Raffles Institution
H2 Mathematics (9758)
Solution for 2022 A-Level Paper 1

Question 1

No.	Suggested Solution	Remarks for Student
	$\mathrm{i}z+2w=-1 \quad ...(1)$ $(2-\mathrm{i})z+\mathrm{i}w=6 \quad ...(2)$ $(2)\times 2\mathrm{i}: \quad 2\mathrm{i}(2-\mathrm{i})z-2w=12\mathrm{i} \quad ...(3)$ $(1)+(3): \quad \mathrm{i}z+4\mathrm{i}z+2z=-1+12\mathrm{i}$ $z=\dfrac{-1+12\mathrm{i}}{2+5\mathrm{i}}$ $z=\dfrac{(-1+12\mathrm{i})(2-5\mathrm{i})}{29}=\dfrac{58+29\mathrm{i}}{29}=2+\mathrm{i}$ Sub back into (1): $w=\dfrac{-1-\mathrm{i}z}{2}=\dfrac{-1-\mathrm{i}(2+\mathrm{i})}{2}=-\mathrm{i}$	This is a non-calculator question. Detailed working needs to be shown.

Question 2

No.	Suggested Solution	Remarks for Student
(a)	$\mathrm{f}(x)=\tan^{-1}\left(\sqrt{2}+x\right)$ $\mathrm{f}'(x)=\dfrac{1}{1+\left(\sqrt{2}+x\right)^2}=\left[1+\left(\sqrt{2}+x\right)^2\right]^{-1}$ $\mathrm{f}''(x)=-\left[1+\left(\sqrt{2}+x\right)^2\right]^{-2}\left(2\left(\sqrt{2}+x\right)\right)=\dfrac{-2\left(\sqrt{2}+x\right)}{\left[1+\left(\sqrt{2}+x\right)^2\right]^2}$	
(b)	$\mathrm{f}(0)=\tan^{-1}\sqrt{2}=0.95532$ $\mathrm{f}'(0)=\dfrac{1}{1+\left(\sqrt{2}\right)^2}=\dfrac{1}{3}=0.33333$ $\mathrm{f}''(0)=\dfrac{-2\sqrt{2}}{[1+2]^2}=\dfrac{-2\sqrt{2}}{9}=-0.31427$ $\mathrm{f}(x)=0.955+0.333x-\dfrac{0.31427}{2}x^2+...$ $=0.955+0.333x-0.157x^2+...$	Note that all calculated values are to be in radians – check that you set calculator to the correct mode. As stated in the question, note the degree of accuracy for this question is 3 s.f.

Question 3

No.	Suggested Solution	Remarks for Student
(a)	$x=\frac{1}{2}\left(e^{3t}+2e^{-3t}\right)=\frac{1}{2}e^{3t}+e^{-3t}\Rightarrow\frac{dx}{dt}=\frac{3}{2}e^{3t}-3e^{-3t}$ $y=\frac{1}{2}\left(e^{3t}-2e^{-3t}\right)=\frac{1}{2}e^{3t}-e^{-3t}\Rightarrow\frac{dy}{dt}=\frac{3}{2}e^{3t}+3e^{-3t}$ $\frac{dy}{dx}=\frac{\frac{3}{2}e^{3t}+3e^{-3t}}{\frac{3}{2}e^{3t}-3e^{-3t}}=\frac{e^{3t}+2e^{-3t}}{e^{3t}-2e^{-3t}}$ $t=\frac{1}{3}\ln 2,\ \frac{dy}{dx}=\frac{2+1}{2-1}=3$ Gradient of normal = $-\frac{1}{3}$	Recall that: $e^{\ln x}=x$ for $x>0$
(b)	$x+y=e^{3t}$ $x-y=2e^{-3t}$ $(x+y)(x-y)=e^{3t}\left(2e^{-3t}\right)$ $x^2-y^2=2\Rightarrow\frac{x^2}{\left(\sqrt{2}\right)^2}-\frac{y^2}{\left(\sqrt{2}\right)^2}=1,\ x\geq\sqrt{2}$	Since question says state the restriction for x, we can use GC to draw $x=\frac{1}{2}\left(e^{3t}+2e^{-3t}\right)$ to get the restriction on x. (x as Y_1 and t as X in GC)

Question 4

No.	Suggested Solution	Remarks for Student
(a)	$\dfrac{\mathrm{d}}{\mathrm{d}x}(\cot x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\cos x}{\sin x}\right)$ $= \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$ (using quotient rule) $= \dfrac{-1}{\sin^2 x}$ (since $\sin^2 x + \cos^2 x = 1$) $= -\operatorname{cosec}^2 x$	
(b)	Note that $\sin 2x = 2\sin x \cos x$ and $\tan x = \dfrac{\sin x}{\cos x}$, Therefore $\sin 2x \tan x = (2\sin x\cos x)\dfrac{\sin x}{\cos x} = 2\sin^2 x$	This is a "show" question, so detailed steps need to be shown.
(c)	$\displaystyle\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec} 6x \cot 3x \,\mathrm{d}x$ $\displaystyle= \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{1}{\sin 6x \tan 3x} \,\mathrm{d}x$ $\displaystyle= \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{1}{2\sin^2 3x} \,\mathrm{d}x$ (from (b)) $\displaystyle= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec}^2 3x \,\mathrm{d}x$ $= -\dfrac{1}{6}\left[\cot 3x\right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$ (from (a)) $= -\dfrac{1}{6}\left[\cot\dfrac{\pi}{3} - \cot\dfrac{\pi}{6}\right]$ $= -\dfrac{1}{6}\left[\dfrac{1}{\sqrt{3}} - \sqrt{3}\right]$ $= -\dfrac{1}{6}\left[\dfrac{1-3}{\sqrt{3}}\right]$ $= \dfrac{1}{3\sqrt{3}}$ or $\dfrac{\sqrt{3}}{9}$	Note that you are expected to simplify the answer. You can also use GC to verify whether your answer is correct.

Question 5

No.	Suggested Solution	Remarks for Student
(a)	To find the intersection between the tangent line and the curve, $(x+8)^2+(mx-14)^2=52$ $x^2+16x+64+m^2x^2-28mx+196=52$ $(m^2+1)x^2+(16-28m)x+208=0...(1)$ line is tangent to curve means (1) has one repeated root, discriminant $=(16-28m)^2-4(m^2+1)(208)=0$ $256-896m+784m^2-832m^2-832=0$ $-48m^2-896m-576=0$ $3m^2+56m+36=0$	
(b)	since tangents intersects at (0,0), equations of both tangents will be $y=m_1x$ and $y=m_2x$, where m_1 and m_2 satisfy the equation in (a), solving $3m^2+56m+36=0$ $3m^2+56m+36=0$ $m=\frac{-56\pm\sqrt{56^2-4(3)(36)}}{6}$ $=\frac{-56\pm 52}{6}$ $=-18$ or $-\frac{2}{3}$ Sub $m=-18$ into (1) in (a), $325x^2+520x+208=0$ $x=-0.8$ (from GC or equation can be reduced to $(x+0.8)^2=0$) $y=-18(-0.8)=14.4$ Sub $m=-\frac{2}{3}$ into (1) in (a), $\frac{13}{9}x^2+\frac{104}{3}x+208=0$ $x=-12$ (from GC or equation can be reduced to $(x+12)^2=0$) $y=-\frac{2}{3}(-12)=8$ Coordinates are $(-12,8)$ and $(-0.8,14.4)$.	

Question 6

No.	Suggested Solution	Remarks for Student
(a)	$\mathrm{f}(x)=\dfrac{ax+k}{x-a}=\dfrac{a(x-a)+a^2+k}{x-a}=a+\dfrac{a^2+k}{x-a}$ $y=\dfrac{1}{x}\xrightarrow{\text{replace } x \text{ by } x-a} y=\dfrac{1}{x-a}\xrightarrow{\text{replace } y \text{ by } \frac{y}{a^2+k}} y=\dfrac{a^2+k}{x-a}\xrightarrow{\text{replace } y \text{ by } y-a} y=a+\dfrac{a^2+k}{x-a}$ Sequence of transformation: 1. Translate a units in the positive x direction 2. Scale the graph by a factor (a^2+k) parallel to the y-axis 3. Translate a units in the positive y direction	
(b)	$y=\dfrac{ax+k}{x-a}$ $yx-ya=ax+k$ $x(y-a)=ay+k$ $x=\dfrac{ay+k}{y-a}$ $\mathrm{f}^{-1}(x)=\dfrac{ax+k}{x-a}=\mathrm{f}(x)$	
(c)	**Method 1** $\mathrm{f}^2(x)=\mathrm{ff}(x)=\mathrm{f}^{-1}\mathrm{f}(x)=x$ **Method 2 (NOT Recommended. Only 1 mark)** $\mathrm{f}^2(x)=\mathrm{f}\left(\dfrac{ax+k}{x-a}\right)$ $=\dfrac{a\left(\dfrac{ax+k}{x-a}\right)+k}{\left(\dfrac{ax+k}{x-a}\right)-a}$ $=\dfrac{a(ax+k)+k(x-a)}{(ax+k)-a(x-a)}$ $=\dfrac{a^2x+kx}{k+a^2}$ $=\dfrac{x(a^2+k)}{k+a^2}=x$	Note that $\mathrm{f}^2(x)\neq(\mathrm{f}(x))^2$
(d)	$\mathrm{f}^{2023}(1)=\mathrm{f}^{2022}\mathrm{f}(1)=\mathrm{f}(1)=\dfrac{a+k}{1-a}$	From (c), $\mathrm{f}^{2022}(x)=\mathrm{f}^2\left(\mathrm{f}^2\left(\cdots\mathrm{f}^2(x)\right)\right)=x$

Question 7

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>

$y = x^{-3}\ln x = \dfrac{\ln x}{x^3}$

$\dfrac{dy}{dx} = \dfrac{x^2 - 3x^2\ln x}{x^6} = \dfrac{1-3\ln x}{x^4}$

$\dfrac{dy}{dx} = 0 \Rightarrow 1-3\ln x = 0 \Rightarrow x = e^{\frac{1}{3}}$

$y = \left(e^{\frac{1}{3}}\right)^{-3}\ln e^{\frac{1}{3}} = \dfrac{1}{3}e^{-1}$

Coordinates are $\left(e^{\frac{1}{3}}, \dfrac{1}{3}e^{-1}\right)$

</td><td></td></tr>
<tr><td>(b)</td><td>

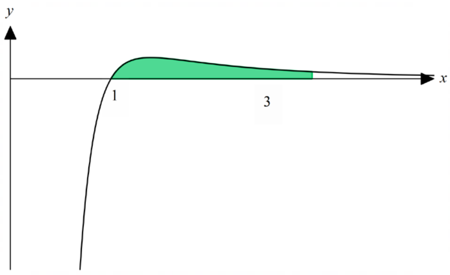
$\int_1^3 x^{-3}\ln x\ dx$

$= \left[-\dfrac{1}{2}x^{-2}\ln x\right]_1^3 + \dfrac{1}{2}\int_1^3 x^{-3}\ dx$

$= -\dfrac{1}{18}\ln 3 - \dfrac{1}{4}\left[x^{-2}\right]_1^3$

$= -\dfrac{1}{18}\ln 3 - \dfrac{1}{36} + \dfrac{1}{4}$

$= \dfrac{2}{9} - \dfrac{1}{18}\ln 3$



</td><td>You can use the GC to check your answer.

Recall that ln(1) = 0.</td></tr>
</table>

Question 8

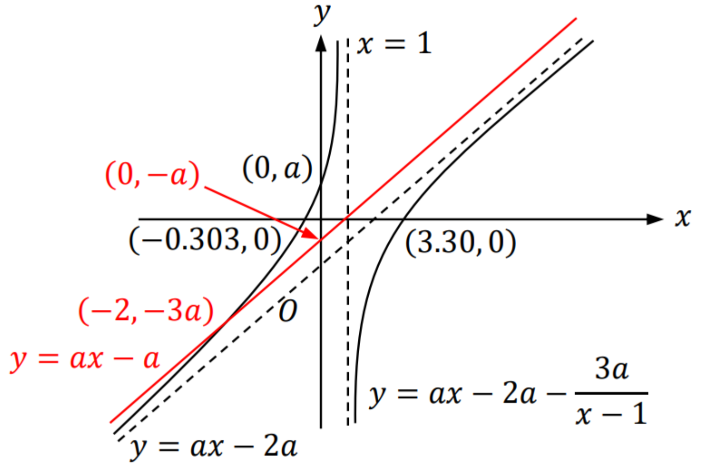
No.	Suggested Solution	Remarks for Student
(a)	**Method 1** $\dfrac{2x-1}{x^2+2x+1}=\dfrac{2x-1}{(x+1)^2}$ $=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}$ $=\dfrac{A(x+1)+B}{(x+1)^2}$ Comparing numerator, $2x-1=A(x+1)+B$ Comparing coefficients, $A=2,\ -1=A+B \Rightarrow B=-3$ Thus, $\int \dfrac{2x-1}{x^2+2x+1}\ \mathrm{d}x$ $=\int \dfrac{2}{x+1}-\dfrac{3}{(x+1)^2}\ \mathrm{d}x$ $=2\ln\|x+1\|+\dfrac{3}{x+1}+c$ **Method 2** $\int \dfrac{2x-1}{x^2+2x+1}\ \mathrm{d}x$ $=\int \dfrac{2x+2-3}{x^2+2x+1}\ \mathrm{d}x$ $=\int \dfrac{2x+2}{x^2+2x+1}\ \mathrm{d}x-\int \dfrac{3}{(x+1)^2}\ \mathrm{d}x,\ x\neq -1$ $=\ln\|x^2+2x+1\|+\dfrac{3}{x+1}+c$ $=\ln(x+1)^2+\dfrac{3}{x+1}+c$	Idea is to apply $\int \dfrac{\mathrm{f}'(x)}{\mathrm{f}'(x)}\mathrm{d}x$

(b)	$\int_0^2 \frac{\lvert 2x-1\rvert}{x^2+2x+1}\ \mathrm{d}x$ $= -\int_0^{\frac{1}{2}} \frac{2x-1}{x^2+2x+1}\ \mathrm{d}x + \int_{\frac{1}{2}}^{2} \frac{2x-1}{x^2+2x+1}\ \mathrm{d}x$ $= -\left[\ln\left(x^2+2x+1\right)+\frac{3}{x+1}\right]_0^{\frac{1}{2}} + \left[\ln\left(x^2+2x+1\right)+\frac{3}{x+1}\right]_{\frac{1}{2}}^{2}$ $= -\left[\ln\frac{9}{4}+2-3\right]+\left[\ln 9+1-\ln\frac{9}{4}-2\right]$ $= \ln\frac{16}{9}$ $= 2\ln\frac{4}{3}$	Note: $\lvert 2x-1\rvert$ $= \begin{cases} 2x-1, & \frac{1}{2} \le x \le 2 \\ -(2x-1), & 0 \le x < \frac{1}{2} \end{cases}$

Question 9

No.	Suggested Solution	Remarks for Student
(a)	$a+2d=ar$ $a+14d=ar^2$ $\frac{a+2d}{a}=\frac{a+14d}{a+2d}$ $a^2+4ad+4d^2=a^2+14ad$ $4d^2-10ad=0$ $d=0$ (rejected since $d\neq 0$) or $d=\frac{5a}{2}$	
(b)	$S_\infty=\frac{\sin\theta}{1-(-\cos\theta)}$ $=\frac{\sin\theta}{1+\cos\theta}$ $=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $=\tan\frac{\theta}{2},\ k=\frac{1}{2}$	
(c)	$\theta=\frac{\pi}{3},\ a=\sin\theta=\frac{\sqrt{3}}{2}$ $r=-\cos\theta=-\frac{1}{2}$ $S_7=\frac{a(1-r^7)}{1-r}$ $=\frac{\frac{\sqrt{3}}{2}\left(1-\left(-\frac{1}{2}\right)^7\right)}{1-\left(-\frac{1}{2}\right)}$ $=\frac{\sqrt{3}}{2}\left(1+\frac{1}{2^7}\right)\times\frac{2}{3}$ $=\frac{\sqrt{3}}{3}\left(\frac{129}{128}\right)$ $=\frac{43\sqrt{3}}{128}$	

Question 10

No.	Suggested Solution	Remarks for Student
(a)	$y = ax + b + \dfrac{a+2b}{x-1}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a - \dfrac{a+2b}{(x-1)^2}$ C has no stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \; a(x-1)^2 - (a+2b) = 0$ has no solution $ax^2 - 2ax - 2b = 0$ has no solution $\therefore$ Discriminant $= 4a^2 - 4a(-2b) < 0$ $a(a+2b) < 0$ Since $a > 0$, then $a + 2b < 0$	
(b)		Given $b = -2a$, $-2b = 4a$. Since $a > 0$, $a < -2b$ From (a), C has no stationary point. Note that the question asked for the equations of the asymptotes and the coordinates of all intercepts. Do indicate them on the graph.
(c)	As shown in diagram for (b)	
(d)	$x - 2 - \dfrac{3}{x-1} \le x - 1$ Comparing with $y = ax + b + \dfrac{a+2b}{x-1} = ax - 2a - \dfrac{3a}{x-1}$, choose $a = 1$ From graphs, $x \le -2$ or $x > 1$	

Question 11

No.	Suggested Solution	Remarks for Student
(a)	$\left\|\overrightarrow{QP}\right\| = 939 \quad \Rightarrow \left\|\begin{pmatrix} 936 \\ 72 \\ p+15 \end{pmatrix}\right\| = 939$ $936^2 + 72^2 + (p+15)^2 = 939^2$ $(p+15)^2 = 441$ $p = 6$ (rejected since $p < -15$) or $p = -36$	
(b)	$\begin{pmatrix} -100 \\ 400 \\ 50 \end{pmatrix} \times \begin{pmatrix} -200 \\ 940 \\ 30 \end{pmatrix} = 50\begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix} \times 10\begin{pmatrix} -20 \\ 94 \\ 3 \end{pmatrix} = 500\begin{pmatrix} -70 \\ -14 \\ -28 \end{pmatrix} = -7000\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ $\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 2560$ Cartesian equation of the plane is $5x + y + 2z = 2560$	Question asked for cartesian equation of the plane
(c)	$l_{PQ} : r = \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix}$ Sub into $5x + y + 2z = 2560$ $5(200 + 312\lambda) + (20 + 24\lambda) + 2(-15 - 7\lambda) = 2560$ $\lambda = 1$ Coordinates are $(512, 44, -22)$	Question asked for coordinates
(d)	Let α be angle of PQ made with vertical $\cos\alpha = \dfrac{\begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix}\right\| \left\|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\|} = \dfrac{7}{313}$ $\alpha = 88.719^\circ$ Thus, the angle of PQ made with horizontal is $90^\circ - \alpha = 1.3^\circ$ (1 d.p.)	The horizontal refers to a plane perpendicular to the z-axes. 178.7^o is also acceptable in this case as question did not specify acute angle.

Question 12

No.	Suggested Solution	Remarks for Student
(a)	$\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP$ $\displaystyle\int \frac{1}{P}\,\mathrm{d}P = \int k\,\mathrm{d}t$ $\ln\lvert P\rvert = kt + C$ $\ln P = kt + C, \quad \text{since } P > 0$ $P = \mathrm{e}^{kt+C} = A\mathrm{e}^{kt}$, where $A = \mathrm{e}^{c}$ Given that $t = 0,\ P = 50 \Rightarrow A = 50$ $\therefore P = 50\mathrm{e}^{kt}$ $t = 10,\ P = 100 \Rightarrow 100 = 50\mathrm{e}^{10k}$ $\mathrm{e}^{10k} = 2 \Rightarrow \mathrm{e}^{k} = 2^{\frac{1}{10}}$ $k = \dfrac{1}{10}\ln 2$ $\therefore P = 50\left(2^{\frac{t}{10}}\right)$	Need to write P in terms of t as stated in the question
(b)	$\dfrac{\mathrm{d}P}{\mathrm{d}t} = \lambda P(500 - P)$ $\displaystyle\int \frac{1}{P(500-P)}\,\mathrm{d}P = \int \lambda\,\mathrm{d}t$ $\displaystyle\int \frac{1}{500P} + \frac{1}{500(500-P)}\,\mathrm{d}P = \lambda t + C$ $\dfrac{1}{500}\ln\left\lvert\dfrac{P}{500-P}\right\rvert = \lambda t + C$ $\dfrac{P}{500-P} = A\mathrm{e}^{500\lambda t},\ A = \pm\mathrm{e}^{500C}$ $P = \dfrac{500A\mathrm{e}^{500\lambda t}}{1 + A\mathrm{e}^{500\lambda t}}$ $t = 0, P = 50 \Rightarrow A = \dfrac{1}{9}$ $P = \dfrac{\frac{500}{9}\mathrm{e}^{500\lambda t}}{1 + \frac{1}{9}\mathrm{e}^{500\lambda t}} = \dfrac{500\mathrm{e}^{500\lambda t}}{9 + \mathrm{e}^{500\lambda t}}$ $t = 10, P = 100$ $\dfrac{1}{4} = \dfrac{1}{9}\mathrm{e}^{5000\lambda} \Rightarrow \mathrm{e}^{\lambda} = \left(\dfrac{9}{4}\right)^{\frac{1}{5000}}$	

	$P=\dfrac{500\left(\frac{9}{4}\right)^{\frac{t}{10}}}{9+\left(\frac{9}{4}\right)^{\frac{t}{10}}}$ or $P=\dfrac{500}{1+9\left(\frac{9}{4}\right)^{-\frac{t}{10}}}$	
(c)	Refined model, $P=\dfrac{500\left(\frac{9}{4}\right)^{\frac{t}{10}}}{9+\left(\frac{9}{4}\right)^{\frac{t}{10}}}=\dfrac{500}{9\left(\frac{9}{4}\right)^{-\frac{t}{10}}+1}\to 500$ as $t\to\infty$ Population approaches 500 in the long run. First model, $P=50\left(2^{\frac{t}{10}}\right)\to\infty$ as $t\to\infty$ which is not realistic in real world as no population could indefinitely approach infinity as resources are limited.	Need to compare the 2 models in the long run