Raffles Institution
H2 Mathematics (9758)
Solution for 2021 A-Level Paper 2

Section A: Pure Mathematics

Question 1

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
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Method 1

$x^3 + 2x^2 + ax + b = 0$

Since all the coefficients of the equation are real, then the conjugate of $1+\frac{1}{2}\mathrm{i}$ is also a root of the equation. That is, $1-\frac{1}{2}\mathrm{i}$ is a root of the equation.

Let the third root be x_3.

$$x^3 + 2x^2 + ax + b = \left(x-1-\frac{1}{2}\mathrm{i}\right)\left(x-1+\frac{1}{2}\mathrm{i}\right)(x-x_3)$$

$$= \left(x^2-2x+\frac{5}{4}\right)(x-x_3)$$

Compare coefficients of

x^2: $\quad 2 = -x_3 - 2 \quad \Rightarrow \quad x_3 = -4$

x: $\quad a = 2x_3 + \frac{5}{4} \quad \Rightarrow \quad a = -\frac{27}{4}$

constants: $\quad b = -\frac{5}{4}x_3 \quad \Rightarrow \quad b = 5$

The other roots of the equations are $1-\frac{1}{2}\mathrm{i}$ and -4. The values of a and b are $-\frac{27}{4}$ and 5 respectively.

Method 2

Since $1+\frac{1}{2}\mathrm{i}$ is a root of $x^3 + 2x^2 + ax + b = 0$, then

$$\left(1+\frac{1}{2}\mathrm{i}\right)^3 + 2\left(1+\frac{1}{2}\mathrm{i}\right)^2 + a\left(1+\frac{1}{2}\mathrm{i}\right) + b = 0$$

$$\frac{1}{4}+\frac{11}{8}\mathrm{i}+2\left(\frac{3}{4}+\mathrm{i}\right)+a\left(1+\frac{1}{2}\mathrm{i}\right)+b=0$$

$$\left(\frac{7}{4}+a+b\right)+\mathrm{i}\left(\frac{27}{8}+\frac{1}{2}a\right)=0$$

Comparing

Imaginary parts: $\quad \frac{27}{8}+\frac{1}{2}a = 0 \quad \Rightarrow \quad a = -\frac{27}{4}$

Real parts: $\quad \frac{7}{4}+a+b=0 \quad \Rightarrow \quad b = 5$

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Since the question did not state that the use of GC is not allowed, we can use the GC to evaluate $\left(1+\frac{1}{2}\mathrm{i}\right)^3$ and $\left(1+\frac{1}{2}\mathrm{i}\right)^2$.

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	The equation becomes $x^3+2x^2-\frac{27}{4}x+5=0$. Using GC to solve the equation, we get the other roots of the equation to be $1-\frac{1}{2}\mathrm{i}$ and -4.	We can use the GC to solve the equation.

Question 2

<table>
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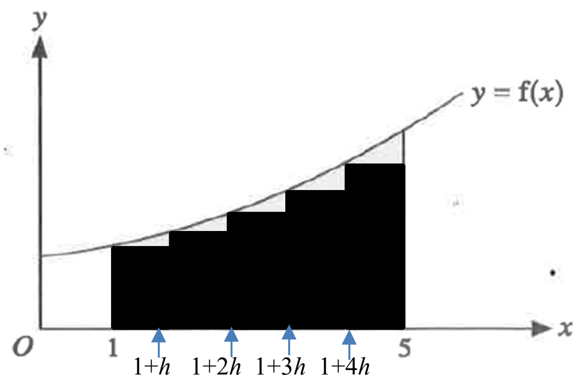
The value of $h=\frac{5-1}{5}=\frac{4}{5}$.



[Note that

$$\sum_{n=0}^{4}\left(\mathrm{f}(1+nh)\right)h = h\mathrm{f}(1)+h\mathrm{f}(1+h)+h\mathrm{f}(1+2h)$$
$$+h\mathrm{f}(1+3h)+h\mathrm{f}(1+4h)$$

$h\mathrm{f}(1)$ represents the area of left-most rectangle in the diagram above. Similarly, $h\mathrm{f}(1+h)$ represents the area of second left-most rectangle in the diagram above, and $h\mathrm{f}(1+4h)$ represents the area of right-most rectangle in the diagram above.

So, $\sum_{n=0}^{4}\left(\mathrm{f}(1+nh)\right)h$ represents the sum of the area of the 5 shaded rectangles. Clearly, the total area of the rectangles is less than the area of A.]

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<tr><td>(b)</td><td>

A similar expression is $\sum_{n=1}^{5}\left(\mathrm{f}(1+nh)\right)h$ or $\sum_{n=0}^{4}\left(\mathrm{f}(1+(n+1)h)\right)h$

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<tr><td>(c)</td><td>

$\mathrm{f}(x)=\frac{1}{20}x^2+1.$

Lower bound for area of $A=\sum_{n=0}^{4}\left(\mathrm{f}(1+nh)\right)h$

$$=\frac{4}{5}\sum_{n=0}^{4}\left(\frac{1}{20}\left(1+\frac{4}{5}n\right)^2+1\right)=5.608$$

Upper bound for area of $A=\sum_{n=1}^{5}\left(\mathrm{f}(1+nh)\right)h$

$$=\frac{4}{5}\sum_{n=1}^{5}\left(\frac{1}{20}\left(1+\frac{4}{5}n\right)^2+1\right)$$
$$=6.568$$

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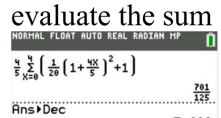
Note that you can use the GC to evaluate the sum



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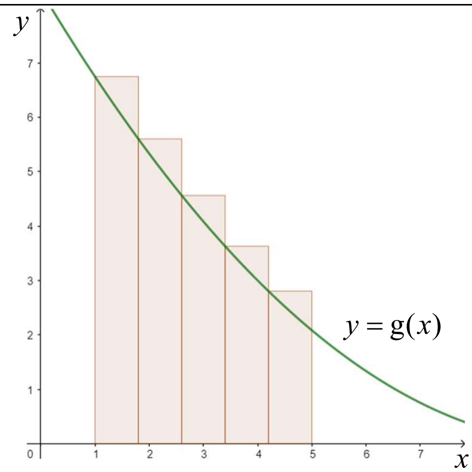
(d)	y 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 x $y = \mathrm{g}(x)$	

Question 3

No.	Suggested Solution	Remarks for Student
(a)(i)	$\mathrm{h}(2)=\frac{1}{2}(2)^2+3=5$ So, $\mathrm{gh}(2)=\mathrm{g}(5)=\frac{6}{24}=\frac{1}{4}$	
(a)(ii)	$\mathrm{g}(x)=1.4$ $\frac{x+1}{5x-1}=1.4$ $x=0.4$	
(b)(i)	$k=-\frac{b}{2}$. The value of x has to be excluded from the domain of f as $\frac{x+a}{2x+b}$ will not be defined at $x=-\frac{b}{2}$.	
(b)(ii)	Let $y=\mathrm{f}(x)$. Then, $y=\frac{x+a}{2x+b}$ $2xy+by=x+a$ $x(2y-1)=a-by$ $x=\frac{a-by}{2y-1}$ $\mathrm{f}^{-1}(x)=\frac{a-bx}{2x-1}$ So, $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\frac{x+a}{2x+b}=\frac{a-bx}{2x-1}$ Comparing, $b=-1$ and $a\in\mathbb{R}$.	Note that if $a=-\frac{1}{2}$ and $b=-1$, then $\mathrm{f}(x)=\frac{x-\frac{1}{2}}{2x-1}=\frac{1}{2},\ x\in\mathbb{R}\setminus\left\{\frac{1}{2}\right\}$ This function f is not one-one and so, f^{-1} would not exist. A more completely correct answer is , $b=-1$ and $a\in\mathbb{R}\setminus\left\{-\frac{1}{2}\right\}$. However, Cambridge says that $b=-1$ and $a\in\mathbb{R}$ is accepted as the above level of detail is not expected.
(b) (iii)	$\mathrm{f}^{-1}(-4)=\frac{a+4b}{-9}=\frac{4-a}{9}$	Alternatively, you can apply $\mathrm{f}^{-1}(-4)=\mathrm{f}(-4)$ from part **(bii)**

Question 4

No.	Suggested Solution	Remarks for Student
(a)	Total time taken $=\dfrac{n}{2}\left(u_1+u_n\right)=\dfrac{10}{2}\left(40+25\right)=325\,\text{s}$	
(b)	Let r be the common ratio in the G.P. (Alfie's programme). Then, $40=25r^9 \qquad \Rightarrow \qquad r^9=\dfrac{8}{5}$ Total time taken by Suzie $=325+25\times10+\dfrac{25\left(\left(\dfrac{8}{5}\right)^{\frac{10}{9}}-1\right)}{\left(\dfrac{8}{5}\right)^{\frac{1}{9}}-1}\approx 894.797\,\text{s}$ Average speed $=\dfrac{30\times35}{894.797}\approx1.17\,\text{m}\,\text{s}^{-1}$	
(c)	$8\,\text{min}=480\,\text{s}$ At the 8th min, Suzie is swimming at the second phase of the programme (i.e. the constant speed for each lap). The time elapsed at the constant speed laps = $480-325=155\,\text{s}$ Note that $6\times25=150\,\text{s}$ So, at the 8th minute, Suzie has completed 6 laps and is 5 s into her 7th lap of the constant speed phase. Suzie is swimming away from her starting point.	

Question 5

No.	Suggested Solution	Remarks for Student
(a)	$\int \tan^2 5x \,\mathrm{d}x = \int \sec^2 5x - 1 \,\mathrm{d}x$ $= \frac{1}{5}\tan 5x - x + C$	Remember to put the arbitrary constant "+*C*" for indefinite integral
(b)	$\int_0^b \sin 2x \sin 3x \,\mathrm{d}x = -\frac{1}{2}\int_0^b \cos 5x - \cos x \,\mathrm{d}x$ $= -\frac{1}{2}\left[\frac{1}{5}\sin 5x - \sin x\right]_0^b$ $= \frac{1}{2}\sin b - \frac{1}{10}\sin 5b$	As this is a definite integral, there should not be "+*C*"
(c)	$\int_a^b \frac{1}{x\ln x}\,\mathrm{d}x = \left[\ln\lvert\ln x\rvert\right]_a^b$ $= \ln\lvert\ln b\rvert - \ln\lvert\ln a\rvert$ $= \ln(\ln b) - \ln(\ln a)$, since $1 < a < b$	Sinec the question stated $1 < a < b$, then $\ln a > 0$ and $\ln b > 0$. You are expected to simplify your answer without the modulus.
(d)	$u = 1 + \mathrm{e}^{2x}$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^{2x} \quad \Rightarrow \quad \mathrm{d}u = 2\mathrm{e}^{2x}\,\mathrm{d}x$ $\int \frac{\mathrm{e}^{2x}}{\left(1+\mathrm{e}^{2x}\right)^3}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\left(1+\mathrm{e}^{2x}\right)^3} 2\mathrm{e}^{2x}\,\mathrm{d}x$ $= \frac{1}{2}\int \frac{1}{u^3}\,\mathrm{d}u$ $= -\frac{1}{4u^2} + C$ $= -\frac{1}{4\left(1+\mathrm{e}^{2x}\right)^2} + C$	Remember to convert back to the original variable.

Section B: Probability and Statistics

Question 6

No.	Suggested Solution	Remarks for Student
	Total probability $= 0.2+0.3+p+p+q=1$ $\Rightarrow \quad 2p+q=0.5$ $\mathrm{E}(X)=0.2+0.6+3p+4p+5q=0.8+7p+5q$ $\mathrm{E}(X^2)=0.2+1.2+9p+16p+25q=1.4+25p+25q$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$ $=1.4+25p+25q-(0.8+7p+5q)^2$ $=1.4+25p+25(0.5-2p)-(0.8+7p+5(0.5-2p))^2$ $=13.9-25p-(3.3-3p)^2=1.61$ Use GC to solve for p to give $p=0.2$ Then $q=0.1$ So, the mean score $=0.8+7(0.2)+5(0.1)=2.7$	

Question 7

No.	Suggested Solution	Remarks for Student
(a)	ABRACADABRA: 5 A, 2 B, 2 R, 1 C, 1 D Total number of different arrangements $=\dfrac{11!}{5!2!2!}=83160$	
(b)	_ BB _ RR _ CD _ The number of different arrangements $=3!\times 2!\times {}^4C_2\times 2!=144$ [BB, RR, CD can be arranged in 3! ways. 2! for CD can be reversed. Notice that we have to split the AAAA and the A since we can only have exactly 4 of the A's next to each other. So, there are ${}^4C_2\times 2!$ ways for AAAA and A to be placed.]	
(c)	The number of arrangements where all 5 A's are together $=\dfrac{7!}{2!2!}=1260$ The required probability $=\dfrac{1260}{83160}=\dfrac{1}{66}$	

Question 8

No.	Suggested Solution	Remarks for Student
(a)	Because the sales manager only wanted to test if the life span of the front tyre is greater than 20 000 miles, so 1-tail test should be used. Null hypothesis, $\mathrm{H}_0: \quad \mu = 20000$ Alternative hypothesis, $\mathrm{H}_1: \quad \mu > 20000$ where μ is the population mean life span (in miles) of the front tyres.	
(b)	$\sum(x-20) = 9.4$ $\bar{x} = 20 + \frac{9.4}{50} = 20.188$ $\frac{1}{49}\left[\sum(x-20)^2 - \frac{\left(\sum(x-20)\right)^2}{50}\right] \approx 0.754955 \simeq 0.755$ The unbiased estimate of the population mean is 20 188 miles. The unbiased estimate of the population variance is $0.755 \times 10^6 = 755000 \text{ miles}^2$	The life span *x* given in this part of the question is in **thousand** miles. Do read the question carefully.
(c)	Perform a one-tail test at 5% level of significance. Under H_0, $\bar{X} \sim \mathrm{N}\left(20000, \frac{754955}{50}\right)$ approximately by Central Limit Theorem, since $n = 50$ is large. p-value $= \mathrm{P}\left(\bar{X} > 20188\right) = 0.063 > 0.05$ Thus, we do not reject H_0 and conclude that there is insufficient evidence at 5% level of significance to say that the population mean life span of the front tyres is more than 20 000 miles.	
(d)	As the population distribution is not known, a sample size of 15 is too small for us to use the Central Limit Theorem. Thus, the test would not be appropriate.	

Question 9

No.	Suggested Solution	Remarks for Student
(a)	No, as according to the scatter diagram, the values of x increases at an increasing rate as t increases.	
(b)(i)		
(b)(ii)	Using GC, $c \approx 2.2927 \approx 2.29$ $d \approx 0.057271 \approx 0.0573$ $r \approx 0.99869 \approx 0.999$	
(c)	The second model is a better model as the points on the second scatter plot seems to lie on a straight line, while the first scatter plot does not seem so. Furthermore, the value of the product moment correlation coefficient for the second model is closer to 1 than that of the first model.	

Question 10

No.	Suggested Solution	Remarks for Student
(a)	Let S be the random variable representing the mass of a seat (in kg), with mean mass μ and variance σ^2. Then, $S \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\mathrm{P}(X < 2.1) = 0.8$ $\mathrm{P}(X < 1.95) = 0.15$ $\mathrm{P}\left(Z < \frac{2.1-\mu}{\sigma}\right) = 0.8$ $\mathrm{P}\left(Z < \frac{1.95-\mu}{\sigma}\right) = 0.15$ $\frac{2.1-\mu}{\sigma} = 0.84162$ $\frac{1.95-\mu}{\sigma} = -1.0364$ $\mu + 0.84162\sigma = 2.1$ $\mu - 1.0364\sigma = 1.95$ Solving, $\mu = 2.0327 \approx 2.03$ $\sigma = 0.079871 \approx 0.0799$	
(b)	Let L be the random variable representing the mass of a leg (in kg). Then, $L \sim \mathrm{N}\left(1.2, 0.02^2\right)$ $\mathrm{P}(L > 1.21) = 0.30854$ Expected number of legs with mass more than 1.21kg in a batch of 500 legs is $500 \times 0.30854 \approx 154.269 \approx 154$	
(c)	Let $X = S + L_1 + L_2 + L_3$. Then, $X \sim \mathrm{N}\left(2.0327 + 3\times1.2, 0.079871^2 + 3\times0.02^2\right)$ i.e. $X \sim \mathrm{N}(5.6327, 0.0075794)$ $\mathrm{P}(5.6 < X < 5.7) = 0.42664 \approx 0.427$	
(d)	Let $Y = 0.91S + L_1 + L_2 + L_3$ Then, $Y \sim \mathrm{N}\left(0.91\times2.0327 + 3\times1.2, 0.91^2\times0.079871^2 + 3\times0.02^2\right)$ i.e. $Y \sim \mathrm{N}(5.44976, 0.0064828)$ $\mathrm{P}(Y < 5.6) = 0.968977 \approx 0.969$	
(e)	Let H be the random variable representing the diameter of the hole (in mm), and D be the random variable representing the diameter of the leg (in mm). Then $H \sim \mathrm{N}\left(31, 0.4^2\right)$ and $D \sim \mathrm{N}\left(30.7, 0.3^2\right)$. $H - D \sim \mathrm{N}\left(31 - 30.7, 0.4^2 + 0.3^2\right)$ i.e. $H - D \sim \mathrm{N}\left(0.3, 0.5^2\right)$ $\mathrm{P}(0 \le H - D < 0.8) = 0.567092$ Probability that the 3 legs can be fitted without any sanding and padding $= (0.567092)^3 = 0.18237 \approx 0.182$	

Question 11

No.	Suggested Solution	Remarks for Student
(a) (i)	The light fittings should be selected randomly, so as to avoid bias.	
(ii)	Total number of faulty light fittings in the 150 working days $= 0\times4+1\times19+2\times38+3\times41+4\times22+5\times16+6\times6+7\times4$ $= 450$ Total number of light fittings checked in the 150 working days $100\times150 = 15000$ The probability that a light fitting is faulty, $p = \dfrac{450}{15000} = 0.03$	
(iii)	Let F be the random variable representing the number of faulty fittings found on each day, out of 100 fittings. $F \sim \mathrm{B}(100, 0.3)$ Expected number of days that 3 faulty fittings are found in a period of 150 working days $=150\times\mathrm{P}(F=3) = 34.121 \approx 34.1$	Cambridge accepts both answers 34 and 34.1.
(b) (i)	Assumptions: • The event of a heating element being faulty is independent of the event of any other heating element being faulty • The probability of a heating element being faulty is constant	
(ii)	$X \sim \mathrm{B}(80, 0.02)$ $\mathrm{P}(1 \le X \le 4) = \mathrm{P}(X \le 4) - \mathrm{P}(X = 0) \approx 0.779$	
(iii)	$\mathrm{P}(X > 3) = 1 - \mathrm{P}(X \le 3) \approx 0.076855$ Let D be the random variable representing the number of days, out of 5 days, that more than 3 elements are faulty. $D \sim \mathrm{B}(5, 0.076855)$ $\mathrm{P}(D \ge 2) = 1 - \mathrm{P}(D \le 1) \approx 0.0505$	
(iv)	For a 5-day working week, a total of $5\times80 = 400$ elements are tested. Let Y be the random variable representing the number of faulty elements, out of 400, that are faulty. $Y \sim \mathrm{B}(400, 0.02)$ $\mathrm{P}(Y \le 8) \approx 0.593$	