1 Using an algebraic method, solve the inequality $\dfrac{4x-3}{4x^2+3x-1} \le 1$. [3]

Hence, find the set of values of x that satisfy $\dfrac{4\ln x-3}{4(\ln x)^2+3\ln x-1} \le 1$. [2]

<table>
<tr><th>No.</th><th>Solution</th></tr>
<tr><td>1</td><td>

$$\frac{4x-3}{4x^2+3x-1} \le 1$$

$$\frac{4x-3}{4x^2+3x-1} - 1 \le 0$$

$$\frac{4x-3-4x^2-3x+1}{(x+1)(4x-1)} \le 0$$

$$\frac{-4x^2+x-2}{(x+1)(4x-1)} \le 0$$

$$\frac{4x^2-x+2}{(x+1)(4x-1)} \ge 0$$

$$4x^2-x+2 = 4\left[\left(x-\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2\right] + 2$$

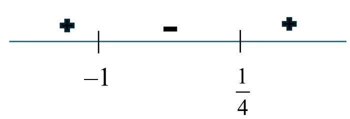
$$= 4\left(x-\frac{1}{8}\right)^2 + \frac{31}{16}$$

Since $\left(x-\frac{1}{8}\right)^2 \ge 0$ for all real values of x, $4\left(x-\frac{1}{8}\right)^2 + \frac{31}{16} > 0$

OR

Since coefficient of x^2 is $4>0$ and the discriminant of $4x^2-x+2$ is $(-1)^2-4(4)(2) = -31 < 0$, $4x^2-x+2>0$ for all $x \in \mathbb{R}$.

We have $(x+1)(4x-1) > 0$.



$x < -1$ or $x > \frac{1}{4}$

Replace x with $\ln x$:

$\ln x < -1$ or $\ln x > \frac{1}{4}$

$0 < x < \mathrm{e}^{-1}$ or $x > \mathrm{e}^{\frac{1}{4}}$

</td></tr>
</table>

2 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x-1}{x}, \quad x \in \mathbb{R},\ x \neq 0,\ x \neq 1.$$

(a) Show that $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$. [3]

(b) Find $\mathrm{f}^3(x)$ in simplified form. [1]

(c) Find $\mathrm{f}^{2030}(5)$. [2]

Functions g and h are defined by

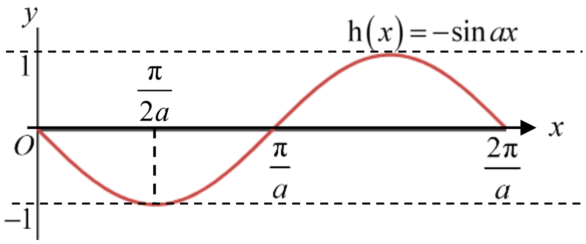
$$\mathrm{g}: x \mapsto \frac{x-1}{x}, \quad x \in \mathbb{R},\ x \geq 1,$$

$$\mathrm{h}: x \mapsto -\sin ax, \quad x \in \mathbb{R},$$

where a is a positive constant.

(d) Find the value of a given that the range of hg is $(-1, 0]$. [2]

2a	Let $y = \frac{x-1}{x} = 1 - \frac{1}{x}$ $\frac{1}{x} = 1 - y$ $x = \frac{1}{1-y}$ Hence $\mathrm{f}^{-1}(x) = \frac{1}{1-x}$ $\mathrm{f}^2(x) = \mathrm{f}\left(\frac{x-1}{x}\right)$ $= \frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}}$ $= \frac{x-1-x}{x-1}$ $= \frac{1}{1-x}$ $\therefore \mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$
2b	$\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$ $\mathrm{f}^3(x) = \mathrm{f}\,\mathrm{f}^2(x)$ $= \mathrm{f}\,\mathrm{f}^{-1}(x)$ $= x$
2c	$\mathrm{f}^{2030}(5) = \mathrm{f}^{3(676)+2}(5) = \mathrm{f}^2(5) = \frac{1}{1-5} = -\frac{1}{4}$

2d	Range of g = $[0,1)$ = new domain of h  Given: range of hg is $(-1,0]$ $\frac{\pi}{2a}=1 \quad \Rightarrow a=\frac{\pi}{2}$

3 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, such that $OC:CA=2:1$. Point D lies on OB, such that $OD:DB=\lambda:\mu$. It is given that the area of triangle ABD is half the area of triangle ABC.

(a) Show the area of triangle ABD is given by $\frac{\mu}{2(\lambda+\mu)}|\mathbf{a}\times\mathbf{b}|$. Hence find the ratio $\lambda:\mu$. [4]

(b) The point E has position vector $\frac{1}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}$. Show that A, E and D are collinear. [3]

It is further given that the angle AOB is $\frac{\pi}{4}$ and O lies on the perpendicular bisector of the line segment AB.

(c) Find the length of projection of $\mathbf{a}$ on $\mathbf{b}$, giving your answer in terms of $|\mathbf{b}|$. Hence find the position vector of the point F, the foot of perpendicular from A to OB. [3]

No.	Solution
3a	Area of triangle $ABD=\frac{1}{2}\left|\overrightarrow{DB}\times\overrightarrow{AB}\right|$ $=\frac{1}{2}\left|\frac{\mu}{\lambda+\mu}\underset{\sim}{b}\times(\underset{\sim}{b}-\underset{\sim}{a})\right|$ $=\frac{\mu}{2(\lambda+\mu)}\left|\underset{\sim}{b}\times\underset{\sim}{b}-\underset{\sim}{b}\times\underset{\sim}{a}\right|$ $=\frac{\mu}{2(\lambda+\mu)}\left|\underset{\sim}{0}+\underset{\sim}{a}\times\underset{\sim}{b}\right|$ $=\frac{\mu}{2(\lambda+\mu)}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|$ Area of triangle $ABC=\frac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{AB}\right|$ $=\frac{1}{2}\left|\frac{1}{3}\underset{\sim}{a}\times(\underset{\sim}{b}-\underset{\sim}{a})\right|$ $=\frac{1}{6}\left|\underset{\sim}{a}\times\underset{\sim}{b}-\underset{\sim}{a}\times\underset{\sim}{a}\right|$ $=\frac{1}{6}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|$

	Area of triangle ABD = 1/2 (area of triangle ABC) $\frac{\mu}{2(\lambda+\mu)}\lvert \underset{\sim}{a}\times\underset{\sim}{b}\rvert=\frac{1}{2}\left(\frac{1}{6}\lvert \underset{\sim}{a}\times\underset{\sim}{b}\rvert\right)$ $\frac{\mu}{2(\lambda+\mu)}=\frac{1}{12}$ $6\mu=\lambda+\mu$ $5\mu=\lambda$ $\lambda:\mu=5:1$
3b	$\overrightarrow{AE}=\overrightarrow{OE}-\overrightarrow{OA}$ $=\frac{1}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}-\mathbf{a}$ $=-\frac{3}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}$ $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$ $=-\mathbf{a}+\frac{5}{6}\mathbf{b}$ $=\frac{4}{3}\left(-\frac{3}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}\right)$ Since $\overrightarrow{AD}=\frac{4}{3}\overrightarrow{AE}$ and A is a common point, A, E and D are collinear.
3c	Length of projection of **a** on **b** $=\lvert\mathbf{a}\ .\ \mathbf{b}\rvert$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\frac{\pi}{4}$ $=\lvert\mathbf{a}\rvert\cos\frac{\pi}{4}$ $=\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert$ $\overrightarrow{OF}=\left(\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert\right)\mathbf{b}$ $=\frac{1}{\sqrt{2}}\mathbf{b}$

4 **(a)** A sequence is such that $u_1 = p$, where p is a constant and $u_{n+1} = \dfrac{5u_n}{8u_n + 1}$, for $n \ldots 1$.

(i) Describe how the sequence behaves when $p = 1$. [2]

(ii) Find the value of p for which $u_6 = \dfrac{3125}{6253}$. [2]

(b) Another sequence v_1, v_2, v_3, … is such that for all $n \ldots 1$, $v_{n+2} - 2v_{n+1} + v_n = k$, where k is a constant.

Let $w_n = v_{n+1} - v_n$ for $n \ldots 1$. Explain why the sequence $\{w_n\}$ is an arithmetic progression. [2]

No.	Solution
4ai	When $p = 1$, $u_1 = 1,\ u_2 = \frac{5}{9},\ u_3 = \frac{25}{49},\ u_4 = \frac{125}{249},\ u_5 = \frac{625}{1245}, \ldots$ The sequence is a decreasing sequence, and it converges to $\frac{1}{2}$.
4aii	$u_{n+1} = \frac{5u_n}{8u_n + 1} \quad \Rightarrow \frac{1}{u_{n+1}} = \frac{8u_n + 1}{5u_n} = \frac{8}{5} + \frac{1}{5u_n}$ $\frac{1}{5u_n} = \frac{1}{u_{n+1}} - \frac{8}{5} = \frac{5 - 8u_{n+1}}{5u_{n+1}}$ $5u_n = \frac{5u_{n+1}}{5 - 8u_{n+1}}$ $u_n = \frac{u_{n+1}}{5 - 8u_{n+1}}$ Using the GC, $u_1 = \frac{1}{5}$ $\therefore p = \frac{1}{5}$
4b	$w_{n+1} - w_n = v_{n+2} - v_{n+1} - (v_{n+1} - v_n)$ $= v_{n+2} - 2v_{n+1} + v_n$ $= k$ (which is a constant) Hence $\{w_n\}$ is an arithmetic progression.

5 The function f is given by

$$\mathrm{f}(x)=\begin{cases}2+\sqrt{2^2-(x-2)^2}, & 0<x\le 4,\\ 2-\sqrt{2^2-(x-6)^2}, & 4<x\le 8.\end{cases}$$

It is given that $\mathrm{f}(x)=\mathrm{f}(x+8)$ for all real values of x.

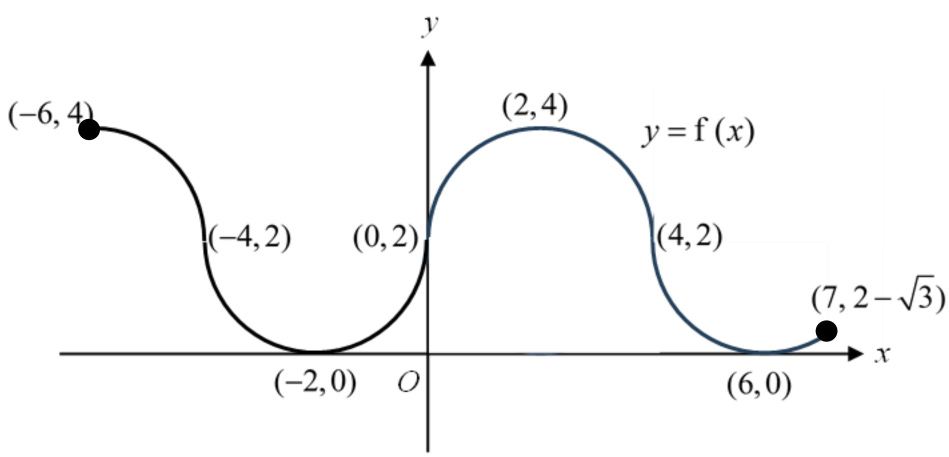
(a) On the diagram in the Printed Answer Book, sketch the graph of $y=\mathrm{f}(x)$ for $-6\le x\le 7$, indicating clearly the coordinates of the end points and the points where the graph cuts the axes. [3]

(b) Without integrating, write down the exact area of the region bounded by $y=\mathrm{f}(x)$, the line $x=4$, the x-axis and the y-axis. [1]

The curve C has equation $\dfrac{(y-3)^2}{a^2}-(x+2)^2=1$, where a is a positive real constant.

(c) State the equations of the asymptotes of C in terms of a. [1]

(d) Determine the range of values of a if there is at most one intersection between C and the graph of $y=\mathrm{f}(x)$. [2]

No.	Solution
5a	
5b	Area $=2\times4+\frac{1}{2}\pi(2^2)=8+2\pi$
5c	Equations of asymptotes: $\dfrac{(y-3)^2}{a^2}=(x+2)^2$ $y-3=\pm a(x+2)$ $y=3+a(x+2)$ and $y=3-a(x+2)$
5d	C is a hyperbola centred at $(-2, 3)$ with vertices at $(-2, 3+a)$ and $(-2, 3-a)$. When $a=3$, there is exactly one intersection at $(-2, 0)$. Hence there is at most one intersection between C and the graph of $y=\mathrm{f}(x)$ when $a\ge 3$.

6 It is given that $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n}$.

(a) Show that $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)}$ is less than $\frac{1}{4}$. [2]

(b) Give a reason why the series $\sum_{r=2}^{\infty} \frac{1}{r\left(r^2-1\right)}$ converges, and write down its value. [2]

(c) Find the smallest value of n for which $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)}$ differs from $\sum_{r=2}^{\infty} \frac{1}{r\left(r^2-1\right)}$ by less than 0.0007. [2]

(d) Find $\sum_{r=m+1}^{N} \frac{1}{(r-m)(r-m+1)(r-m+2)}$, where m and N are integers with $N > m > 0$.
(There is no need to express your answer as a single algebraic fraction.) [2]

No.	Solution
6a	$\frac{1}{2(n+1)} - \frac{1}{2n} = \frac{n-(n+1)}{2n(n+1)} = \frac{-1}{2n(n+1)} < 0 \quad \text{for all} \quad n \geq 2$ $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n} < \frac{1}{4}$ **Alternative** $2(n+1) > 2n \quad$ for all $n \geq 2$ $\frac{1}{2(n+1)} < \frac{1}{2n}$ $\frac{1}{2(n+1)} - \frac{1}{2n} < 0$ $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n} < \frac{1}{4}$
6b	As $n \to \infty$, $\frac{1}{2(n+1)} \to 0$ and $\frac{1}{2n} \to 0$, $\therefore \sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)} \to \frac{1}{4}$, a constant. Hence $\sum_{r=2}^{\infty} \frac{1}{r\left(r^2-1\right)}$ converges and $\sum_{r=2}^{\infty} \frac{1}{r\left(r^2-1\right)} = \frac{1}{4}$.

6c

$$\left|\sum_{r=2}^{n}\frac{1}{r\left(r^2-1\right)}-\sum_{r=2}^{\infty}\frac{1}{r\left(r^2-1\right)}\right|<0.0007$$

$$\left|\frac{1}{4}+\frac{1}{2(n+1)}-\frac{1}{2n}-\frac{1}{4}\right|<0.0007$$

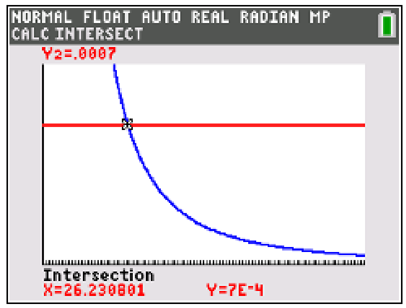
$$\left|\frac{1}{2(n+1)}-\frac{1}{2n}\right|<0.0007$$

Using GC:

n	$\left\lvert\frac{1}{2(n+1)}-\frac{1}{2n}\right\rvert$
26	$0.000712>0.0007$
27	$0.000661<0.0007$

smallest value of $n=27$

Alternative



$n>26.231 \Rightarrow$ smallest value of $n=27$

6d

$$\sum_{r=m+1}^{N}\frac{1}{(r-m)(r-m+1)(r-m+2)}$$

$$=\frac{1}{1(2)(3)}+\frac{1}{2(3)(4)}+\ldots+\frac{1}{(N-m)(N-m+1)(N-m+2)}$$

$$=\sum_{r=2}^{N-m+1}\frac{1}{(r-1)(r)(r+1)}$$

$$=\sum_{r=2}^{N-m+1}\frac{1}{r\left(r^2-1\right)}$$

$$=\frac{1}{4}+\frac{1}{2(N-m+2)}-\frac{1}{2(N-m+1)}$$

<u>Alternative</u>

$$\sum_{r=m+1}^{N}\frac{1}{(r-m)(r-m+1)(r-m+2)}$$

$$=\sum_{r+m-1=m+1}^{r+m-1=N}\frac{1}{(r-1)(r)(r+1)}$$

$$=\sum_{r=2}^{N-m+1}\frac{1}{(r)\left(r^2-1\right)}$$

$$=\frac{1}{4}+\frac{1}{2(N-m+2)}-\frac{1}{2(N-m+1)}$$

7 Do not use a calculator in answering this question.

(a) Find the complex number z which satisfies the equation $\dfrac{4|z|}{15-z^*}=5\mathrm{i}$. [3]

(b) The complex number w is such that $(w-\mathrm{i})^3=-\mathrm{i}$.

(i) Given that one possible value of w is $2\mathrm{i}$, find the two other possible values of w. Give your answers in cartesian form $a+b\mathrm{i}$. [4]

The points W_1, W_2 and W_3 on the Argand diagram represent the three roots of the equation $(w-\mathrm{i})^3=-\mathrm{i}$, and the point A represents the complex number $k\mathrm{i}$, where k is a positive real number.

(ii) Show that the points W_1, W_2 and W_3 lie on a circle with centre A on the Argand diagram for some value of k, stating the value of k. [2]

No.	Solution
7a	Let $z=a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. $\dfrac{4\lvert z\rvert}{15-z^*}=5\mathrm{i}$ $\dfrac{4\sqrt{a^2+b^2}}{15-(a-b\mathrm{i})}=5\mathrm{i}$ $4\sqrt{a^2+b^2}=75\mathrm{i}-5a\mathrm{i}-5b=-5b+(75-5a)\mathrm{i}$ Comparing real and imaginary parts: $\begin{cases}4\sqrt{a^2+b^2}=-5b & ---(1)\\ 75-5a=0\Rightarrow a=15 & ---(2)\end{cases}$ Substitute (2) into (1): $4\sqrt{15^2+b^2}=-5b$ $4^2\left(15^2+b^2\right)=25b^2$ $9b^2=3600$ $b^2=400$ $b=-20 \qquad (\because b<0 \text{ by eqn. } (1))$ $\therefore z=15-20\mathrm{i}$

7bi	$(w-\mathrm{i})^3=-\mathrm{i}$ $w^3+3w^2(-\mathrm{i})+3w(-\mathrm{i})^2+(-\mathrm{i})^3+\mathrm{i}=0$ $w^3-3\mathrm{i}w^2-3w+2\mathrm{i}=0$ Since 2i is a root, $w^3-3\mathrm{i}w^2-3w+2\mathrm{i}=0=(w-2\mathrm{i})(Aw^2+Bw+C)$ By comparing coefficients, $A=1,C=-1$ $C-2\mathrm{i}B=-3\Rightarrow B=\dfrac{-1+3}{2\mathrm{i}}=-\mathrm{i}$ $w^2-\mathrm{i}w-1=0$ $w=\dfrac{\mathrm{i}\pm\sqrt{(-\mathrm{i})^2-4(-1)}}{2}$ $w=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}$ or $-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}$
7bii	Let W_1, W_2, W_3 and A represent 2i, $\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}$, $-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}$ and ki respectively. For $W_1A=W_2A$, i.e. $\left\|\dfrac{\sqrt{3}+\mathrm{i}}{2}-k\mathrm{i}\right\|=\|2\mathrm{i}-k\mathrm{i}\|$, $\left(\dfrac{\sqrt{3}}{2}\right)^2+\left(\dfrac{1}{2}-k\right)^2=(2-k)^2\Rightarrow\dfrac{3}{4}+1-k+k^2=4-4k+k^2\Rightarrow k=1$. So, we have $W_1A=\|2\mathrm{i}-\mathrm{i}\|=1$ and $W_2A=\left\|\dfrac{\sqrt{3}+\mathrm{i}}{2}-\mathrm{i}\right\|=1$. Check that $W_3A=1$ too: $\left\|\dfrac{-\sqrt{3}+\mathrm{i}}{2}-\mathrm{i}\right\|=\left\|\dfrac{-\sqrt{3}-\mathrm{i}}{2}\right\|=\sqrt{\left(\dfrac{-\sqrt{3}}{2}\right)^2+\left(\dfrac{-1}{2}\right)^2}=1$ Since the distances are all equal to 1, W_1, W_2 and W_3 lie on a circle with radius 1, centred at A representing the complex number i.

8 **(a)** A curve C with equation $y = \mathrm{f}(x)$ undergoes in succession, the following transformations.

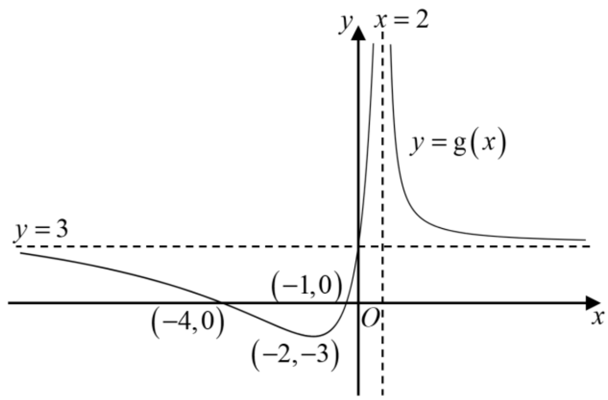
A: A reflection in the x-axis.

B: A stretch parallel to the x-axis with scale factor $\frac{1}{2}$, with the y-axis invariant.

The resulting curve has equation $y = ax^2 + \frac{b}{x}$, where a and b are real constants.

Given that $\left(-1, \frac{1}{3}\right)$ is a turning point of $y = \frac{1}{\mathrm{f}(x)}$, find the values of a and b and state the equation of C. [5]

(b) The diagram below shows the curve of $y = \mathrm{g}(x)$. The curve has a minimum point at $(-2, -3)$ and crosses the x-axis at $(-1, 0)$ and $(-4, 0)$. The line $x = 2$ is the vertical asymptote and the line $y = 3$ is the horizontal asymptote.



(i) Sketch the graph of $y = \mathrm{g}'(x)$, labelling the coordinates of all relevant point(s) and state the equations of any asymptotes. [2]

(ii) Find the area of the region bounded by the graph of $y = \mathrm{g}'(x)$, the lines $x = -4$, $x = -2$ and the x-axis. [2]

No.	Solution
8a	$y = ax^2 + \frac{b}{x}$ $\downarrow$ Before B $\left(\text{replace } x \text{ with } \frac{x}{2}\right)$ $y = a\left(\frac{x}{2}\right)^2 + \frac{b}{x/2}$ $y = \frac{ax^2}{4} + \frac{2b}{x}$ $\downarrow$ Before A (replace y with $-y$) $y = -\frac{ax^2}{4} - \frac{2b}{x} = \mathrm{f}(x)$ $\left(-1, \frac{1}{3}\right)$ turning point on $y = \frac{1}{\mathrm{f}(x)}$ $\Rightarrow$ $(-1, 3)$ turning point on $y = \mathrm{f}(x)$: $-\frac{a}{4} + 2b = 3$ -------------- (1)

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}ax + \frac{2b}{x^2}$

Since when $x = -1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\frac{1}{2}a + 2b = 0$. -------------- (2)

Solving (1), (2) using a GC, $a = -4,\ b = 1$. Therefore, $y = \mathrm{f}(x) = x^2 - \frac{2}{x}$.

Alternative

$y = \mathrm{f}(x)$

$\downarrow$ A (replace y with $-y$)

$y = -\mathrm{f}(x)$

$\downarrow$ B (replace x with $2x$)

$y = -\mathrm{f}(2x) = ax^2 + \frac{b}{x}$

$\mathrm{f}(2x) = -\frac{a(2x)^2}{4} - \frac{2b}{2x} \quad \Rightarrow \quad \mathrm{f}(x) = -\frac{ax^2}{4} - \frac{2b}{x}$

$y = \frac{1}{\mathrm{f}(x)} = \frac{-x}{\frac{a}{4}x^3 + 2b}$

Subst. $\left(-1, \frac{1}{3}\right)$ into $y = \frac{1}{\mathrm{f}(x)}$: $\frac{1}{3} = \frac{1}{-\frac{a}{4} + 2b} \Rightarrow -\frac{a}{4} + 2b = 3$ -------------- (1)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(\frac{a}{4}x^3 + 2b\right)(-1) - (-x)\left(\frac{3a}{4}x^2\right)}{\left(\frac{a}{4}x^3 + 2b\right)^2}$$

When $x = -1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\left(\frac{a}{4}(-1)^3 + 2b\right)(-1) - (-(-1))\left(\frac{3a}{4}(-1)^2\right) = 0$.

$\frac{a}{2} + 2b = 0$ -------------- (2)

Solving (1), (2) using a GC, $a = -4,\ b = 1$. Therefore, $y = \mathrm{f}(x) = x^2 - \frac{2}{x}$.

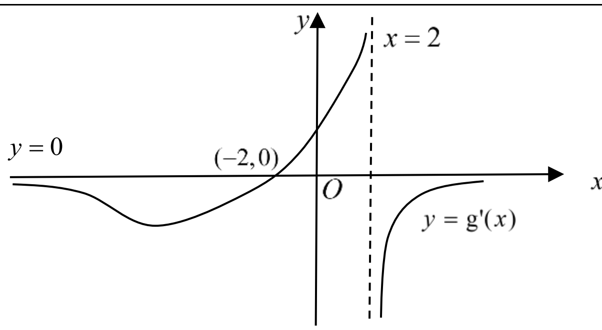
Alternative

Since $\left(-1, \frac{1}{3}\right)$ is a turning point of $y = \frac{1}{\mathrm{f}(x)}$, $(-1, 3)$ is a turning point of $y = \mathrm{f}(x)$.

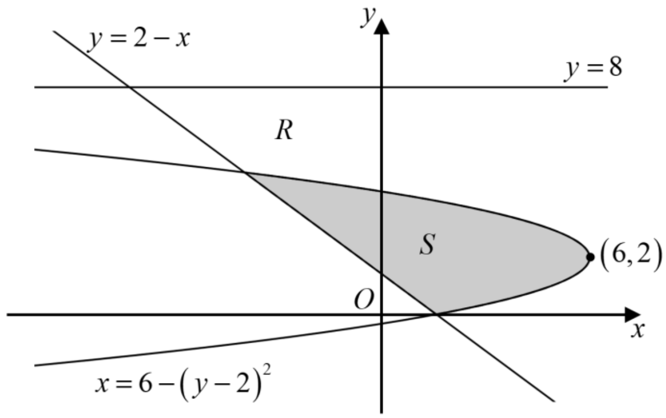
$(-1,3) \xrightarrow[A]{} (-1,-3) \xrightarrow[B]{} \left(-\frac{1}{2}, -3\right)$, a turning point on $y = ax^2 + \frac{b}{x}$

$y = ax^2 + \frac{b}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2ax - \frac{b}{x^2}$

Subst. $\left(-\frac{1}{2}, -3\right)$ into $y = ax^2 + \frac{b}{x}$: $\frac{a}{4} - 2b = -3$ -------------- (1)

	When $x=-\frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$: $-a-4b=0$. -------------- (2) Solving (1), (2) using a GC, $a=-4,\ b=1$. Therefore, the transformed graph has equation $y=-4x^2+\frac{1}{x}$. $y=-4x^2+\frac{1}{x}\xrightarrow[B^{-1}]{}y=-x^2+\frac{2}{x}\xrightarrow[A^{-1}]{}C: y=x^2-\frac{2}{x}$
8bi	
8bii	Required area $=-\int_{-4}^{-2}\mathrm{g}'(x)\ \mathrm{d}x=-\left[\mathrm{g}(x)\right]_{-4}^{-2}$ $=-\left[\mathrm{g}(-2)-\mathrm{g}(-4)\right]$ (from graph in the question) $=-\left[-3-0\right]=3$ units^2

9 In the diagram below, the region R is bounded by the curve C with equation $x=6-(y-2)^2$, the lines $y=8$, $y=2-x$ and the y-axis. The shaded region S is bounded by C and the line $y=2-x$.



(a) Find the exact area of region R. [5]

(b) Find the volume of the solid of revolution formed when S is rotated through 360° about the x-axis, leaving your answer to 2 decimal places. [3]

No.	Solution
9a	For point of intersection of C and $y=2-x$: $2-y=6-(y-2)^2$ $2-y=6-y^2+4y-4$ $y^2-5y=0$ $y(y-5)=0$ $y=0$ or $y=5$ When $x=0,\ 6-(y-2)^2=0 \Rightarrow y=2\pm\sqrt{6}$. Area of R $=-\left[\int_{2+\sqrt{6}}^{5} 6-(y-2)^2\,\mathrm{d}y+\int_5^8 2-y\ \mathrm{d}y\right]$ OR $\dfrac{6+3}{2}\times 3-\int_{2+\sqrt{6}}^{5} 6-(y-2)^2\,\mathrm{d}y$ $=-\left\{\left[6y-\dfrac{(y-2)^3}{3}\right]_{2+\sqrt{6}}^{5}+\left[2y-\dfrac{y^2}{2}\right]_5^8\right\}$ $=-\left[(30-9)-\left(6(2+\sqrt{6})-\dfrac{(\sqrt{6})^3}{3}\right)+\left(16-\dfrac{64}{2}\right)-\left(10-\dfrac{25}{2}\right)\right]$ $=\dfrac{9}{2}+4\sqrt{6}$ units^2 Alternative method: For point of intersection of C and $y=2-x$: $2-y=6-(y-2)^2$ $2-y=6-y^2+4y-4$ $y^2-5y=0$ $y(y-5)=0$ $y=0$ or $y=5$ $\therefore x=-3$ or 2

	When $y=8,\ 8=2-x \Rightarrow x=-6$. Area of R $=(6)(8)-\int_{-6}^{-3}(2-x)\,\mathrm{d}x-\int_{-3}^{0}\left(2+\sqrt{6-x}\right)\mathrm{d}x$ $=48-\left[2x-\frac{x^2}{2}\right]_{-6}^{-3}-\left[2x-\frac{(6-x)^{\frac{3}{2}}}{3/2}\right]_{-3}^{0}$ $=48-\frac{39}{2}-\left(-\frac{2}{3}(6)^{\frac{3}{2}}-\left(-6-\frac{2}{3}(9)^{\frac{3}{2}}\right)\right)$ $=\frac{57}{2}-\left(24-\frac{2}{3}\left(6\sqrt{6}\right)\right)$ $=\frac{9}{2}+4\sqrt{6}$
9b	$x=6-(y-2)^2$ $y=2\pm\sqrt{6-x}$ Volume of solid $=\pi\int_{-3}^{6}\left(2+\sqrt{6-x}\right)^2\mathrm{d}x-\pi\int_{-3}^{2}(2-x)^2\,\mathrm{d}x-\pi\int_{2}^{6}\left(2-\sqrt{6-x}\right)^2\mathrm{d}x$ $=327.25$ units^3

10 **(a)** **(i)** Express $1+4x$ in the form $A(2x-2)+B$, where A and B are constants to be determined. [1]

(ii) Hence, find $\int \frac{1+4x}{x^2-2x+5}\,\mathrm{d}x$. [4]

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} x\sin 2x\,\mathrm{d}x$. [3]

No.	Solution
10ai	$1+4x=A(2x-2)+B$ $2Ax-2A+B=4x+1$ $2A=4 \Rightarrow A=2$ and $B-2A=1 \Rightarrow B=5$ $1+4x=2(2x-2)+5$
10a ii	$\int \frac{1+4x}{x^2-2x+5}\,\mathrm{d}x=\int \frac{2(2x-2)+5}{x^2-2x+5}\,\mathrm{d}x$ $=\int \frac{2(2x-2)}{x^2-2x+5}\,\mathrm{d}x+\int \frac{5}{x^2-2x+5}\,\mathrm{d}x$ $=2\int \frac{2x-2}{x^2-2x+5}\,\mathrm{d}x+5\int \frac{1}{(x-1)^2+4}\,\mathrm{d}x$ $=2\ln(x^2-2x+5)+\frac{5}{2}\tan^{-1}\frac{x-1}{2}+C$ Note: $x^2-2x+5>0$ for all real x.
10b	$\int_0^{\frac{\pi}{3}} x\sin 2x\,\mathrm{d}x=\left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{3}}+\frac{1}{2}\int_0^{\frac{\pi}{3}}\cos 2x\,\mathrm{d}x$ $=\left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{3}}+\frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{3}}$ $\boxed{u=x \quad \frac{\mathrm{d}v}{\mathrm{d}x}=\sin 2x \quad \frac{\mathrm{d}u}{\mathrm{d}x}=1 \quad v=-\frac{1}{2}\cos 2x}$ $=\left[-\frac{1}{2}\left(\frac{\pi}{3}\right)\cos\frac{2\pi}{3}\right]+\frac{1}{4}\left[\sin 2\left(\frac{\pi}{3}\right)\right]$ $=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$

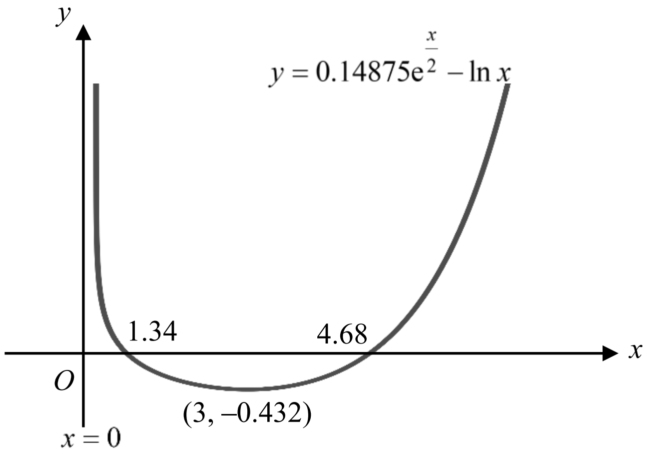
11 It is given that $2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2}{x}-\ln x=y$.

(a) Use the substitution $y=\ln\left(\frac{u}{x}\right)$ to show that the differential equation can be reduced to $\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{f}(u)$, where the function $\mathrm{f}(u)$ is to be found. [3]

(b) Given that y has a minimum value at $x=3$, solve the differential equation $2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2}{x}-\ln x=y$, to find the particular solution for y in terms of x. [5]

(c) Sketch the graph of this particular solution. [2]

No.	Solution
11a	$y=\ln\left(\frac{u}{x}\right) \Rightarrow y=\ln u-\ln x \qquad --------(1)$ Differentiating w.r.t. x: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{1}{x} \qquad ---(2)$ Subst. (1) and (2) into given DE $2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2}{x}-\ln x=y$: $2\left(\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{1}{x}\right)+\frac{2}{x}-\ln x=\ln u-\ln x$ $2\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}=\ln u$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}u\ln u$
11b	$\int\frac{\frac{1}{u}}{\ln u}\,\mathrm{d}u=\int\frac{1}{2}\mathrm{d}x$ $\ln\left(\lvert\ln u\rvert\right)=\frac{x}{2}+C$ $\lvert\ln u\rvert=B\mathrm{e}^{\frac{x}{2}}$, where $B=\mathrm{e}^{C}$ $\ln u=A\mathrm{e}^{\frac{x}{2}}$ From (1): $y+\ln x=A\mathrm{e}^{\frac{x}{2}} \Rightarrow y=A\mathrm{e}^{\frac{x}{2}}-\ln x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{A}{2}\mathrm{e}^{\frac{x}{2}}-\frac{1}{x}$ When $x=3$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$: $0=\frac{A}{2}\mathrm{e}^{\frac{3}{2}}-\frac{1}{3}$. $A=\frac{2}{3}\mathrm{e}^{-\frac{3}{2}}=0.14875$ $\therefore y=0.149\mathrm{e}^{\frac{x}{2}}-\ln x$ **<u>Alternative</u>** When $x=3$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. Sub into DE: $y=0+\frac{2}{3}-\ln 3=-0.43195$ Sub into (3): $-0.43195=A\mathrm{e}^{\frac{3}{2}}-\ln 3$ $A=\left(\ln 3-0.43195\right)\mathrm{e}^{-\frac{3}{2}}=0.14875$ $\therefore y=0.149\mathrm{e}^{\frac{x}{2}}-\ln x$

11c	

12 A chemical processing plant uses two types of automated dosing pumps, Pump A and Pump B, to regulate the flow of a catalyst into a reaction chamber. Each pump is removed from the plant and tested over a 10-hour trial period, and the volume of liquid dosed per hour is recorded.

The following data were collected.

- Pump A: The volume of liquid dosed was 4.5 litres in the first hour, and for each subsequent hour, the volume of liquid dosed decreased by a constant percentage of $r\%$.
- Pump B: The volume of liquid dosed was 4.7 litres in the first hour, and for each subsequent hour, the volume of liquid dosed decreased by 0.1 litres.

(a) Show that the total volume of liquid dosed by Pump A at the end of the trial period is

$$\frac{450}{r}\left[1-\left(1-\frac{r}{100}\right)^{k}\right] \text{ litres,}$$

where k is a constant to be determined. [3]

(b) It would be ideal if at the end of the trial period, the volume of liquid dosed by Pump A is equal to that of Pump B. Find the value of r for this ideal situation. [2]

After the trial period, the two pumps were cleaned and reset, and were subsequently installed in the actual plant. Assume that the two pumps perform consistently as what was observed during the trial period.

(c) Pump A is installed for dosing in Reaction Line 1. To prevent underdosing, Pump A will cease to operate if the volume dosed in the next hour falls below 4.0 litres. With the assumption that $r = 1$, find the number of complete hours that Pump A should be in operation. [2]

Before operations begin in Reaction Line 2, Pump A breaks down.

(d) To minimise disruption, the company considers buying a new motor for Pump A from a third-party supplier. The supplier claims that with the new motor, Pump A can dose at least 360 litres of liquid in total if it ran continuously without stopping. Determine the range of values of r for which the supplier's claim is invalid. [2]

(e) The company eventually decides to use Pump B in Reaction Line 2. Using an algebraic method, determine if Pump B can dose at least 150 litres of liquid if left to run continuously without stopping. [3]

No.	Solution
12a	Total volume of liquid dosed by Pump A at the end of 10 hours $=4.5+4.5\left(1-\frac{r}{100}\right)+\ldots+4.5\left(1-\frac{r}{100}\right)^9$ $=\frac{4.5\left[1-\left(1-\frac{r}{100}\right)^{10}\right]}{1-\left(1-\frac{r}{100}\right)}$ $a=4.5$, $r=1-\frac{r}{100}$, $n=10$ $=\frac{450}{r}\left[1-\left(1-\frac{r}{100}\right)^{10}\right]$ Hence $k=10$.
12b	$\frac{450}{r}\left[1-\left(1-\frac{r}{100}\right)^{10}\right]=\frac{10}{2}\left[2(4.7)+9(-0.1)\right]=42.5$ By GC, $r=1.2771=1.28\ (\text{to 3 s.f.})$.
12c	Let n be the number of hours pump A has operated. $4.5(0.99)^{n-1}\geq 4.0$ n : $4.5(0.99)^{n-1}$ 11 : 4.07 12 : 4.03 13 : 3.99 It should operate 12 complete hours. **<u>Alternative</u>** $(n-1)\ln(0.99)\geq\ln\left(\frac{4.0}{4.5}\right)$ $n-1\leq\ln\left(\frac{4.0}{4.5}\right)\Big/\ln(0.99)$ $n\leq 12.719$ It should operate 12 complete hours.
12d	For the supplier's claim to be invalid, $\frac{4.5}{1-\left(1-\frac{r}{100}\right)}<360$ $\frac{450}{r}<360$ $r>1.25$

12e	Total volume of liquid by Pump B $=\frac{n}{2}\left[2(4.7)+(n-1)(-0.1)\right]=4.75n-0.05n^2$ $4.75n-0.05n^2 \geq 150 \Rightarrow 0.05n^2-4.75n+150 \leq 0$ Since Discriminant $=(-4.75)^2-4(0.05)(150)=-7.43<0$ and Coeff of $n^2>0$, there is no real solution for n. Thus it is not possible for Pump B to dose at least 150 litres of liquid if run continuously. **<u>Alternative</u>** Let $4.7+(n-1)(-0.1)=0$ $n=48$ Hence Pump B can only operate at most 47 hours. Total volume dosed at the end of 47 hours $=\frac{47}{2}\left[2(4.7)+(47-1)(-0.1)\right]$ $=112.8<150$ Thus it is not possible for Pump B to dose at least 150 litres of liquid if run continuously.