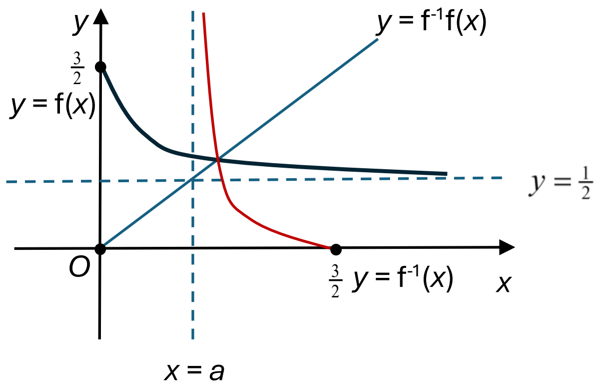
Section A: Pure Mathematics Solutions

1(a)	$y=\frac{1}{2}\ln\left(1+e^{3x}\right)$ $e^{2y}=1+e^{3x}$ Diff. wrt x, $2e^{2y}\frac{dy}{dx}=3e^{3x}$ (Shown) -- (1) <u>Alternative solution:</u> $y=\ln\sqrt{1+e^{3x}}$ $\frac{dy}{dx}=\frac{3e^{3x}\left(1+e^{3x}\right)^{\frac{-1}{2}}}{2\left(1+e^{3x}\right)^{\frac{1}{2}}}=\frac{3e^{3x}}{2\left(1+e^{3x}\right)}$ $2\left(1+e^{3x}\right)\frac{dy}{dx}=3e^{3x}$ $2e^{2y}\frac{dy}{dx}=3e^{3x}$ $\left(\begin{array}{l}\because y=\ln\sqrt{1+e^{3x}}=\frac{1}{2}\ln\left(1+e^{3x}\right)\\ \text{i.e. } 2y=\ln\left(1+e^{3x}\right)\Rightarrow e^{2y}=1+e^{3x}\end{array}\right)$
(b)	Diff. wrt x, $2e^{2y}\frac{d^2y}{dx^2}+4e^{2y}\left(\frac{dy}{dx}\right)^2=9e^{3x}$ ---- (2) When $x=0, y=\frac{1}{2}\ln(1+1)=\frac{1}{2}\ln 2$, $e^{2y}=e^{\ln 2}=2$ Sub into (1): $\frac{dy}{dx}=\frac{3}{4}$ Sub into (2): $2(2)\frac{d^2y}{dx^2}+8\left(\frac{3}{4}\right)^2=9$ => $\frac{d^2y}{dx^2}=\frac{9}{8}$ $y=\frac{1}{2}\ln 2+\frac{3}{4}x+\frac{9}{16}x^2+\ldots$
(c)	$\ln\left(\frac{\sqrt{1+e^{-3x}}}{2}\right)=\ln\left(\sqrt{1+e^{3(-x)}}\right)-\ln 2$ $=-\ln 2+\frac{1}{2}\ln 2+\frac{3}{4}(-x)+\frac{9}{16}(-x)^2+\ldots$ $=-\frac{1}{2}\ln 2-\frac{3}{4}x+\frac{9}{16}x^2+\ldots$

2(a)	$\sum_{r=2}^{2n} \frac{2}{(2r-1)(2r+1)}$ Replace r by $r+1$, $= \sum_{r+1=2}^{r+1=2n} \frac{2}{(2(r+1)-1)(2(r+1)+1)}$ $= \sum_{r=1}^{r=2n-1} \frac{2}{(2r+1)(2r+3)}$ $= \sum_{r=2}^{2n-1} \frac{2}{(2r+1)(2r+3)} + \frac{2}{(2+1)(2+3)}$ $= \frac{1}{5} - \frac{1}{2(2n-1)+3} + \frac{2}{15}$ $= \frac{1}{3} - \frac{1}{4n+1}$
(b)(i)	$x_{n+1} = \sqrt{4+5x_n} \quad \text{for } n = 1, 2, 3, \ldots.$ Since sequence converges to L, as $n \to \infty,\ x_n \to L,\ x_{n+1} \to L.$ $L = \sqrt{4+5L}$ $L^2 = 4+5L \qquad \text{--- (*)}$ $L^2 - 5L - 4 = 0$ $L = \frac{5 \pm \sqrt{41}}{2}$ Since x_n are positive numbers, $L = \frac{5+\sqrt{41}}{2}$.
(ii)	$x_{n+1}{}^2 - L^2 = (4+5x_n) - (5L+4) \quad \text{(fr (*))}$ $= 5(x_n - L)$
(iii)	If $x_n > L$, $x_{n+1}{}^2 - L^2 = 5(x_n - L) > 0$ $(x_{n+1} - L)(x_{n+1} + L) > 0$ $\Rightarrow x_{n+1} - L > 0 \quad \text{since } x_{n+1} > 0,\ L > 0, \text{ so } x_{n+1} + L > 0$ $\Rightarrow x_{n+1} > L$

3(a)

$\mathrm{f}: x \mapsto \frac{1}{2} + \mathrm{e}^{-x},\ x \geq 0$ and a is a positive constant.

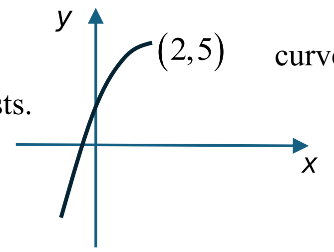


(b)

$\mathrm{g}: x \mapsto 1 + 4x - x^2,\ x \leq 2$

Every horizontal line will cut the curve at most once.

Hence g is 1-1. Therefore, g^{-1} exists.



(c)

$y = \mathrm{f}(x)$

$y = 5 - (x-2)^2$

$(x-2)^2 = 5 - y$

$x = 2 \pm \sqrt{5-y}$

Since $x \leq 2$, $\mathrm{g}^{-1}(x) = 2 - \sqrt{5-x}$

$\mathrm{D}_{\mathrm{g}^{-1}} = \mathrm{R}_{\mathrm{g}} = (-\infty, 5]$

(d)

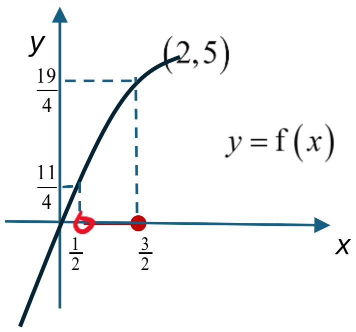
From graph of f in (i), $\mathrm{R_f} = \left(\frac{1}{2}, \frac{3}{2}\right] \subseteq (-\infty, 2] = D_g$

$\therefore$ gf exists.

(e)

$\mathrm{R_{gf}} = \left(5 - \left(\tfrac{1}{2} - 2\right)^2, 5 - \left(\tfrac{3}{2} - 2\right)^2\right]$

$= \left(\frac{11}{4}, \frac{19}{4}\right]$



4(a)	Circumference of base = length of arc AB $2\pi r = a\theta$ $r = \frac{a}{2\pi}\theta$
(b)	(i) Let h = height of cone $h^2 = a^2 - r^2$ $= a^2 - \frac{a^2}{4\pi^2}\theta^2$ $= \frac{a^2}{4\pi^2}\left(4\pi^2 - \theta^2\right)$ $V = \frac{1}{3}\pi r^2 h$ $V^2 = \frac{1}{9}\pi^2 r^4 h^2$ $= \frac{1}{9}\pi^2\left(\frac{a^4}{16\pi^4}\right)\cdot\frac{a^2}{4\pi^2}\left(4\pi^2 - \theta^2\right)$ $= \frac{a^6}{576\pi^4}\theta^4\left(4\pi^2 - \theta^2\right)$
(c)	**Method 1** $\frac{\mathrm{d}V^2}{\mathrm{d}\theta} = \frac{a^6}{576\pi^4}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(4\pi^2\theta^4 - \theta^6\right)$ $= \frac{a^6}{576\pi^4}\left(16\pi^2\theta^3 - 6\theta^5\right)$ $= \frac{a^6}{576\pi^4}\cdot 2\theta^3\left(8\pi^2 - 3\theta^2\right)$ $\frac{\mathrm{d}V^2}{\mathrm{d}\theta} = 0$ when $\theta^3\left(8\pi^2 - 3\theta^2\right) = 0$ $\theta^2 = \frac{8\pi^2}{3} \quad \Rightarrow \quad \theta = 2\pi\sqrt{\frac{2}{3}}$ $\frac{\mathrm{d}^2V^2}{\mathrm{d}\theta^2} = \frac{a^6}{576\pi^4}\left(48\pi^2\theta^2 - 30\theta^4\right)$ When $\theta = 2\pi\sqrt{\frac{2}{3}}$, $\frac{\mathrm{d}^2V^2}{\mathrm{d}\theta^2} = \frac{a^6}{576\pi^4}\left(48\pi^2\left(\frac{8\pi^2}{3}\right) - 30\left(\frac{64\pi^4}{9}\right)\right) = -\frac{4a^6}{27} < 0$ Therefore V is max when $\theta = 2\pi\sqrt{\frac{2}{3}}$ **Method 2**

	$V^2 = \frac{a^6}{576\pi^4}\theta^4\left(4\pi^2-\theta^2\right)$ Differentiating w.r.t. θ, $2V\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{a^6}{576\pi^4}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(4\pi^2\theta^4-\theta^6\right)$ $= \frac{a^6}{576\pi^4}\left(16\pi^2\theta^3-6\theta^5\right)$ $= \frac{a^6}{576\pi^4}\cdot 2\theta^3\left(8\pi^2-3\theta^2\right)$ $\frac{\mathrm{d}V}{\mathrm{d}\theta}=0$ when $\theta^3\left(8\pi^2-3\theta^2\right)=0$ $\theta^2 = \frac{8\pi^2}{3} \quad \Rightarrow \theta = 2\pi\sqrt{\frac{2}{3}}$ $(\because \theta > 0)$ $2V\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}+2\left(\frac{\mathrm{d}V}{\mathrm{d}\theta}\right)^2 = \frac{a^6}{576\pi^4}\left(48\pi^2\theta^2-30\theta^4\right)$ When $\theta = 2\pi\sqrt{\frac{2}{3}}$, $\frac{\mathrm{d}V}{\mathrm{d}\theta}=0$ $2V\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = \frac{a^6}{576\pi^4}\left(48\pi^2\left(\frac{8\pi^2}{3}\right)-30\left(\frac{64\pi^4}{9}\right)\right) = -\frac{4a^6}{27} < 0$ Since $V > 0$, $\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} < 0$ Therefore V is max when $\theta = 2\pi\sqrt{\frac{2}{3}}$
(d)	Note that: $\theta = 2\pi\sqrt{\frac{2}{3}}$ $r = \frac{a}{2\pi}\theta = \frac{a}{2\pi}\left(2\pi\sqrt{\frac{2}{3}}\right) = a\sqrt{\frac{2}{3}}$ $h^2 = a^2 - r^2 = a^2 - \left(a\sqrt{\frac{2}{3}}\right)^2 = \frac{1}{3}a^2 \quad \Rightarrow h = \frac{1}{\sqrt{3}}a$ $\frac{r}{h} = a\sqrt{\frac{2}{3}} \times \frac{\sqrt{3}}{a} = \sqrt{2} \quad \Rightarrow \quad r = \sqrt{2}h$ $W = \frac{1}{3}\pi r_w^2 h_w = \frac{1}{3}\pi\left(\sqrt{2}y\right)^2 y = \frac{2\pi y^3}{3}$
(e)	$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$

	$-\dfrac{\pi}{15} = 2\pi y^2 \times \dfrac{dy}{dt}$ $\dfrac{dy}{dt} = -\dfrac{1}{30y^2}$ When $y = 2$, $\dfrac{dy}{dt} = -\dfrac{1}{120}$ cm s^{-1} Rate of change of y is $-\dfrac{1}{120}$ cm s^{-1}.

Section B: Probability and Statistics [60 marks]

5	**[Solution]**
a)	Required Probability = P(Dave hits 2 targets and Rafael hits 0 or 1)+ P(Dave hits 1 and Rafael 0) $=\left(\frac{7}{10}\right)^2\left[\left(\frac{8}{10}\right)\left(\frac{2}{10}\right)(2)+\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)\right]+\left(\frac{7}{10}\right)\left(\frac{3}{10}\right)(2)\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)$ $=0.1932$
b)	P(Day was bad given Dave wins) $=\frac{(0.3)(0.39)}{(0.7)(0.1932)+(0.3)(0.39)}$ $=0.464 \quad \text{(3s.f.)}$

6	**[Solution]**
(a)(i)	Required number of ways = $\binom{12}{5}\binom{5}{1}$=3960
(a)(ii)	Required number of ways =$\binom{6}{2}\binom{6}{3}\binom{5}{1}=1500$ **Alternative method:** Case 1: 2 not identical at 3rd storey $\binom{6}{2}\times 2!\binom{6}{3}=600$ Case 2: 2 identical at 3rd storey $\binom{6}{2}\binom{6}{3}\frac{3!}{2!}=900$
(a)(iii)	Case 1: 2 in 1st storey, 1 in 2nd storey, 2 in 3rd storey Number of ways = $\binom{3}{2}\binom{3}{1}\binom{6}{2}\binom{5}{1}=675$ Case 2: 1 in 1st storey, 2 in 2nd storey, 2 in 3rd storey Number of ways = $\binom{3}{1}\binom{3}{2}\binom{6}{2}\binom{5}{1}=675$ Case 3: 1 in 1st storey, 1 in 2nd storey, 3 in 3rd storey Number of ways = $\binom{3}{1}\binom{3}{1}\binom{6}{3}\binom{5}{1}=900$ Total number of ways = 2250
(b)	Number of ways to arrange the 10 participants = $(5-1)!(2!)^5=768$ Number of ways to slot in the 2 game masters = $\binom{5}{2}$2!=20 Required number = $768\times 20=15360$

7	**Solution**
(a)	Let X be the number of faulty bowls in a box of 20 bowls. $X \sim \text{B}(20, 0.08)$ $\text{P}(X > 2) = 1 - \text{P}(X \le 2)$ $= 1 - 0.78795$ $= 0.21205$ (5 s.f.) $= 0.212$ (3 s.f.)
(b)	Let W be the number of faulty bowls in a carton (of 12 boxes). $W \sim \text{B}(12 \times 20, 0.08)$, i.e. $W \sim \text{B}(240, 0.08)$ $\text{P}(W < 15) = \text{P}(W \le 14) = 0.12933$ (5 s.f.) Required Probability $= {}^3C_2 (0.12933)^2 (1 - 0.12933)$ $= 0.0437$ (3 s.f.) **OR** Let A be the number of cartons out of 3 contains fewer than 15 faulty bowls each. $A \sim \text{B}(3, 0.12933)$ Required Probability = $\text{P}(A = 2) = 0.0437$ (3 s.f.)
(c)	$Y \sim \text{B}(12, 0.21205)$ $\text{E}(Y) = np = 12 \times 0.21205 = 2.5446 = 2.54$ (3 s.f.) $\text{Var}(Y) = np(1-p) = 12 \times 0.21205 \times (1 - 0.21205)$ $= 2.0050$ (5 s.f.) $= 2.01$ (3s.f.)
(d)	$Y \sim \text{B}(12, 0.21205)$ Let $T = Y_1 + ... + Y_{35}$. Since sample size 35 is large, by Central Limit Theorem, $T \sim \text{N}(35 \times 2.5446, 35 \times 2.0050)$ approximately i.e. $T \sim \text{N}(89.061, 70.175)$ approximately Required probability = $\text{P}(T \le 85) = 0.314$ (3 s.f.)

[Solutions]

8(a)

Let W and L be the marks obtained in the written paper and lab-based practical respectively.

$W \sim \mathrm{N}(62,\ \sigma^2)$ and $L \sim \mathrm{N}(56,\ 12^2)$

$\mathrm{P}(W \le 85) = 0.95$

$\mathrm{P}\left(Z \le \dfrac{85-62}{\sigma}\right) = 0.95$

Using GC, $\dfrac{85-62}{\sigma} = 1.64485$

$\sigma = 13.983$ (5 s.f.)

(b)

$W - L \sim \mathrm{N}(62-56,\ 13.983^2 + 12^2)$

$W - L \sim \mathrm{N}(6,\ 339.524)$

$\mathrm{P}\left(|W-L| \ge 9\right) = 1 - \mathrm{P}\left(-9 < W-L < 9\right) = 0.643$ (3 s.f.)

(c)

$T = 0.6W + 0.4L \sim \mathrm{N}\left(0.6\times 62 + 0.4\times 56,\ 0.6^2 \times 13.983^2 + 0.4^2 \times 12^2\right)$

$T \sim \mathrm{N}\left(59.6,\ 93.42874\right)$

$\mathrm{P}\left(T < 60\right) = 0.517$

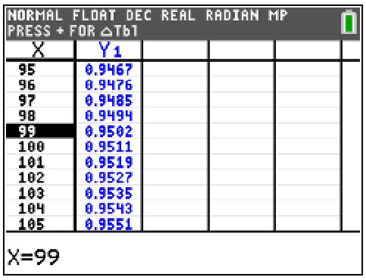
(d)

$\bar{T} = \dfrac{T_1 + T_2 + \ldots + T_n}{n} \sim \mathrm{N}\left(59.6,\ \dfrac{93.42874}{n}\right)$

$\mathrm{P}\left(\bar{T} \ge 58\right) \ge 0.95$

<u>Method 1</u>: From GC,

n	$\mathrm{P}\left(\bar{T} \ge 58\right)$
98	$0.9494 < 0.95$
99	$0.9502 > 0.95$



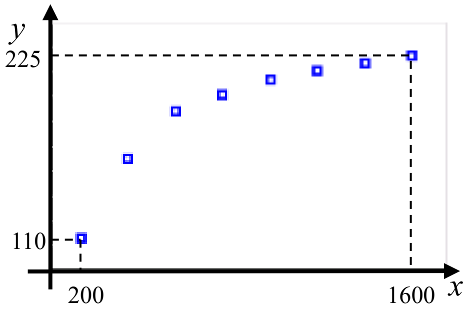
Therefore, the least n is 99.

<u>Method 2 (More tedious)</u>

	$\mathrm{P}\left(\bar{T} \geq 58\right) \geq 0.95$ $P\left(Z \geq \dfrac{58-59.6}{\sqrt{\dfrac{93.42874}{n}}}\right) \geq 0.95$ $\Rightarrow \dfrac{58-59.6}{\sqrt{\dfrac{93.42874}{n}}} \leq -1.64485$ $\Rightarrow \dfrac{-1.6\sqrt{n}}{\sqrt{93.42874}} \leq -1.64485$ $\Rightarrow \dfrac{1.6\sqrt{n}}{\sqrt{93.42874}} \geq 1.64485$ $\Rightarrow n \geq 98.746$ Therefore, the least n is 99.
(e)	The assumption is that a candidate's mark for the written paper and lab-based practical are independent. This may not be valid in practice as the a candidate who does well for the written paper is more likely to do better in the lab-based practical too.

9(a)	Assume the queuing time follows normal distribution.
(b)	Using GC, $\bar{x} = 4.24$ (exact) Let μ be the population mean queuing time and Y be the queuing time of a randomly chosen student. Null hypothesis $H_0 : \mu = 4.5$ Alternative hypothesis $H_1 : \mu < 4.5$ (manager's claim) At 5 % significance level. Under H_0, $\bar{X} \sim N\left(4.5, \frac{0.48}{10}\right)$ Test statistic: $Z = \frac{\bar{X} - 4.5}{\sqrt{\frac{0.48}{10}}} \sim N(0,1)$ From GC, p-value $= 0.118 > 0.05$ $\therefore$ do not reject H_0 There is insufficient evidence at 5% level of significance to conclude that the manager's claim is justified.
(c)	The sample is not random because (one of the following reasons): • Not all the students from the school were given equal chance to be selected. • Students who are friends or from the same class may visit the same food stall together. Thus they are not selected independently.
(d)	$s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$ $= \frac{1}{49}\left(1068 - \frac{(4.6 \times 50)^2}{50}\right)$ $= 0.20408 \approx 0.204$
(e)	Assumption in (a) is no longer necessary because sample size 50 is large, by Central Limit Theorem, the sample mean queuing time now follows normal distribution approximately.
(g)	Null hypothesis $H_0 : \mu = \lambda$ Alternative hypothesis $H_1 : \mu > \lambda$ (student leader's claim) At 2 % significance level. Under H_0, $\bar{X} \sim N\left(\lambda, \frac{0.20408}{50}\right)$ approximately by Central limit theorem since $n = 50$ large.

	Test statistic: $Z = \dfrac{\overline{X} - \lambda}{\sqrt{\dfrac{0.20408}{50}}} \sim \mathrm{N}(0,1)$ Critical region = $\{z \in \mathbb{R} : z \geq 2.05375\}$ since upper tail test To reject H_0, $\dfrac{4.6 - \lambda}{\sqrt{\dfrac{0.20408}{50}}} \geq 2.05375$ $0 < \lambda \leq 4.47$

10(a)	
(b)	$r = 0.894$ (3 s.f.) Even though the r-value indicates a strong positive linear correlation between x and y, the scatter diagram indicates that a curvilinear correlation exists between x and y. Therefore the linear model may not be the best.
(d)	Model I: As x increases, y increases at an increasing rate. Model II: As x increases, y increases at a decreasing rate. Model III: As x increases, y decreases. Hence, only Model II can be used to model the relationship between x and y as it is the only one matches the trend of data shown in the scatter diagram. <u>Alternative solution:</u> Since the graph of $y = a + b\ln x$ is concave downwards whereas the graph of $y = a + bx^2$ and $y = a + be^{-x}$ are concave upwards, the graph of $y = a + b\ln x$ will be more suitable to describe the scatter diagram of s and y. Hence Model II is more suitable.
(e)	The regression line of y on $\ln x$ is $y = 54.269\ln x - 168.22$, i.e. $y = 54.3\ln x - 168$ (to 3 s.f.)
(f)	$y = 54.269\ln 700 - 168.22 = 187.30 \approx 187$ (3 s.f.) The input data $x = 700$ is within the data range **[200, 1600]** of the sample. Furthermore, there is a strong positive linear correlation as indicated by $r = 0.984$. Therefore the estimation is reliable.
(g)	It will remain the same.