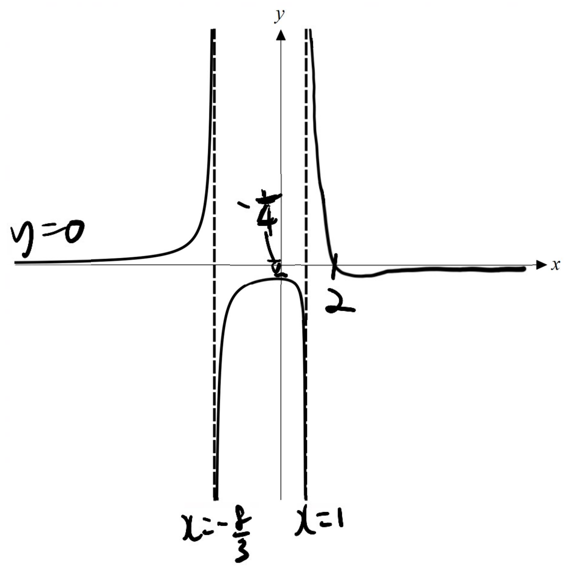
Raffles Institution
H2 Mathematics (9758)
Solution for 2019 A-Level Paper 2

Section A: Pure Mathematics

Question 1

No.	Suggested Solution	Remarks for Student
(i)	$I = -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\int (1-x)^{\frac{3}{2}}\ \mathrm{d}x$ $= -\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + c$	Don't forget to include the arbitrary constant of integration.
(ii)	$x = 1-u^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -2u$ $I = \int (1-u^2)\sqrt{u^2}\,(-2u)\ \mathrm{d}u$ $= 2\int (u^2-1)u^2\ \mathrm{d}u$ $= 2\int (u^4-u^2)\ \mathrm{d}u$ $= \frac{2}{5}u^5 - \frac{2}{3}u^3 + d$ $= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + d$	$x = 1-u^2$ $u = (1-x)^{\frac{1}{2}}$ Final answer in terms of x.
(iii)	$\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + d - \left(-\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + c\right)$ $= \frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + \frac{4}{15}(1-x)^{\frac{5}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} + d - c$ $= \frac{2}{3}(1-x)^{\frac{3}{2}}(x-1) + \frac{10}{15}(1-x)^{\frac{5}{2}} + d - c$ $= -\frac{2}{3}(1-x)^{\frac{3}{2}}(1-x) + \frac{2}{3}(1-x)^{\frac{5}{2}} + d - c$ $= -\frac{2}{3}(1-x)^{\frac{5}{2}} + \frac{2}{3}(1-x)^{\frac{5}{2}} + d - c$ $= d - c$	Show difference in answer in parts (i) and (ii) do not depend on x.

Question 2

No.	Suggested Solution	Remarks for Student
(i)	$y=\dfrac{2-x}{3x^2+5x-8}$ $=\dfrac{2-x}{(3x+8)(x-1)}$ 	
(ii)	$x<-\dfrac{8}{3}$ or $1<x<2$	
(iii)	$\dfrac{x-2}{(3x+8)(x-1)}>0$ $\dfrac{2-x}{(3x+8)(x-1)}<0$ $\therefore -\dfrac{8}{3}<x<1$ or $x>2$	

Question 3

No.	Suggested Solution	Remarks for Student
(i)	$900 = 2\pi r^2 + 2\pi rh$ $450 = \pi r^2 + \pi rh$ $h = \dfrac{450 - \pi r^2}{\pi r}$ $V = \pi r^2 h = \pi r^2 \left(\dfrac{450 - \pi r^2}{\pi r}\right) = 450r - \pi r^3$ $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 450 - 3\pi r^2 = 0 \quad \Rightarrow \pi r^2 = 150$ $\Rightarrow r = \sqrt{\dfrac{150}{\pi}} \ (r > 0)$ $\dfrac{\mathrm{d}^2V}{\mathrm{d}r^2} = -6\pi r < 0$ for all $r > 0$, that is, $r = \sqrt{\dfrac{150}{\pi}}$ gives max V. $\max V = \pi r^2 h$ $= 150\left(\dfrac{450 - \pi r^2}{\pi r}\right)$ $= 150\left(\dfrac{450 - 150}{\pi\sqrt{\dfrac{150}{\pi}}}\right)$ $= \dfrac{300\sqrt{150}}{\sqrt{\pi}}$ $h = \dfrac{450 - \pi r^2}{\pi r}$ $\dfrac{1}{h} = \dfrac{\pi r}{450 - \pi r^2}$ $\dfrac{r}{h} = \dfrac{\pi r^2}{450 - \pi r^2} = \dfrac{150}{450 - 150} = \dfrac{1}{2}$ $\therefore r : h = 1 : 2$	

Question 4

No.	Suggested Solution	Remarks for Student
(i)	$\mathrm{f}'(x) = 2\sec 2x \tan 2x$ $\mathrm{f}''(x) = 2(2\sec 2x \tan 2x)\tan 2x + 2(2\sec^2 2x)\sec 2x$ $= 4\sec 2x \tan^2 2x + 4\sec^3 2x$ $\mathrm{f}(0) = 1$ $\mathrm{f}'(0) = 0$ $\mathrm{f}''(0) = 4$ $\mathrm{f}(x) \approx 1 + 2x^2$	
(ii)	$\int_0^{0.02} (1+2x^2)\,\mathrm{d}x = 0.0200053333 \approx 0.02001$ (5 d.p.)	
(iii)	$\int_0^{0.02} \sec 2x \,\mathrm{d}x = 0.0200053355 \approx 0.02001$ (5 d.p.)	
(iv)	The approximation is good for small values of x. Part (ii) uses Maclaurin series with polynomial of degree 2 to approximate the integral with $x = 0.02$ which only differ from actual value from 9^{th} d.p. This already provides a good approximation.	
(v)	We would need g and all derivatives of g to be defined in order to apply Macluarin series. However $\operatorname{cosec} 2x = \dfrac{1}{\sin 2x}$ is undefined at $x = 0$.	

Question 5

No.	Suggested Solution	Remarks for Student
(i)	$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}$ $= \underset{\sim}{b} + \lambda\overrightarrow{BD}$ $= \underset{\sim}{b} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right)$ $= \underset{\sim}{b} + \lambda\left(\underset{\sim}{b} + 5\underset{\sim}{a} - \underset{\sim}{b}\right)$ $= \underset{\sim}{b} + 5\lambda\underset{\sim}{a}$ $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ $= \underset{\sim}{a} + \mu\overrightarrow{AC}$ $= \underset{\sim}{a} + \mu\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$ $= \underset{\sim}{a} + \mu\left(2\underset{\sim}{a} + 4\underset{\sim}{b} - \underset{\sim}{a}\right)$ $= (1+\mu)\underset{\sim}{a} + 4\mu\underset{\sim}{b}$ $\underset{\sim}{b} + 5\lambda\underset{\sim}{a} = (1+\mu)\underset{\sim}{a} + 4\mu\underset{\sim}{b}$ $4\mu = 1 \Rightarrow \mu = \frac{1}{4}$ $5\lambda = 1 + \mu = \frac{5}{4} \Rightarrow \lambda = \frac{1}{4}$ $\overrightarrow{OX} = \underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}$	Another way to understand the solution is that *X* is the point of intersection between the line *BD* and line *AC*. Equation of line *BD* is $\mathbf{r} = \overrightarrow{OB} + \lambda\overrightarrow{BD}$ Likewise, the eqn of the line *AX* is $\mathbf{r} = \overrightarrow{OA} + \mu\overrightarrow{AC}$
(ii)	$\overrightarrow{OY} = \alpha\overrightarrow{OD} + (1-\alpha)\overrightarrow{OC}$ $= \alpha\left(\underset{\sim}{b} + 5\underset{\sim}{a}\right) + (1-\alpha)\left(2\underset{\sim}{a} + 4\underset{\sim}{b}\right)$ $\overrightarrow{OY} = \beta\overrightarrow{OX}$ $= \beta\left(\underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}\right)$ $\alpha\left(\underset{\sim}{b} + 5\underset{\sim}{a}\right) + (1-\alpha)\left(2\underset{\sim}{a} + 4\underset{\sim}{b}\right) = \beta\left(\underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}\right)$ Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are nonzero nonparallel vectors, then Coeff of $\underset{\sim}{b}$: $\beta = \alpha + 4(1-\alpha) \Rightarrow \beta + 3\alpha = 4$ ...(1) Coeff of $\underset{\sim}{a}$: $\frac{5}{4}\beta = 5\alpha + 2(1-\alpha) \Rightarrow \frac{5}{4}\beta - 3\alpha = 2$ ...(2) $(1)+(2)$: $\frac{9}{4}\beta = 6$ $\therefore \beta = \frac{24}{9} = \frac{8}{3}$ $\overrightarrow{OY} = \beta\overrightarrow{OX} = \frac{24}{9}\left(\underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}\right) = \frac{8}{3}\underset{\sim}{b} + \frac{10}{3}\underset{\sim}{a}$ $OX : OY = 3 : 8$	Similar to **(i)**, *Y* is the point of intersection between the line *CD* and the line *OX* Eqn of line *CD*: $\mathbf{r} = \overrightarrow{OC} + \alpha\overrightarrow{CD}$ Eqn of line *OX*: $\mathbf{r} = \beta\overrightarrow{OX}$

Section B: Statistics

Question 6

No.	Suggested Solution	Remarks for Student
(i)	These 22 clubs form the population as they are ALL the clubs in Division One whose approaches to training she is interested to find out.	
(ii)	**How** Assuming no special treatment with regard to facilities for supporters of different divisions, he could randomly pick a certain number of clubs out of the 100, say 10, to do a thorough investigation. **Why** This is to avoid bias and it would also be more cost effective and manageable.	
(iii)	${}^{22}C_5 \times {}^{24}C_5 \times {}^{26}C_5 \times {}^{28}C_5 = 7.24 \times 10^{18}$	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	The event that one mug is faulty is independent of each other. The probability of a mug being faulty remains constant at 0.08.	
(ii)	$F \sim \text{B}(50, 0.08)$ $\text{P}(F \geq 7) = 1 - \text{P}(F \leq 6)$ $= 0.10187$ ≈ 0.102	
(iii)	Let W denote number of days out of 5 with at least 7 faulty mugs. $W \sim \text{B}(5, 0.10187)$ $\text{P}(W \leq 2) = 0.99098$ ≈ 0.991	
(iv)	${}^{10}C_2 p^2 (1-p)^8 = 45p^2(1-p)^8$	
(v)	P(no fault) + P(one fault) $= 0.92^2(1-p)^2 + \left[\text{P}(\text{fault with one mug}) + \text{P}(\text{fault with one saucer})\right]$ $= 0.8464(1-p)^2 + \left[2(0.92)(0.08)(1-p)^2 + 2(0.92^2)p(1-p)\right]$ $= 0.8464(1-p)^2 + 0.1472(1-p)^2 + 1.6928p(1-p)$ $= 0.9936(1-p)^2 + 1.6928p(1-p)$ $0.9936(1-p)^2 + 1.6928p(1-p) = 0.97$ $p = 0.0689$	

Question 8

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td></td><td>
<table>
<tr><th></th><th>Orange</th><th>Yellow</th><th>Green</th><th>White</th><th>Total</th></tr>
<tr><td>Horse</td><td>1</td><td>1</td><td>3</td><td>4</td><td>9</td></tr>
<tr><td>Rider</td><td>1</td><td>1</td><td>7</td><td>5</td><td>14</td></tr>
<tr><td>Dog</td><td>3</td><td>7</td><td>1</td><td>6</td><td>17</td></tr>
<tr><td>Bird</td><td>4</td><td>5</td><td>6</td><td>1</td><td>16</td></tr>
<tr><td>Total</td><td>9</td><td>14</td><td>17</td><td>16</td><td>56</td></tr>
</table>
</td><td>You can just create the “total” column on the question booklet itself in the A level.</td></tr>
<tr><td>(i)
(a)</td><td>$\frac{9+14}{56}=\frac{23}{56}$</td><td>Just count relevant cells from the table above</td></tr>
<tr><td>(b)</td><td>$\frac{17+16-7}{56}=\frac{13}{28}$</td><td></td></tr>
<tr><td>(ii)
(a)</td><td>$\frac{8\times 7}{56\times 55}=\frac{1}{55}$</td><td></td></tr>
<tr><td>(b)</td><td>Case 1: Dog is yellow, the other item is a non-yellow Horse/Rider/Bird
Probability $=\frac{7}{56}\times\frac{32}{55}\times 2$
Case 1: Dog is not yellow, the other item is a yellow Horse/Rider/Bird
Probability $=\frac{10}{56}\times\frac{7}{55}\times 2$
Required probability $=\frac{7}{56}\times\frac{32}{55}\times 2+\frac{10}{56}\times\frac{7}{55}\times 2=\frac{21}{110}$</td><td></td></tr>
<tr><td>(iii)</td><td>$\frac{a}{56}\times\frac{b}{55}\times 2=\frac{1}{77}$
where a,b refer to the number of his first and second favourites
$\therefore ab=20=1\times 20$ or 2×10 or 4×5
Reference to the table, only 4,5 is possible
Thus, possible combinations are as follow
<table>
<tr><th>4</th><th>5</th></tr>
<tr><td>White Horse</td><td>White Rider</td></tr>
<tr><td>White Horse</td><td>Yellow Bird</td></tr>
<tr><td>Orange Bird</td><td>White Rider</td></tr>
<tr><td>Orange Bird</td><td>Yellow Bird</td></tr>
</table>
</td><td></td></tr>
</table>

Question 9

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(i)</td><td>Since the manager wants to check “whether the mean resistance is in fact 750 ohms”, significant variation of both more or less than 750 is not acceptable. Thus, he should carry out a 2-tail test to see if the mean resistance differs from 750 ohms.

Null hypothesis, $H_0 : \mu = 750$

Alternative hypothesis, $H_1 : \mu \neq 750$

where μ is the population mean resistance of resistors rated at 750 ohms.</td><td></td></tr>
<tr><td>(ii)</td><td>Let X denote the resistance (in ohms) of a resistor rated at 750 ohms.

Using GC, $\bar{x} = 756$

Perform an 2-tailed test at 5% significance level.

Under H_0, $\bar{X} \sim N\left(750, \frac{100}{8}\right)$

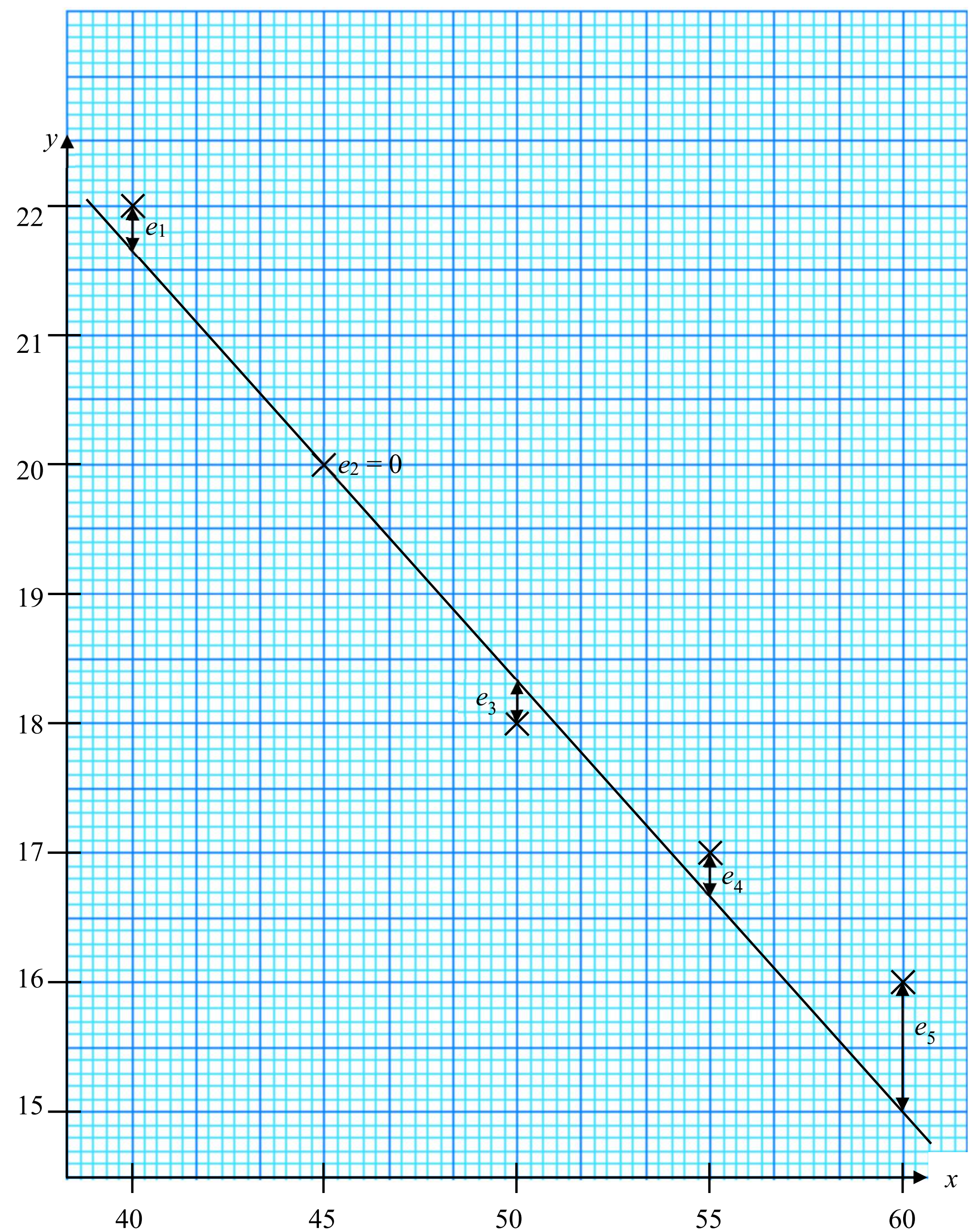
p-value $= 2P\left(\bar{X} > 756\right) = 0.0897 > 0.05$, hence we do not reject $H_0 : \mu = 750$.

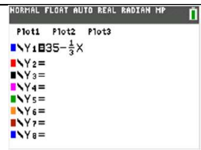
The manager does not have sufficient evidence at 5% level of significance to claim that the mean resistance is not 750 ohms.</td><td>Since X follows a Normal distribution, then so does $\bar{X}$.

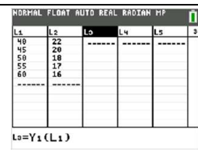
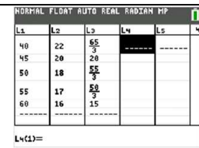
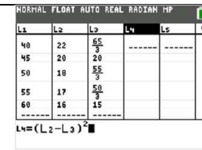
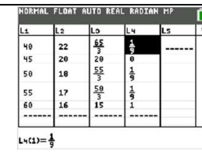
The population variance is given in the question so we use it.

Answer in context with reference to the alternative hypothesis.</td></tr>
<tr><td>(iii)</td><td>As the distribution of the resistance of the population of resistors is unknown, we need a large sample (of at least 30) so that we can use te Central Limit Theorem to approximate the probability distribution of $\bar{X}$ to follow a normal distribution.</td><td></td></tr>
</table>

Question 10

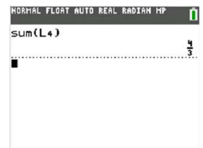
<table>
<tr><th>No.</th><th>Suggested Solution</th></tr>
<tr><td>(i)
(a)</td><td>
</td></tr>
<tr><td>(b)</td><td>The residuals are the distances labelled $e_1, \ldots, e_5$ in the diagram.</td></tr>
<tr><td>(c)</td><td>$\sum_{i=1}^{5}(e_i)^2 = \frac{1}{9}+0+\frac{1}{9}+\frac{1}{9}+1=\frac{4}{3}$</td></tr>
</table>



Key the eqn in Y_1 Define $L_3 = Y_{estimated}$ Define $L_4 = e_i^2$

Optional : Using List to calculate the Sum



Go to List (Type 2nd → stat → MATH) Sum all the entries in L_4

(d) Residuals can be positive or negative since they are calculated by taking (*the observed y-value*) – (*the estimated y-value*). Taking the sum of these residuals will result in them cancelling out and thus losing important information. Taking the sum of the squares of the residuals will avoid this problem.

(ii) Sum of the squares of the residuals for Bhani's model = $1 < \dfrac{4}{3}$ (Sum of the squares of the residuals for Abi's model), Bhani's model is a better fit.

(iii) $(\bar{x}, \bar{y}) = (50, 18.6)$

(iv) Regression line: $y = -0.3x + 33.6$

$r = -0.985$ (3 s.f.)

(v) $x = 30$, $y = 24.6$ (kilometres per litre)

$x = 30$ is outside the given range of x-values. The estimate obtained using extrapolation is unreliable.

(vi) Sum of the squares of the residuals = 0 implies that ALL data points lie on the regression line, which is highly unlikely in a real-life experiment.

Question 11

No.	Suggested Solution	Remarks for Student
(i)	Let W denote mass of a white ball in grams. Carbon Let B denote mass of a black ball in grams. Hydrogen $W \sim \mathrm{N}\left(110, 4^2\right),\ B \sim \mathrm{N}\left(55, 2^2\right)$ $W_1 + W_2 + W_3 + W_4 \sim \mathrm{N}\left(440, 64\right)$ $\mathrm{P}\left(W_1 + W_2 + W_3 + W_4 > 425\right) = 0.96960 \approx 0.970$	
(ii)	$W + B \sim \mathrm{N}\left(165, 20\right)$ $\mathrm{P}\left(161 < W + B < 175\right) = 0.80178 \approx 0.802$	
(iii)	$W_1 + W_2 + B_1 + B_2 + B_3 \sim \mathrm{N}\left(385, 44\right)$ $\mathrm{P}\left(W_1 + W_2 + B_1 + B_2 + B_3 < M\right) = 0.271$ $\therefore M = 380.955 \approx 381$	
(iv)	Let W' denote mass of a drilled white ball in grams. Let B' denote mass of a drilled black ball in grams. Let R denote mass of a rod in grams. $W' \sim \mathrm{N}\left(77, 7.84\right)$ $B' \sim \mathrm{N}\left(49.5, 3.24\right)$ $R \sim \mathrm{N}\left(20, 0.81\right)$ Let X denote mass of the model in grams $X = W' + B_1' + B_2' + B_3' + B_4' + R_1 + R_2 + R_3 + R_4$ $X \sim \mathrm{N}\left(355, 24.04\right)$ $\mathrm{P}\left(X > 350\right) = 0.84608 \approx 0.846$	