

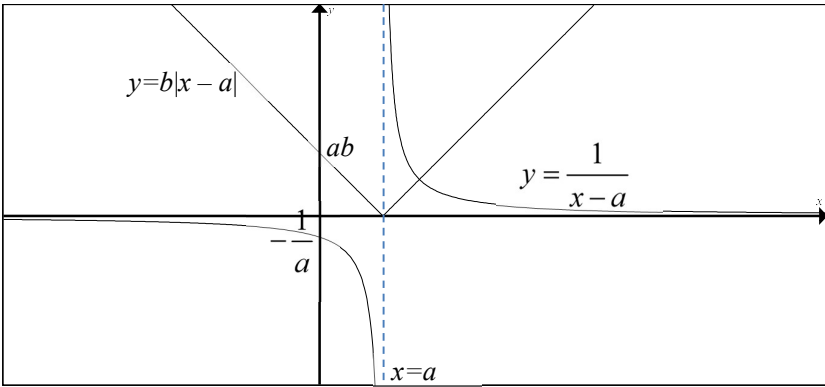


Raffles Institution
H2 Mathematics (9758)
Solution for 2017 A-Level Paper 1

Question 1

No.	Suggested Solution	Remarks for Student
	$e^{2x} \ln(1+ax)$ $= \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots \right) \left(ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \dots \right), \quad -1 < ax \leq 1$ $= ax - \frac{(ax)^2}{2} + 2ax^2 + \frac{(ax)^3}{3} - x(ax)^2 + \frac{ax(2x)^2}{2} + \dots$ $= ax + \left(\frac{4a - a^2}{2} \right) x^2 + \left(\frac{a^3 - 3a^2 + 6a}{3} \right) x^3 + \dots$ No term in x^2, we have $4a - a^2 = 0 \Rightarrow a = 0 \text{ (rejected since } a \neq 0 \text{) or } a = 4.$	

Question 2

No.	Suggested Solution	Remarks for Student
(i)		
(ii)	To solve $\frac{1}{x-a} < b x-a$, we consider the graphs $y = \frac{1}{x-a}$ and $y = b x-a$ drawn in (i), noting they intersect at a point where $x > a$. $\frac{1}{x-a} = b(x-a), \quad x > a.$ $(x-a)^2 = \frac{1}{b} \Rightarrow x = a + \sqrt{\frac{1}{b}}, \left(\text{reject } x = a - \sqrt{\frac{1}{b}} \text{ since } x > a \right)$ $\frac{1}{x-a} < b x-a $ $x > a + \sqrt{\frac{1}{b}} \text{ or } x < a$	You need to refer to (i) if you are using “Hence”.

Question 3

No.	Suggested Solution	Remarks for Student
(i)	$y^2 - 2xy + 5x^2 - 10 = 0 \dots (1)$ Differentiate (1) with respect to x: $2y \frac{dy}{dx} - 2x \frac{dy}{dx} - 2y + 10x = 0$ $y \frac{dy}{dx} - x \frac{dy}{dx} - y + 5x = 0 \dots (2)$ When $\frac{dy}{dx} = 0$, $y = 5x$ Sub into (1): $25x^2 - 10x^2 + 5x^2 - 10 = 0 \Rightarrow 20x^2 = 10. \therefore x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$	
(ii)	Differentiate (2) with respect to x: $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - x \frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{dy}{dx} + 5 = 0 \dots (3)$ When $x = \frac{1}{\sqrt{2}}$, $y = \frac{5}{\sqrt{2}}$ and $\frac{dy}{dx} = 0$. Sub into (3): $\frac{5}{\sqrt{2}} \frac{d^2y}{dx^2} - \frac{1}{\sqrt{2}} \frac{d^2y}{dx^2} + 5 = 0 \Rightarrow \frac{d^2y}{dx^2} = -\frac{5\sqrt{2}}{4} < 0$ Thus, $\left(\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ is a maximum turning point.	Use 2nd Derivative Test here. 1st derivative test cannot be used as $\frac{dy}{dx}$ depends on both x and y. Note that exact values with working are required as no calculator is allowed for this question.

Question 4

No.	Suggested Solution	Remarks for Student
(i)	$y = \frac{4x+9}{x+2} = \frac{4(x+2)+1}{x+2} = 4 + \frac{1}{x+2}$ Note that C is defined on all values of x except -2. $\frac{dy}{dx} = -\frac{1}{(x+2)^2} < 0 \text{ for all } x \neq -2$ Thus, gradient of C is negative for all points on C.	
(ii)	$y = 4 + \frac{1}{x+2}$ Asymptotes are $x = -2, y = 4$	
(iii)	A translation of 2 units in the positive x direction follow by a translation of 4 units in the negative y direction.	Replace x by $(x - 2)$ follow by replacing y by $(y + 4)$ is not acceptable.

Question 5

No.	Suggested Solution	Remarks for Student
(i)	$f(x) = x^3 + ax^2 + bx + c$ $f(1) = 8 \Rightarrow 1 + a + b + c = 8 \Rightarrow a + b + c = 7 \quad \dots(1)$ $f(2) = 12 \Rightarrow 8 + 4a + 2b + c = 12 \Rightarrow 4a + 2b + c = 4 \quad \dots(2)$ $f(3) = 25 \Rightarrow 27 + 9a + 3b + c = 25 \Rightarrow 9a + 3b + c = -2 \quad \dots(3)$ Solving (1), (2) and (3), $a = -\frac{3}{2}, b = \frac{3}{2}, c = 7$	
(ii)	$f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7$ We know that in general a cubic function has 1, 2 or 3 x-intercepts. $f'(x) = 3x^2 - 3x + \frac{3}{2}$ $= 3\left(x^2 - x\right) + \frac{3}{2}$ $= 3\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) + \frac{3}{2}$ $= 3\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \text{ for all } x \text{ since } \left(x - \frac{1}{2}\right)^2 \geq 0 \text{ for all } x$ $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. Since gradient of curve is always positive, curve is always increasing. (in particular, f is 1-1) So, $f(x) = 0$ has only one root. Using GC, the root is -1.33 (3 s.f.)	
(iii)	$f'(x) = 2 \Rightarrow 3\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = 2$ $\Rightarrow \left(x - \frac{1}{2}\right)^2 = \frac{5}{12}$ $x = \frac{1}{2} \pm \sqrt{\frac{5}{12}}$	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	It is a line passing through the point A with position vector $\underline{a}$ and is parallel to the vector $\underline{b}$.	
(ii)	It is a plane with normal vector $\underline{n}$ such that the dot product of the position vector of any point on the plane with the normal has the value d . Since $\underline{n}$ is a unit vector, the magnitude of d is the distance from the origin to the plane.	
(iii)	$(\underline{a} + t\underline{b}) \cdot \underline{n} = d \Rightarrow \underline{a} \cdot \underline{n} + t\underline{b} \cdot \underline{n} = d$ $\because \underline{b} \cdot \underline{n} \neq 0, \quad t = \frac{d - \underline{a} \cdot \underline{n}}{\underline{b} \cdot \underline{n}}$ $\therefore \underline{r} = \underline{a} + \left(\frac{d - \underline{a} \cdot \underline{n}}{\underline{b} \cdot \underline{n}} \right) \underline{b}$ which is the position vector of the point of intersection of the line in (i) and the plane in (ii).	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	$\int \sin 2mx \sin 2nx \, dx$ $= -\frac{1}{2} \int -2 \sin 2mx \sin 2nx \, dx$ $= -\frac{1}{2} \int \cos(2m+2n)x - \cos(2m-2n)x \, dx$ $= -\frac{1}{4m+4n} \sin(2m+2n)x + \frac{1}{4m-4n} \sin(2m-2n)x + C$	
(ii)	$\int_0^\pi (f(x))^2 \, dx$ $= \int_0^\pi (\sin 2mx + \sin 2nx)^2 \, dx$ $= \int_0^\pi (\sin^2 2mx + \sin^2 2nx + 2 \sin 2mx \sin 2nx) \, dx$ $= \int_0^\pi \left(\frac{1 - \cos 4mx}{2} + \frac{1 - \cos 4nx}{2} + 2 \sin 2mx \sin 2nx \right) \, dx$ $= \left[\frac{1}{2}x - \frac{1}{8m} \sin 4mx \right]_0^\pi + \left[\frac{1}{2}x - \frac{1}{8n} \sin 4nx \right]_0^\pi$ $+ \left[-\frac{1}{2m+2n} \sin(2m+2n)x + \frac{1}{2m-2n} \sin(2m-2n)x \right]_0^\pi$ $= \pi \quad (\because \sin k\pi = 0 \text{ for } k \in \mathbb{Z})$	

Question 8

No.	Suggested Solution	Remarks for Student
(a)	$z^2(1-i) - 2z + (5+5i) = 0$ $z = \frac{2 \pm \sqrt{2^2 - 4(1-i)5(1+i)}}{2(1-i)}$ $z = \frac{2 \pm \sqrt{4-40}}{2(1-i)}$ $z = \frac{2 \pm 6i}{2(1-i)}$ $z = \frac{2+6i}{2(1-i)} \times \frac{1+i}{1+i} \quad \text{or} \quad z = \frac{2-6i}{2(1-i)} \times \frac{1+i}{1+i}$ $z = \frac{2+2i+6i-6}{4} \quad \text{or} \quad z = \frac{2+2i-6i+6}{4}$ $z = -1+2i \quad \text{or} \quad z = 2-i$	You can also divide through by $(1-i)$ before applying the quadratic formula Note that detailed working is required as calculator is not allowed.
(b) (i)	$w = 1-i$ $w^2 = (1-i)^2 = -2i$ $w^3 = -2i(1-i) = -2-2i$ $w^4 = (w^2)^2 = (-2i)^2 = -4$ $w^4 + pw^3 + 39w^2 + qw + 58 = 0$ $-4 + p(-2-2i) + 39(-2i) + q(1-i) + 58 = 0$ Compare Real and Imaginary parts: $-4 - 2p + q + 58 = 0 \quad \dots(1)$ $-2p - 78 - q = 0 \quad \dots(2)$ $(1) + (2): \quad -4p - 24 = 0 \Rightarrow p = -6$ $\text{Sub into (2): } 12 - 78 - q = 0 \Rightarrow q = -66$	
(ii)	$w^4 - 6w^3 + 39w^2 - 66w + 58 = 0$ Note that $(1-i)$ is a root and coefficients are real, thus $(1+i)$ is also a root One quadratic factor is $(w-(1-i))(w-(1+i)) = w^2 - 2w + 2$ $\therefore w^4 - 6w^3 + 39w^2 - 66w + 58 = (w^2 - 2w + 2)(w^2 + pw + q)$ By comparing constant terms, $58 = 2q \Rightarrow q = 29$ By comparing linear term, $-66 = -2(29) + 2p \Rightarrow p = -4$ $\therefore w^4 - 6w^3 + 39w^2 - 66w + 58 = (w^2 - 2w + 2)(w^2 - 4w + 29)$	

	Alternatively $ \begin{array}{r} w^2 - 2w + 2 \overline{) w^4 - 6w^3 + 39w^2 - 66w + 58} \\ \underline{-(w^4 - 2w^3 + 2w^2)} \\ -4w^3 + 37w^2 - 66w + 58 \\ \underline{-(-4w^3 + 8w^2 - 8w)} \\ 29w^2 - 58w + 58 \\ \underline{-(29w^2 - 58w + 58)} \\ 0 \end{array} $	
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Question 9

No.	Suggested Solution	Remarks for Student
(a)	$S_n = \sum_{r=1}^n u_r = An^2 + Bn$	
(i)	$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= An^2 + Bn - A(n-1)^2 - B(n-1) \\ &= A(2n-1) + B \end{aligned}$	
(ii)	$\begin{aligned} u_{10} &= 48 \Rightarrow 19A + B = 48 \\ u_{17} &= 90 \Rightarrow 33A + B = 90 \\ \text{Solving, } A &= 3, B = -9 \end{aligned}$	
(b)	$\begin{aligned} r^2(r+1)^2 - (r-1)^2 r^2 &= r^2 \left[(r+1)^2 - (r-1)^2 \right] \\ &= r^2 \left[(r+1) - (r-1) \right] \left[(r+1) + (r-1) \right] \\ &= r^2 (2)(2r) \\ &= 4r^3 \end{aligned}$ $\begin{aligned} \sum_{r=1}^n r^3 &= \frac{1}{4} \sum_{r=1}^n \left(r^2(r+1)^2 - (r-1)^2 r^2 \right) \\ &= \frac{1}{4} \left[\cancel{(1^2)(2^2)} - \cancel{(0^2)(1^2)} \right. \\ &\quad + \cancel{(2^2)(3^2)} - \cancel{(1^2)(2^2)} \\ &\quad + \cancel{(3^2)(4^2)} - \cancel{(2^2)(3^2)} \\ &\quad + \dots \\ &\quad + \cancel{(n-1)^2 n^2} - \cancel{(n-2)^2 (n-1)^2} \\ &\quad \left. + n^2 (n+1)^2 - (n-1)^2 n^2 \right] \\ &= \frac{1}{4} n^2 (n+1)^2 \end{aligned}$	
(c)	Let $a_n = \frac{x^n}{n!}$. $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \frac{x}{n+1}$ $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = \lim_{n \rightarrow \infty} \frac{ x }{n+1} = 0 < 1 \text{ for each } x.$ Therefore $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges. $\sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x$ from MF26.	

Question 10

No.	Suggested Solution	Remarks for Student
	$l_C : \vec{r} = \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_N : \vec{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}, \mu \in \mathbb{R}$	
(i)	$\lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}$ $3\lambda = 1 + 4\mu \Rightarrow 3\lambda - 4\mu = 1 \quad \dots(1)$ $\lambda = 2 + 5\mu \Rightarrow \lambda - 5\mu = 2 \quad \dots(2)$ $-2\lambda = -1 + \mu(a+1) \dots(3)$ Solving (1) and (2): By GC, $\mu = -\frac{5}{11}$ and $\lambda = -\frac{3}{11}$ $\therefore \frac{6}{11} = -1 - \frac{5}{11}(a+1) \Rightarrow a = -4.4$	
(ii)	$\overrightarrow{OR} = \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \text{ for some } \lambda \in \mathbb{R}, \text{ since } R \text{ lies on } C.$ If angle PRQ is 90°, then $\overrightarrow{RP} \cdot \overrightarrow{RQ} = 0 \Rightarrow \begin{pmatrix} 1-3\lambda \\ 2-\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 5-3\lambda \\ 7-\lambda \\ -3+2\lambda \end{pmatrix} = 0$ $(5-15\lambda-3\lambda+9\lambda^2) + (14-7\lambda-2\lambda+\lambda^2) + (3-6\lambda-2\lambda+4\lambda^2) = 0$ $14\lambda^2 - 35\lambda + 22 = 0$ Discriminant $= 35^2 - 4(14)(22) = -7 < 0 \Rightarrow$ no solution. Therefore not possible for angle PQR to be 90°.	
(iii)	Length PR is as small as possible implies PR is perpendicular to C. $\overrightarrow{RP} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1-3\lambda \\ 2-\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0$ $3-9\lambda+2-\lambda+2-4\lambda=0 \Rightarrow \lambda = \frac{1}{2}$ Coordinates of R is $\left(\frac{3}{2}, \frac{1}{2}, -1\right)$ Exact minimum length $\overrightarrow{RP} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 0^2} = \frac{1}{2}\sqrt{10}$	

Question 11

No.	Suggested Solution	Remarks for Student
(i) (a)	$\frac{dv}{dt} = c$	
(b)	$v = ct + d$ $v = 4$ when $t = 0 \Rightarrow d = 4$ $v = 29$ when $t = 2.5 \Rightarrow 29 = 2.5c + 4 \Rightarrow c = 10$ $\therefore v = 10t + 4$	
(ii)	$\frac{dv}{dt} = 10 - kv$ $\int \frac{1}{10 - kv} dv = \int dt$ $-\frac{1}{k} \ln(10 - kv) = t + b$ $10 - kv = e^{-kt - kb} = Ae^{-kt}$, where $A = e^{-kb}$ is a positive constant $v = \frac{10 - Ae^{-kt}}{k}$ $v = 0$ when $t = 0 : A = 10$ $\therefore v = \frac{10}{k}(1 - e^{-kt})$	
(iii)	$v = \frac{10}{k}(1 - e^{-kt})$ $v \rightarrow 40$ as $t \rightarrow \infty \Rightarrow 40 = \frac{10}{k} \Rightarrow k = \frac{1}{4}$ $\therefore v = 40\left(1 - e^{-\frac{1}{4}t}\right)$ When $v = 0.9 \times 40 = 36$, $36 = 40\left(1 - e^{-\frac{1}{4}t}\right) \Rightarrow e^{-\frac{1}{4}t} = 0.1 \Rightarrow t = 9.21 \text{ s.}$	