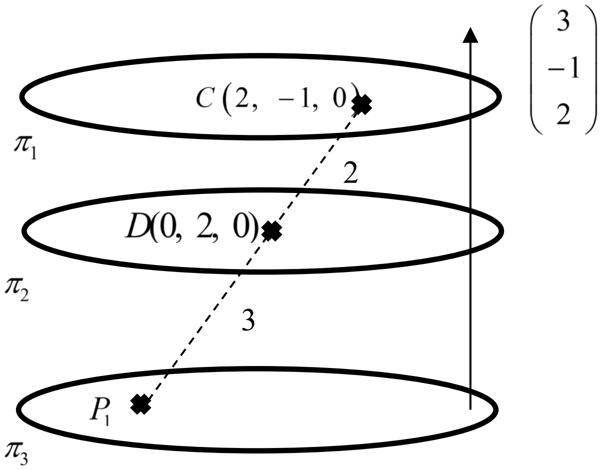
Let R be the reflection of $(2,\ -1,\ 1)$ about π_1

Using ratio theorem,

$$\overrightarrow{OF}=\frac{\overrightarrow{OR}+\begin{pmatrix}2\\-1\\1\end{pmatrix}}{2}\Rightarrow \overrightarrow{OR}=2\begin{pmatrix}\frac{11}{7}\\-\frac{6}{7}\\\frac{5}{7}\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}\frac{8}{7}\\-\frac{5}{7}\\\frac{3}{7}\end{pmatrix}$$

Equation of line l': $\mathbf{r}=\begin{pmatrix}\frac{8}{7}\\-\frac{5}{7}\\\frac{3}{7}\end{pmatrix}+\gamma\begin{pmatrix}2\\-4\\-5\end{pmatrix}$, $\gamma\in\mathbb{R}$

(c) Since there is a non-zero distance between π_2 and plane π_3, π_3 is parallel to π_2 (and π_1).



Hence a normal to π_3 is $\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$C(2,\ -1,\ 0)$ is on π_1,

and $D(0,\ 2,\ 0)$ is on π_2

<u>Case 1: π_2 is between π_1 and π_3</u>

Let P_1 be a point on π_3 such that

$CD:DP_1=2:3$

Using ratio theorem,

$$\begin{pmatrix}0\\2\\0\end{pmatrix}=\frac{2\overrightarrow{OP_1}+3\begin{pmatrix}2\\-1\\0\end{pmatrix}}{5}$$

$$\Rightarrow 2\overrightarrow{OP_1}=5\begin{pmatrix}0\\2\\0\end{pmatrix}-3\begin{pmatrix}2\\-1\\0\end{pmatrix}\Rightarrow \overrightarrow{OP_1}=\begin{pmatrix}-3\\\frac{13}{2}\\0\end{pmatrix}$$

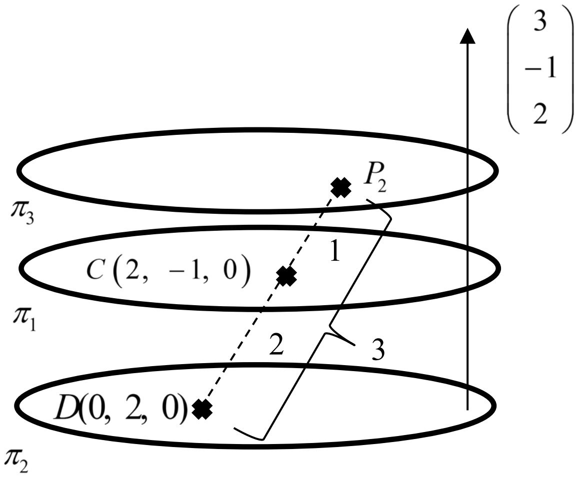
Equation of π_3: $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}-3\\\frac{13}{2}\\0\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=-9-\frac{13}{2}+0=-\frac{31}{2}$

Cartesian equation of π_3: $3x-y+2z=-\frac{31}{2}$

<u>Case 2: π_1 is between π_2 and π_3</u>

Let P_2 be a point on π_3 such that

$CD:CP_2=2:1$

	Using ratio theorem, $\begin{pmatrix}2\\-1\\0\end{pmatrix}=\dfrac{2\overrightarrow{OP_2}+\begin{pmatrix}0\\2\\0\end{pmatrix}}{3}$ $\Rightarrow 2\overrightarrow{OP_2}=3\begin{pmatrix}2\\-1\\0\end{pmatrix}-\begin{pmatrix}0\\2\\0\end{pmatrix}$ $\Rightarrow \overrightarrow{OP_2}=\begin{pmatrix}3\\-\frac{5}{2}\\0\end{pmatrix}$  Equation of π_3: $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}3\\-\frac{5}{2}\\0\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=9+\dfrac{5}{2}+0=\dfrac{23}{2}$ Cartesian equation of π_3: $3x-y+2z=\dfrac{23}{2}$

Source of Question: DHS CT2 9758/2023/01/Q11

6 Some computer games involve the locating of targets and planning actions on what to do with these targets.

In a particular computer-simulated scenario, a steep slope is modelled by the equation

$$p\colon\ \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix},$$

where λ and μ are real numbers.

(a) Express p in the form $\mathbf{r}.\begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = c$, where the values of a, b, and c are to be determined. [2]

(b) In this game, a first target is located on the slope such that it has the shortest distance to the point with coordinates (3, 8, 6). To destroy the target, a missile is set with a flight path given by the equation $l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} + \alpha \begin{pmatrix} h \\ s \\ t \end{pmatrix}$, where h, s and t are real values with s and t being positive, and α is a real parameter. (By setting the values of h, s and t, the direction of travel of the missile can be altered.)

(i) Determine the coordinates of the first target on the slope. [2]

(ii) Explain why it is not possible for the missile to approach the first target in the direction normal to the surface of the slope when h is set to be $s-t$. [2]

(iii) State a set of values of h, s and t such that makes it is possible for the missile to approach the first target in the direction normal to the surface of the slope. [1]

(c) A second target is located at position (3, 1, 0) on the slope. A surveillance drone is sent to capture close up pictures of the target. It is set to fly along the flight path given by equation $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ m \\ 1 \end{pmatrix}$, where m is a constant, $m \neq 1$, and β is a real parameter. In order to capture the best view of the target, the drone should ideally get as close to the target as possible.

Given that the drone flies at an angle $60°$ to the slope, determine the coordinates of the position of the drone where it should stop to capture the best view of the target. [5]

Solution:

6(a)	$p:\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}+\mu\begin{pmatrix}4\\1\\2\end{pmatrix}$ $\begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}2\\-4\\-2\end{pmatrix}=2\begin{pmatrix}1\\-2\\-1\end{pmatrix}$ $p:\mathbf{r}.\begin{pmatrix}1\\-2\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}.\begin{pmatrix}1\\-2\\-1\end{pmatrix}=1$ $a=-2,\ b=-1, c=1$
(b)(i)	Since 1st target is the point on the slope nearest to (3,8,6), 1st target is the foot of perpendicular from (3,8,6) to slope. $\mathbf{r}=\begin{pmatrix}3\\8\\6\end{pmatrix}+\alpha\begin{pmatrix}1\\-2\\-1\end{pmatrix}=\begin{pmatrix}3+\alpha\\8-2\alpha\\6-\alpha\end{pmatrix}$ $\begin{pmatrix}3+\alpha\\8-2\alpha\\6-\alpha\end{pmatrix}.\begin{pmatrix}1\\-2\\-1\end{pmatrix}=1$ $3+\alpha-2(8-2\alpha)-(6-\alpha)=1$ $\alpha=\frac{10}{3}$ $\mathbf{r}=\begin{pmatrix}3\\8\\6\end{pmatrix}+\frac{10}{3}\begin{pmatrix}1\\-2\\-1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}19\\4\\8\end{pmatrix}$ Coordinates of first target is $\left(\frac{19}{3}, \frac{4}{3}, \frac{8}{3}\right)$.
(ii)	When $h=s-t$, direction of the path is $\mathbf{d}=\begin{pmatrix}s-t\\s\\t\end{pmatrix}$ For the missile to have the shortest possible flight distance, it must take a perpendicular path to the point on the slope, ie, $\mathbf{d}\ //\ \mathbf{n}$, $\begin{pmatrix}1\\-2\\-1\end{pmatrix}=k\begin{pmatrix}s-t\\s\\t\end{pmatrix}$ for some real *k*.

	Equating the x, y and z components, $1 = k\left(s-t\right)\cdots(1)$ $-2 = ks\cdots(2)$ $-1 = kt\cdots(3)$ Solving (2) and (3), $ks - kt = -1$ RHS of (1) = $ks - kt = -1$ $\neq$ LHS of (1) Equations (1), (2) and (3) has no solution. Hence, it is not possible for the missile to hit the target in the shortest possible flight distance when h is set to be $s-t$.
(iii)	$\begin{pmatrix} h \\ s \\ t \end{pmatrix} // \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ with s and t positive. Set of values is $\{h, s, t \in \mathbb{R} : h = -n, s = 2n, t = n \text{ where } n \in \mathbb{R}\}$
(c)	$\sin 60^\circ = \dfrac{\left\lvert \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} . \begin{pmatrix} 0 \\ m \\ 1 \end{pmatrix} \right\rvert}{\sqrt{6}\sqrt{m^2+1}} = \dfrac{\lvert -2m-1 \rvert}{\sqrt{6}\sqrt{m^2+1}}$ $\dfrac{\sqrt{3}}{2} = \dfrac{\lvert 2m+1 \rvert}{\sqrt{6}\sqrt{m^2+1}}$ $\dfrac{3}{4} = \dfrac{4m^2+4m+1}{6\left(m^2+1\right)}$ $m^2 - 8m + 7 = 0$ $\left(m-1\right)\left(m-7\right) = 0$ $m = 7 \quad \left(\text{Since given m} \neq 1\right)$ Hence, $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$. $\left[\left[\begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}\right] - \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}\right] \bullet \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = 0$ $7(7+7\beta) + 6 + \beta = 0$ $55 + 50\beta = 0$ $\beta = -1.1$ $\mathbf{r} = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} - 1.1 \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0.3 \\ 4.9 \end{pmatrix}$ $\therefore$ Drone should stop at (3, 0.3, 4.9)