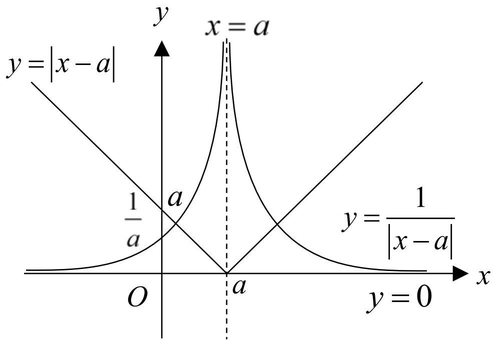
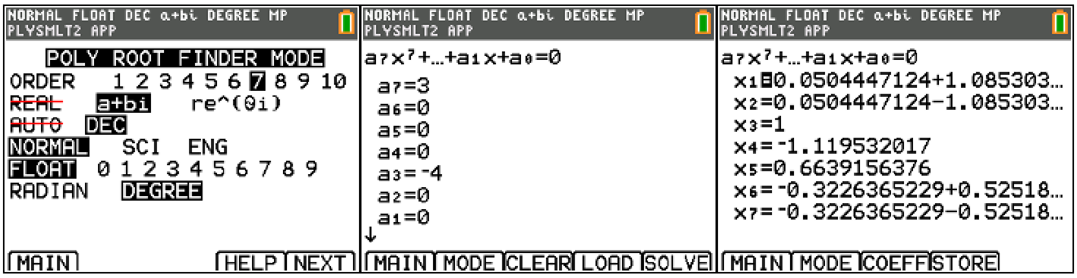
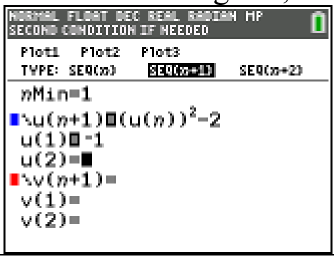
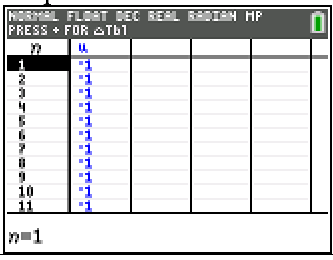
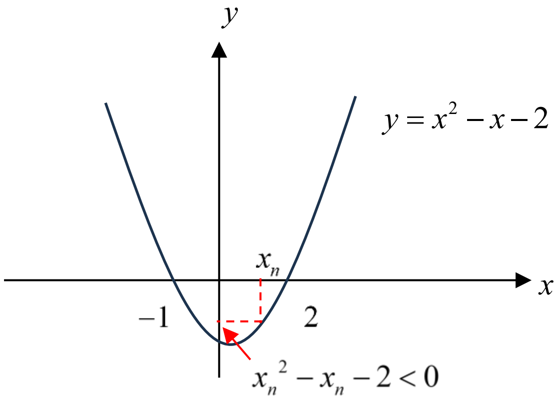
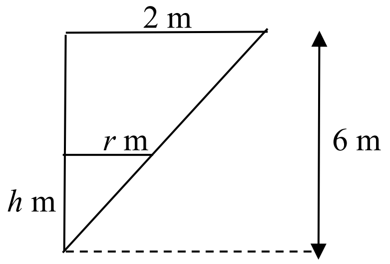
| Qn | Solution |
|---|---|
| **1** | **Graphing Techniques, Equations and Inequalities** |
| **(a)** |  |
| **(b)** | Points of intersection:<br>$\lvert x-a\rvert=\frac{1}{\lvert x-a\rvert}$<br>Since $x\neq a$, $\lvert x-a\rvert>0$<br>$(x-a)^2=1$<br>$x=a\pm1$<br>For $\lvert x-a\rvert>\frac{1}{\lvert x-a\rvert}$, from the graph,<br>$x<a-1$ or $x>a+1$ |

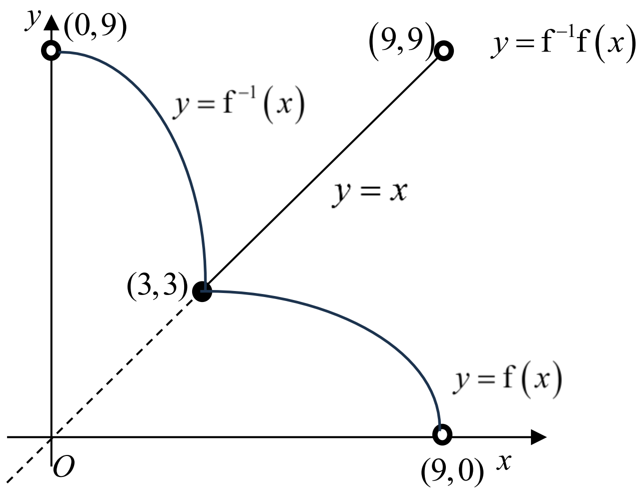
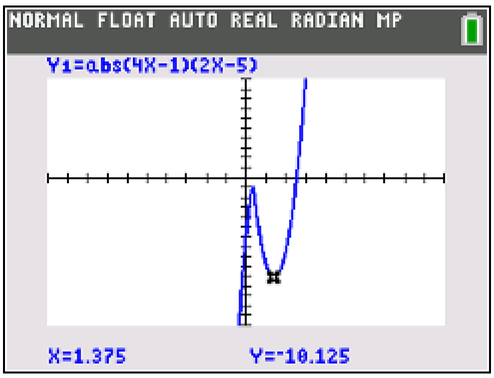
| Qn | Solution |
|---|---|
| **2** | **Differentiation** |
| **(a)** | $y^2 = x + e^{2x} y \qquad --(1)$<br>Differentiate with respect to $x$,<br>$2y\frac{dy}{dx} = 1 + e^{2x}\frac{dy}{dx} + y\left(2e^{2x}\right)$<br>$\frac{dy}{dx}\left(2y - e^{2x}\right) = 1 + 2ye^{2x}$ (shown) $\quad --(2)$ |
| **(b)** | Subt $x = 0$ into $(1)$<br>$y^2 = y$<br>$y^2 - y = 0$<br>$y(y-1) = 0$<br>$y = 0$ or $y = 1$<br><br>Subt $(0,0)$ into $(2)$<br>$\frac{dy}{dx} = -1$<br>$\therefore$ the equation of the tangent is $y = -x$<br>Subt $(0,1)$ into $(2)$<br>$\frac{dy}{dx} = 3$<br>$\therefore$ the equation of the tangent is<br>$y - 1 = 3(x - 0)$<br>$y = 3x + 1$ |

| Qn | Solution |
|---|---|
| 3 | **Techniques of Integration** |
| (a) | $\int \frac{x^2-3x-1}{x^2-4x+1}\,\mathrm{d}x$<br>$=\int 1+\frac{x-2}{x^2-4x+1}\,\mathrm{d}x$<br>$=\int 1\,\mathrm{d}x+\frac{1}{2}\int \frac{2(x-2)}{x^2-4x+1}\,\mathrm{d}x$<br>$=\int 1\,\mathrm{d}x+\frac{1}{2}\int \frac{2x-4}{x^2-4x+1}\mathrm{d}x$<br>$=x+\frac{1}{2}\ln\left\|x^2-4x+1\right\|+C$ |
| (b) | $\int x\ln(kx)\,\mathrm{d}x$<br>$=\left[\left(\frac{x^2}{2}\right)(\ln(kx))\right]-\int\left(\frac{x^2}{2}\right)\frac{k}{kx}\,\mathrm{d}x$<br>$=\frac{x^2}{2}\ln(kx)-\frac{1}{2}\int x\,\mathrm{d}x$<br>$=\frac{x^2}{2}\ln(kx)-\frac{1}{2}\left(\frac{x^2}{2}\right)+C$<br>$=\frac{x^2}{2}\ln(kx)-\frac{x^2}{4}+C$ |
| (c) | $x=\cos\theta$<br>$\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta$<br>$\int \frac{\sin\theta}{2\cos 2\theta+1}\,\mathrm{d}\theta$<br>$=\int \frac{\sin\theta}{2\left(2\cos^2\theta-1\right)+1}\,\mathrm{d}\theta$<br>$=\int \frac{1}{4\cos^2\theta-1}(\sin\theta)\,\mathrm{d}\theta$<br>$=-\int \frac{1}{4x^2-1}\,\mathrm{d}x$<br>$=\int \frac{1}{1-(2x)^2}\,\mathrm{d}x$<br>$=\frac{1}{2}\left(\frac{1}{2(1)}\right)\ln\left\|\frac{1+2x}{1-2x}\right\|+C$<br>$=\frac{1}{4}\ln\left\|\frac{1+2\cos\theta}{1-2\cos\theta}\right\|+C$ |

| Qn | Solution |
|---|---|
| **4** | **A&GS, Sequences & Series** |
| **(a)** | $a+6d=br^2 \quad -- \quad (1)$<br>$a+9d=br^5 \quad -- \quad (2)$<br>$a+10d=br^9 \quad -- \quad (3)$<br>$(2)-(1):$<br>$3d=br^2\left(r^3-1\right) \quad -- \quad (4)$<br>$(3)-(2):$<br>$d=br^5\left(r^4-1\right) \quad -- \quad (5)$<br>Substitute (5) into (4),<br>$3br^5\left(r^4-1\right)=br^2\left(r^3-1\right)$<br>$3r^3\left(r^4-1\right)=\left(r^3-1\right) \quad (\text{since } b,r\neq 0)$<br>$3r^7-4r^3+1=0 \quad (\text{shown})$<br>By GC, since the series is convergent, $\lvert r\rvert<1$,<br>$r=0.6639$ (4 d.p.)<br><br> |
| **(b)** | New Series: $br, br^3, br^5, \ldots$<br>$\left\lvert \frac{br}{1-r^2}-S\right\rvert \geq 1.78b$<br>$\left\lvert \frac{br}{1-r^2}-\frac{b}{1-r}\left(1-r^n\right)\right\rvert \geq 1.78b$<br>$\left\lvert \frac{0.6639b}{1-0.6639^2}-\frac{b}{1-0.6639}\left(1-0.6639^n\right)\right\rvert \geq 1.78b$<br>$\left\lvert \frac{0.6639}{1-0.6639^2}-\frac{1}{1-0.6639}\left(1-0.6639^n\right)\right\rvert \geq 1.78 \quad (\because b>0)$<br>Using GC,<br>When $n=14, \left\lvert \frac{0.6639}{1-0.6639^2}-\frac{1}{1-0.6639}\left(1-0.6639^n\right)\right\rvert = 1.7785<1.78$<br>When $n=15, \left\lvert \frac{0.6639}{1-0.6639^2}-\frac{1}{1-0.6639}\left(1-0.6639^n\right)\right\rvert = 1.7818\geq 1.78$<br>Therefore, least $n=15$. |

| Qn | Solution |
|---|---|
| **5** | **Recurrence Relations** |
| **(a)** | Let $x_n \to L$ as $n \to \infty$. Then $x_{n+1} \to L$.<br>$L = L^2 - 2$<br>$L^2 - L - 2 = 0$<br>$(L+1)(L-2) = 0$<br>$L = -1$ or $L = 2$ (shown) |
| **(b)** | **Method 1:** From the limits,<br>Since $L = -1$, the sequence remains constant at $-1$.<br><br>**Method 2:** Using GC, the sequence remains constant at $-1$.<br> |
| **(c)** | $x_{n+1} - x_n = x_n{}^2 - 2 - x_n$<br>$= x_n{}^2 - x_n - 2$<br>If $-1 < x_n < 2$, $x_n{}^2 - x_n - 2 < 0$ from the graph.<br><br>Hence $x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$. |

| Qn | Solution |
|---|---|
| 6 | **Application of Differentiation** |
| **(a)** | <br>By similar triangles,<br>$\frac{r}{h}=\frac{2}{6}$<br>$\Rightarrow r=\frac{1}{3}h$<br>$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h=\frac{1}{3}\pi\left(\frac{1}{9}h^2\right)h=\frac{1}{27}\pi h^3$ |
| **(b)** | $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{in}}{\mathrm{d}t}-\frac{\mathrm{d}V_{out}}{\mathrm{d}t}$<br>$=\frac{\mathrm{d}V_{in}}{\mathrm{d}t}-0.03$<br>Since $\frac{\mathrm{d}h}{\mathrm{d}t}=0.1$,<br>$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$<br>$=\frac{1}{9}\pi h^2\times 0.1$<br>$=\frac{1}{9}\pi(3)^2\times 0.1$<br>$=0.1\pi$<br>$\therefore \frac{\mathrm{d}V_{in}}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}t}+0.03=0.1\pi+0.03=0.344$<br>Hence, water is pouring in at a rate of 0.344 m/s |

| Qn | Solution |
|---|---|
| 7 | **Functions Graphing Techniques Equations and Inequalities** |
| (a) | For $\mathrm{f}^{-1}$ to exist, f must be one-one function.<br>Least value of $k=3$ |
| (b) | Let $y=\frac{1}{2}\sqrt{36-(x-3)^2}$<br>$2y=\sqrt{36-(x-3)^2}$<br>$4y^2=36-(x-3)^2$<br>$(x-3)^2=36-4y^2$<br>$x-3=\pm\sqrt{36-4y^2}$<br>$x=3-\sqrt{36-4y^2}$ or $x=3+\sqrt{36-4y^2}$<br>(Rejected, $x\geq 3$)<br>$\mathrm{f}^{-1}(x)=3+\sqrt{36-4x^2}$<br>$=3+2\sqrt{9-x^2}$ |
| (c) |  |
| (d) | Range of f = (0,3]<br>Domain of g = $(0,\infty)$<br>Since $R_f \subseteq D_g$, gf exists |
| (e) | $D_f=[3,9)\xrightarrow{f}R_f=(0,3]\xrightarrow{g}R_{gf}=\left[-\frac{81}{8},11\right]$<br><br>$R_{gf}=\left[-\frac{81}{8},11\right]$ |

| Qn | Solution |
|---|---|
| **8** | **Definite Integrals** |
| **(a)** | $x = 2t + 3$<br>$\frac{\mathrm{d}x}{\mathrm{d}t} = 2$<br>$y = 5 - 4t^2$<br>$\frac{\mathrm{d}y}{\mathrm{d}t} = -8t$<br>$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-8t}{2} = -4t$<br>At $(2,4)$,<br>$x = 2t + 3 \Rightarrow 2 = 2t + 3 \Rightarrow t = -\frac{1}{2}$<br>$\frac{\mathrm{d}y}{\mathrm{d}x} = -4\left(-\frac{1}{2}\right) = 2$<br>Equation of normal, $N$:<br>$y - 4 = -\frac{1}{2}(x - 2)$<br>$y = -\frac{1}{2}x + 5$ |
| **(b)** | $x = 2t + 3 \Rightarrow t = \frac{x-3}{2}$<br>Sub $t$ into $y$:<br>$y = 5 - 4t^2$<br>$= 5 - 4\left(\frac{x-3}{2}\right)^2$<br>$= 5 - (x-3)^2$<br><br>**Alternatively,**<br>$y = 5 - 4t^2 \Rightarrow t = \pm\sqrt{\frac{5-y}{4}}$<br>Sub $t$ into $x$:<br>$x = 2\left(\pm\frac{\sqrt{5-y}}{2}\right) + 3 = 3 \pm \sqrt{5-y}$ |

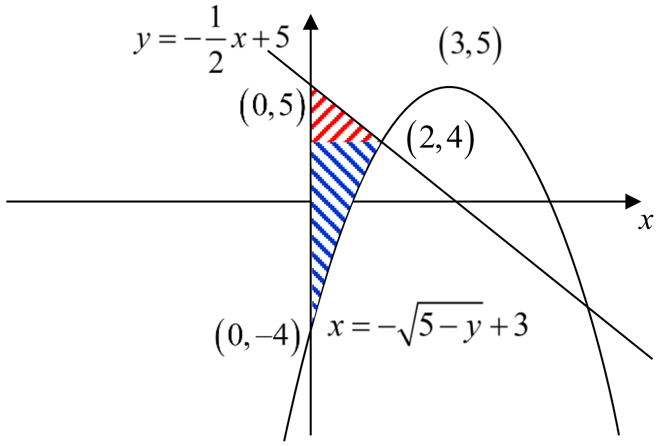
**(c)**

$y=5-(x-3)^2$

$x=3\pm\sqrt{5-y}$

Since $x\le 3$, $x=3-\sqrt{5-y}$

$$
\begin{aligned}
x^2&=\left(3-\sqrt{5-y}\right)^2\\
&=9-6\sqrt{5-y}+5-y\\
&=14-y-6\sqrt{5-y}
\end{aligned}
$$




Volume of solid of revolution about $y$-axis

$$
\begin{aligned}
&=\pi\int_{-4}^{4}x^2\ \mathrm{d}y\ +\ \text{Volume of cone}\\
&=\pi\int_{-4}^{4}\left(3-\sqrt{5-y}\right)^2\ \mathrm{d}y+\frac{1}{3}\pi(2)^2(1)\\
&=\pi\int_{-4}^{4}3^2-6\sqrt{5-y}+(5-y)\ \mathrm{d}y+\frac{4}{3}\pi\\
&=\pi\int_{-4}^{4}14-y-6\sqrt{5-y}\ \mathrm{d}y+\frac{4}{3}\pi\\
&=\pi\left[14y-\frac{1}{2}y^2-\frac{6(5-y)^{\frac{3}{2}}}{\frac{3}{2}(-1)}\right]_{-4}^{4}+\frac{4}{3}\pi\\
&=\pi\left[14y-\frac{1}{2}y^2+4(5-y)^{\frac{3}{2}}\right]_{-4}^{4}+\frac{4}{3}\pi\\
&=\pi(56-8+4+56+8-108)+\frac{4}{3}\pi\\
&=\frac{28\pi}{3}
\end{aligned}
$$

| Qn | Solution |
|---|---|
| 9 | **Complex Numbers** |
| (a) | **Method 1:**<br>Since the **equation has real coefficients**, $-2+2\mathrm{i}$ is a root<br>$\Rightarrow -2-2\mathrm{i}$ is also a root.<br>Consider<br>$\left[z-(-2+2\mathrm{i})\right]\left[z-(-2-2\mathrm{i})\right]=\left[(z+2)-2\mathrm{i}\right]\left[(z+2)+2\mathrm{i}\right]$<br>$=(z+2)^2-[2\mathrm{i}]^2$<br>$=z^2+4z+4+4$<br>$=z^2+4z+8$<br><br>$z^3+az^2+bz-16\sqrt{2}=\left(z^2+4z+8\right)(pz+q)$<br>coeff of $z^3$ : $p=1$<br>constant: $q=-2\sqrt{2}$<br>$z^3+az^2+bz-16\sqrt{2}=\left(z^2+4z+8\right)\left(z-2\sqrt{2}\right)$<br>coeff of $z^2$ : $a=-2\sqrt{2}+4$<br>coeff of $z$: $b=-8\sqrt{2}+8$<br>$\therefore a=4-2\sqrt{2},\ b=8-8\sqrt{2}$ and the roots are $-2+2\mathrm{i}, -2-2\mathrm{i}$ and $2\sqrt{2}$.<br><br>**Method 2:**<br>Since $z=-2+2\mathrm{i}$ is a root, $(-2+2\mathrm{i})^3+a(-2+2\mathrm{i})^2+b(-2+2\mathrm{i})-16\sqrt{2}=0$<br>$16+16\mathrm{i}+a(-8\mathrm{i})+b(-2+2\mathrm{i})-16\sqrt{2}=0$<br>$16-2b-16\sqrt{2}+16\mathrm{i}-8a\mathrm{i}+2b\mathrm{i}=0$<br>Comparing real and imaginary parts,<br>$16-2b-16\sqrt{2}=0$ and $16-8a+2b=0$<br>$b=8-8\sqrt{2}$     $8a=16+2b$<br><br>Subst $b=8-8\sqrt{2}$ into $8a=16+2b$,<br>$8a=16+2\left(8-8\sqrt{2}\right)$<br>$\therefore a=4-2\sqrt{2}$<br>$\therefore b=8-8\sqrt{2}$<br><br>Since the equation has real coefficients, $-2+2\mathrm{i}$ is a root<br>$\Rightarrow -2-2\mathrm{i}$ is also a root.<br>Consider<br>$\left[z-(-2+2\mathrm{i})\right]\left[z-(-2-2\mathrm{i})\right]=\left[(z+2)-2\mathrm{i}\right]\left[(z+2)+2\mathrm{i}\right]$<br>$=(z+2)^2-[2\mathrm{i}]^2$<br>$=z^2+4z+4+4$<br>$=z^2+4z+8$ |

$z^3+\left(4-2\sqrt{2}\right)z^2+\left(8-8\sqrt{2}\right)z-16\sqrt{2}=\left(z^2+4z+8\right)\left(pz+q\right)$

coeff of $z^3$ : $p=1$

constant: $q=-2\sqrt{2}$

$z^3+\left(4-2\sqrt{2}\right)z^2+\left(8-8\sqrt{2}\right)z-16\sqrt{2}=\left(z^2+4z+8\right)\left(z-2\sqrt{2}\right)$

The roots are $-2+2\mathrm{i},-2-2\mathrm{i}$ and $2\sqrt{2}$.

**(b)**

Let $z_1=-2+2\mathrm{i}\Rightarrow z_1$ is in 2nd quadrant

$$|z_1|=\sqrt{(-2)^2+(2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\arg(z_1)=\pi-\tan^{-1}\left(\frac{2}{2}\right)=\pi-\tan^{-1}(1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

Let $z_2=-2-2\mathrm{i}\Rightarrow z_2$ is in 3rd quadrant

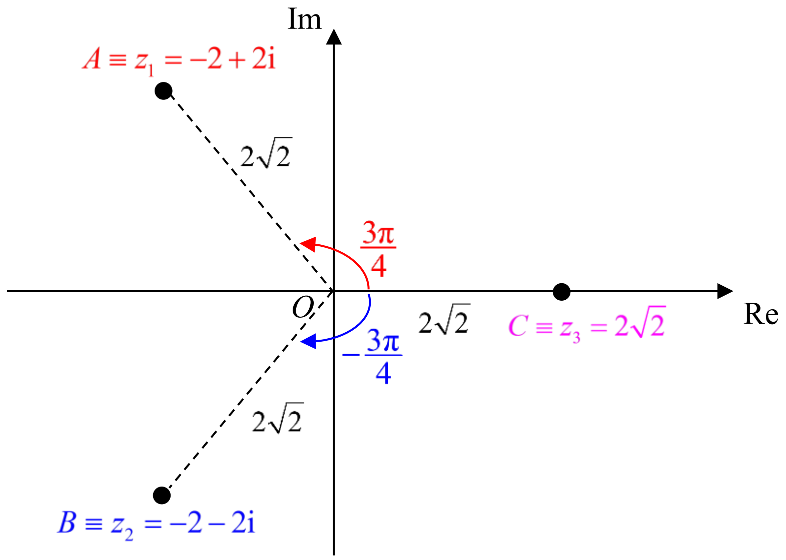
$$|z_2|=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\arg(z_2)=-\pi+\tan^{-1}\left(\frac{2}{2}\right)=-\pi+\tan^{-1}(1)=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}$$

Let $z_3=2\sqrt{2}\Rightarrow z_2$ is in 1st quadrant

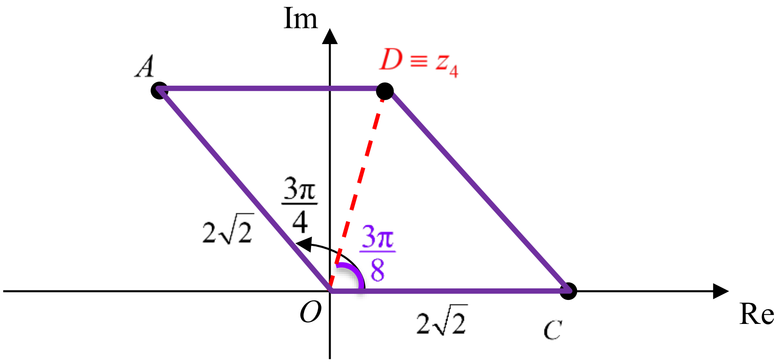
$$|z_3|=2\sqrt{2}$$

$$\arg(z_3)=0$$




*B* is the reflection of *A* **about the real axis**. OR
*B* is the $90^0$ anti-clockwise rotation of *A* **about *O***.

| | |
|---|---|
| **(c)** | **Method 1:** Since $OA = OC = 2\sqrt{2}$, we consider $D$ be the point such that $OADC$ is a rhombus. |



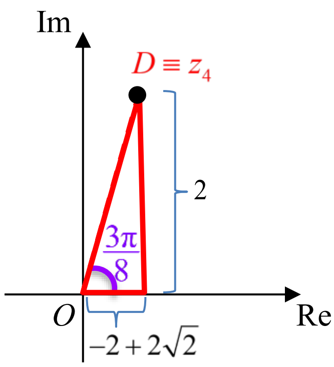

$OD$ bisects $\angle AOC$

Thus, $\angle DOC = \frac{1}{2}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{8}$

Let $D \equiv z_4$ where

$$z_4 = -2 + 2\text{i} + 2\sqrt{2}$$
$$= -2 + 2\sqrt{2} + 2\text{i}$$

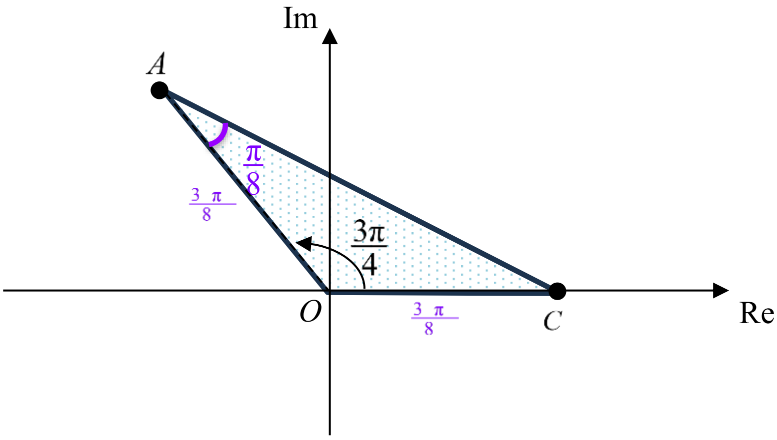
Since $z_4$ lies in 1st quadrant,



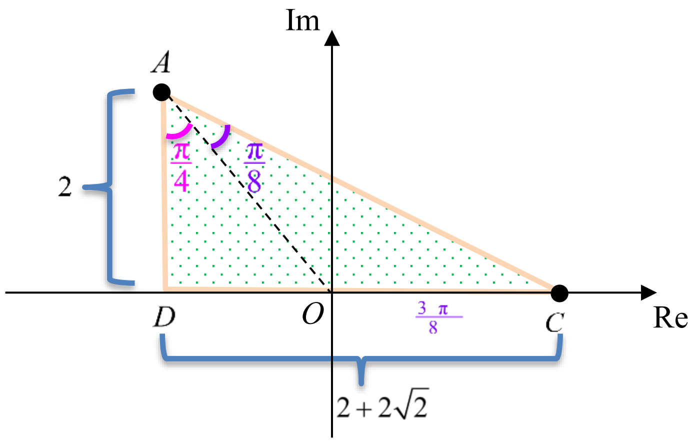

$$\arg(z_4) = \frac{3\pi}{8}$$

$$\tan\frac{3\pi}{8} = \frac{2}{-2+2\sqrt{2}}$$
$$= \frac{1}{-1+\sqrt{2}} \times \frac{-1-\sqrt{2}}{-1-\sqrt{2}}$$
$$= \frac{-1-\sqrt{2}}{(-1)^2 - \left(\sqrt{2}\right)^2}$$
$$= 1+\sqrt{2} \text{ (shown)}$$

**Method 2:**
Consider triangle $OAC$ and observe that triangle $OAC$ is an isosceles triangle as $OA = OC = 2\sqrt{2}$.




$\angle OAC = \frac{1}{2}\left(\pi - \frac{3\pi}{4}\right) = \frac{\pi}{8}$




Let $D$ be the point that represent $-2+0\text{i}$

$\angle OAD = \angle AOD = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

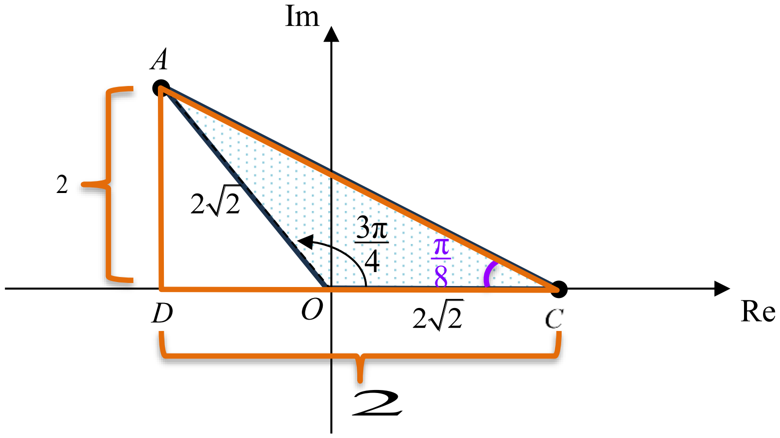
$\angle DAC = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$

Using triangle $DAC$

$$\tan\frac{3\pi}{8} = \frac{2+2\sqrt{2}}{2}$$

$$= 1+\sqrt{2} \text{ (shown)}$$

**Method 3:**
Consider triangle $OAC$ and observe that triangle $OAC$ is an isosceles triangle as $OA = OC = 2\sqrt{2}$.

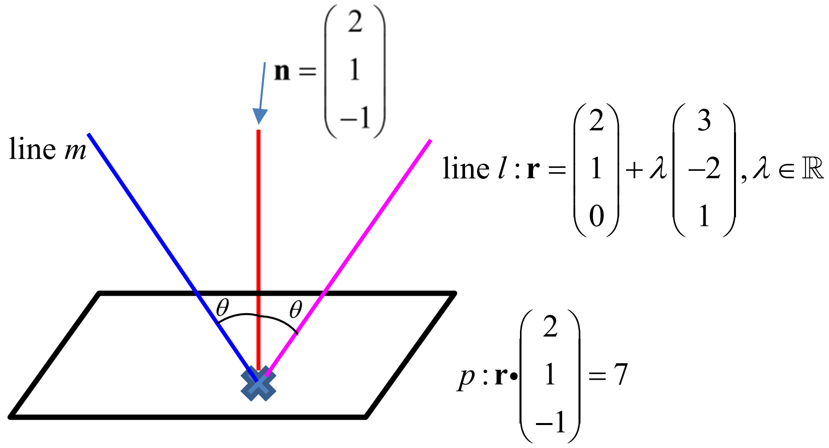


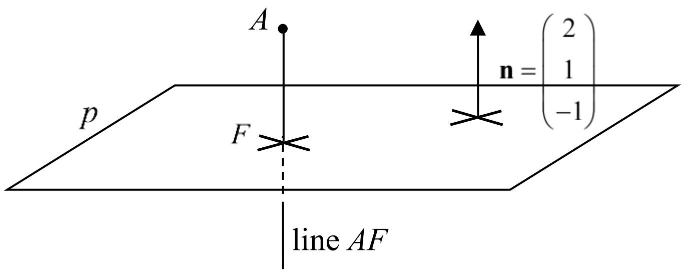
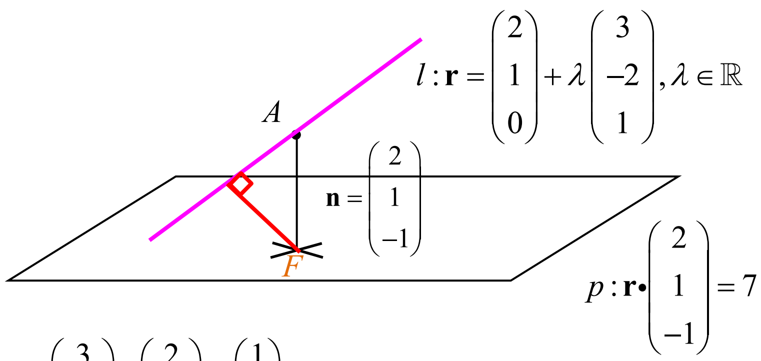

$\angle OCA = \frac{1}{2}\left(\pi - \frac{3\pi}{4}\right) = \frac{\pi}{8}$

Let $D$ be the point that represent $-2+0\text{i}$
Using triangle $DAC$

$$\tan\frac{\pi}{8} = \frac{2}{2+2\sqrt{2}} = \frac{1}{1+\sqrt{2}}$$

$$\begin{aligned}\tan\frac{3\pi}{8} &= \tan\left(\frac{\pi}{4}+\frac{\pi}{8}\right)\\ &= \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{8}}\\ &= \frac{1+\left(\frac{1}{1+\sqrt{2}}\right)}{1-(1)\left(\frac{1}{1+\sqrt{2}}\right)}\\ &= \frac{1+\sqrt{2}+1}{1+\sqrt{2}-1}\\ &= \frac{2+\sqrt{2}}{\sqrt{2}}\\ &= \frac{2}{\sqrt{2}}+1\\ &= 1+\sqrt{2} \text{ (shown)}\end{aligned}$$

| Qn | Solution |
| --- | --- |
| **10** | **3D Vectors** |
| **(a)** | Let $\theta$ be the acute angle between the direction vector of the flight path and normal vector of the rooftop.<br><br>$\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rvert}{\sqrt{14}\sqrt{6}} = \dfrac{3}{\sqrt{14}\sqrt{6}}$<br>The angle between the original flight path and the new flight path will be $2\theta$.<br>$\cos 2\theta = 2\cos^2\theta - 1$<br>$= 2\left(\dfrac{3}{\sqrt{14}\sqrt{6}}\right)^2 - 1$<br>$= -\dfrac{11}{14}$ |

| | |
|---|---|
| **(b)** | Let the foot of perpendicular be $F$.<br><br><br>$l_{AF}:\mathbf{r}=\begin{pmatrix}2\\1\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix},\ \mu\in\mathbb{R}$<br>$\begin{pmatrix}2+2\mu\\1+\mu\\-\mu\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}=7$<br>$4+4\mu+1+\mu+\mu=7$<br>$6\mu+5=7$<br>$\mu=\frac{1}{3}$<br>Coordinates of $F$ is $\left(\frac{8}{3},\frac{4}{3},-\frac{1}{3}\right)$. |
| **(c)** | <br><br>$\mathbf{d}=\begin{pmatrix}3\\-2\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\7\end{pmatrix}$<br>$l_s:\mathbf{r}=\begin{pmatrix}\frac{8}{3}\\\frac{4}{3}\\-\frac{1}{3}\end{pmatrix}+\alpha\begin{pmatrix}1\\5\\7\end{pmatrix},\ \alpha\in\mathbb{R}$ |

$x=\frac{8}{3}+\alpha \Rightarrow \alpha=\frac{3x-8}{3}$

$y=\frac{4}{3}+5\alpha \Rightarrow \alpha=\frac{3y-4}{15}$

$z=-\frac{1}{3}+7\alpha \Rightarrow \alpha=\frac{3z+1}{21}$

$\therefore \frac{3x-8}{3}=\frac{3y-4}{15}=\frac{3z+1}{21}$

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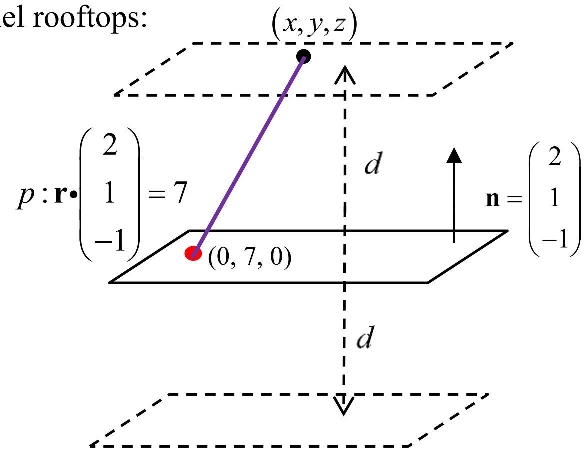
Let $(0,7,0)$ be a point on the first rooftop and $(x,y,z)$ be a point on the second rooftop.

Distance between 2 parallel rooftops:




$$\frac{\left|\left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}0\\7\\0\end{pmatrix}\right]\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}{\sqrt{6}}=d$$

$$\frac{\left|\begin{pmatrix}x\\y-7\\z\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}{\sqrt{6}}=d$$

$|2x+y-7-z|=\sqrt{6}d$

$2x+y-7-z=\sqrt{6}d$ or $2x+y-7-z=-\sqrt{6}d$

$2x+y-z=\sqrt{6}d+7$ or $2x+y-z=-\sqrt{6}d+7$

**Alternatively,**

$2x+y-z=7$

$$\mathbf{r}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=7$$

$$\mathbf{r}\cdot\frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\sqrt{6}}=\frac{7}{\sqrt{6}}$$

Let the second rooftop be:

$$\mathbf{r}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=D$$

$$\mathbf{r}\cdot\frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\sqrt{6}}=\frac{D}{\sqrt{6}}$$

Distance between 2 parallel rooftops:

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| | $\left\lvert\frac{7}{\sqrt{6}}-\frac{D}{\sqrt{6}}\right\rvert=d$<br>$\frac{7}{\sqrt{6}}-\frac{D}{\sqrt{6}}=d$ or $-\frac{7}{\sqrt{6}}+\frac{D}{\sqrt{6}}=d$<br>$D=7-\sqrt{6}d$ or $D=7+\sqrt{6}d$<br>$\mathbf{r}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=\sqrt{6}d+7$ or $\mathbf{r}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=-\sqrt{6}d+7$<br>$2x+y-z=\sqrt{6}d+7$ or $2x+y-z=-\sqrt{6}d+7$ |

| Qn | Solution |
|---|---|
| **11** | **Definite Integrals, Maclaurin Series, Application of Differentiation** |
| **(a)** | $\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$ <br> When $x=0$, <br> $\left(\frac{0+y}{2}\right)^2+\left(\frac{y-0}{4}\right)^2=1$ <br> $\frac{5y^2}{16}=1$ <br> $y=\pm\frac{4}{\sqrt{5}}$ <br><br> Since the region lies in the first quadrant, $y>0$, hence we take $y=\frac{4}{\sqrt{5}}$. <br> **Method 1:** <br> $\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$ <br> $4(x+y)^2+(y-x)^2=16$ <br> $4x^2+8xy+4y^2+y^2-2xy+x^2=16$ <br> $5x^2+6xy+5y^2=16$ <br> Differentiate w.r.t. $x$, <br> $10x+6y+6x\frac{\mathrm{d}y}{\mathrm{d}x}+10y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ <br> Differentiate w.r.t. $x$, <br> $10+6\frac{\mathrm{d}y}{\mathrm{d}x}+6\frac{\mathrm{d}y}{\mathrm{d}x}+6x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+10\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+10y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ <br> $10+12\frac{\mathrm{d}y}{\mathrm{d}x}+6x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+10\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+10y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ <br><br> When $x=0$, $y=\frac{4}{\sqrt{5}}$, <br> $10(0)+6\left(\frac{4}{\sqrt{5}}\right)+6(0)\frac{\mathrm{d}y}{\mathrm{d}x}+10\left(\frac{4}{\sqrt{5}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ <br> $\left(\frac{40}{\sqrt{5}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{24}{\sqrt{5}}$ <br> $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{5}$ <br> $10+12\left(-\frac{3}{5}\right)+6(0)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+10\left(-\frac{3}{5}\right)^2+10\left(\frac{4}{\sqrt{5}}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ <br> $\left(\frac{40}{\sqrt{5}}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{32}{5}$ <br> $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{4\sqrt{5}}{25}$ <br> $y=\frac{4}{\sqrt{5}}-\frac{3}{5}x-\frac{2}{5\sqrt{5}}x^2+\dots$ |

**Method 2:**

$$\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$$

Differentiate w.r.t. $x$,

$$2\left(\frac{x+y}{2}\right)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{1}{2}\right)+2\left(\frac{y-x}{4}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)\left(\frac{1}{4}\right)=0$$

$$\frac{1}{2}(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{y-x}{8}\left(\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)=0$$

Differentiate w.r.t. $x$,

$$\frac{1}{2}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{1}{2}(x+y)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\frac{1}{8}\left(\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)^2+\frac{1}{8}(y-x)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0$$

**Method 3:**

$$\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$$

$$4(x+y)^2+(y-x)^2=16$$

$$4x^2+8xy+4y^2+y^2-2xy+x^2=16$$

$$5x^2+6xy+5y^2=16$$

$$5y^2+6xy+5x^2-16=0$$

$$y=\frac{-6x\pm\sqrt{(6x)^2-4(5)(5x^2-16)}}{2(5)}$$

$$=\frac{-6x\pm\sqrt{320-64x^2}}{10}$$

Since $y>0$,

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| | $y=\dfrac{-6x+\sqrt{320-64x^2}}{10}$<br>$=\dfrac{-6x+8\sqrt{5-x^2}}{10}$<br>$=\dfrac{-3x+4\sqrt{5-x^2}}{5}$<br>$=-\dfrac{3}{5}x+\dfrac{4}{5}\left(5-x^2\right)^{\frac{1}{2}}$<br>$=-\dfrac{3}{5}x+\dfrac{4\sqrt{5}}{5}\left(1-\dfrac{x^2}{5}\right)^{\frac{1}{2}}$<br>$=-\dfrac{3}{5}x+\dfrac{4\sqrt{5}}{5}\left(1-\left(\dfrac{1}{2}\right)\left(\dfrac{x^2}{5}\right)+...\right)$<br>$\approx-\dfrac{3}{5}x+\dfrac{4\sqrt{5}}{5}-\dfrac{2\sqrt{5}}{25}x^2$<br>$\approx\dfrac{4\sqrt{5}}{5}-\dfrac{3}{5}x-\dfrac{2\sqrt{5}}{25}x^2$<br><br>Required Estimate $=\int_0^{0.5} y\ \mathrm{d}x$<br>$\approx\int_0^{0.5}\dfrac{4}{\sqrt{5}}-\dfrac{3}{5}x-\dfrac{2}{5\sqrt{5}}x^2\mathrm{d}x$<br>$\approx 0.812\mathrm{m}^2\ (3\text{ s.f.})$ |
| **(b)** | $L=\sqrt{(\cos t-2\sin t)^2+(\cos t+2\sin t)^2}$<br>$=\sqrt{\cos^2 t-4\sin t\cos t+4\sin^2 t+\cos^2 t+4\sin t\cos t+4\sin^2 t}$<br>$=\sqrt{2\cos^2 t+8\sin^2 t}$<br>$=\sqrt{2+6\sin^2 t}$<br>$\dfrac{\mathrm{d}L}{\mathrm{d}t}=\dfrac{12\sin t\cos t}{2\sqrt{2+6\sin^2 t}}$<br>$=\dfrac{6\sin t\cos t}{\sqrt{2+6\sin^2 t}}$<br>Let $\dfrac{\mathrm{d}L}{\mathrm{d}t}=0$.<br>$6\cos t\sin t=0$<br>$\cos t=0$ or $\sin t=0$<br>$t=-\dfrac{\pi}{2}$ or 0 or $\dfrac{\pi}{2}$ or $\pi$.<br><br>When $t=\pm\dfrac{\pi}{2}$, $L=2\sqrt{2}$.<br>When $t=0$ or $\pi$, $L=\sqrt{2}$.<br>$\therefore$ Maximum $L$ is $2\sqrt{2}$.<br>$\therefore$ Minimum $L$ is $\sqrt{2}$. |

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| **(c)** | $\frac{x^2}{8}+\frac{y^2}{2}=1$ **OR** $\frac{x^2}{2}+\frac{y^2}{8}=1$<br><br>$\text{Required Area} = 4\int_0^{\sqrt{8}} \sqrt{2-\frac{x^2}{4}}\ \mathrm{d}x$<br>$\approx 12.6\text{m}^2\ (3\text{ s.f.})$<br>**OR**<br>$\text{Required Area} = 4\int_0^{\sqrt{2}} \sqrt{8-4x^2}\ \mathrm{d}x$<br>$\approx 12.6\text{m}^2\ (3\text{ s.f.})$ |