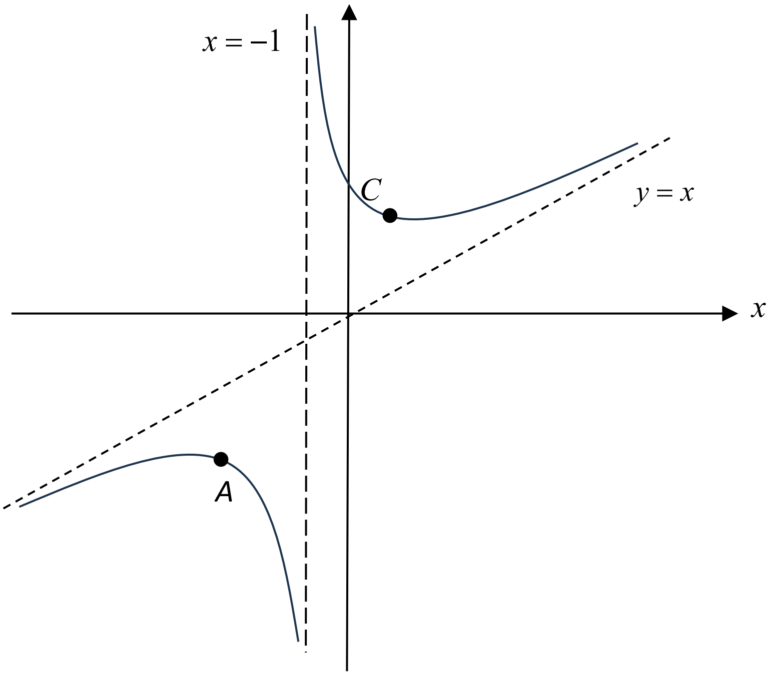
2025 Year 6 H2 Math Prelim Exam P2
Suggested Solutions

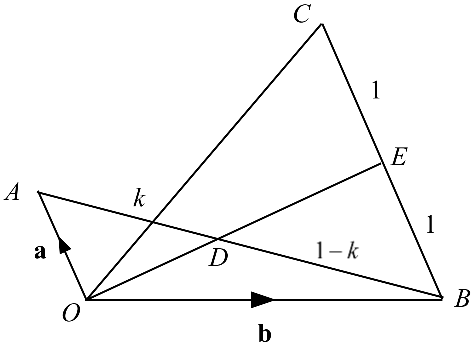
Section A: Pure Mathematics [40 marks]

Qn	Suggested Solutions	Comments
1(a)	$y = x + \frac{9}{x+1}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{9}{(x+1)^2}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{18}{(x+1)^3}$ C concaves upwards $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{18}{(x+1)^3} > 0$ $(x+1)^3 > 0 \Rightarrow x > -1$	When a function $y = \mathrm{f}(x)$ concaves upwards, the gradient of its gradient function is positive; i.e. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$.
(b)	 Stationary points: $A(-4,-7)$, $B(2,5)$ y-intercept: $C(0,9)$	For $y = x + \frac{a}{x+b}$, where a & b are constants, there will be a vertical asymptote ($x = -b$) & an oblique asymptote ($y = x$). For any axial intercept(s), they can be found by letting x & y be zero respectively. For any stationary point(s), they can be found from $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ or using the GC.
		Total marks: 5

Qn	Suggested Solution	Comments
2(a)	$\int \mathrm{e}^{2x}\tan^{-1}\left(\mathrm{e}^{x}\right)\mathrm{d}x$ $=\int t^{2}\tan^{-1}(t)\ \frac{\mathrm{d}t}{t}$ $=\int t\tan^{-1}(t)\ \mathrm{d}t$ (shown) $\boxed{\begin{array}{l} t=\mathrm{e}^{x}\Rightarrow \mathrm{d}t=\mathrm{e}^{x}\mathrm{d}x \\ \therefore \mathrm{d}x=\frac{\mathrm{d}t}{\mathrm{e}^{x}}=\frac{\mathrm{d}t}{t} \end{array}}$	• Clear working needs to be shown to get full credit.
(b)	$\int t\tan^{-1}(t)\ \mathrm{d}t$ $=\frac{t^{2}}{2}\tan^{-1}t-\frac{1}{2}\int\frac{t^{2}}{1+t^{2}}\ \mathrm{d}t$ $\boxed{\begin{array}{ll} u=\tan^{-1}\mathrm{t} & \frac{\mathrm{d}v}{\mathrm{d}t}=t \\ \frac{\mathrm{du}}{\mathrm{d}t}=\frac{1}{1+t^{2}} & v=\frac{t^{2}}{2} \end{array}}$ $=\frac{t^{2}}{2}\tan^{-1}t-\frac{1}{2}\int\left(\frac{1+t^{2}}{1+t^{2}}-\frac{1}{1+t^{2}}\right)\ \mathrm{d}t$ $=\frac{t^{2}}{2}\tan^{-1}t-\frac{1}{2}\int\left(1-\frac{1}{1+t^{2}}\right)\ \mathrm{d}t$ $=\frac{t^{2}}{2}\tan^{-1}t-\frac{1}{2}\left(t-\tan^{-1}t\right)+C$ $=\frac{\tan^{-1}t}{2}\left(t^{2}+1\right)-\frac{t}{2}+C$ $\int_{0}^{1}\mathrm{e}^{2x}\tan^{-1}\left(\mathrm{e}^{x}\right)\mathrm{d}x$ $\boxed{\begin{array}{l} x=0,\ t=\mathrm{e}^{0}=1 \\ x=1,\ t=\mathrm{e} \end{array}}$ $=\left[\frac{\tan^{-1}t}{2}\left(t^{2}+1\right)-\frac{t}{2}\right]_{1}^{\mathrm{e}}$ $=\left(\frac{\tan^{-1}\mathrm{e}}{2}\left(\mathrm{e}^{2}+1\right)-\frac{\mathrm{e}}{2}\right)-\left(\tan^{-1}1-\frac{1}{2}\right)$ $=\frac{\tan^{-1}\mathrm{e}}{2}\left(\mathrm{e}^{2}+1\right)-\frac{\mathrm{e}}{2}-\frac{\pi}{4}+\frac{1}{2}$	• Use LIATE to decide on the choice of term to differentiate. • $\frac{t^{2}}{1+t^{2}}$ can be made proper by splitting the numerator or by long div • Note that $\tan^{-1}1=\frac{\pi}{4}$
		Total marks: 7

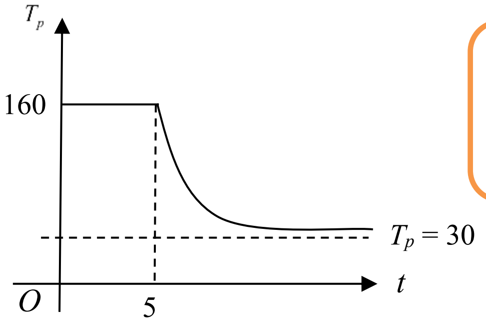
	Suggested Solution	**Comments**
3(a)	$x^3+y^3-5xy=0$ Differentiate wrt x : $3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}-5\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)=0$ $\left(3y^2-5x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=5y-3x^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5y-3x^2}{3y^2-5x}$	Note that product rule have to be applied $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y(1)$
(b)	Gradient of normal = $-\frac{3y^2-5x}{5y-3x^2}$ For normal to be parallel to the y-axis, gradient of normal is undefined. Normal // y-axis → Tangent // x-axis → $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ → $5y-3x^2=0$ Hence, $5y-3x^2=0$ $y=\frac{3}{5}x^2$ ---- (1) Note making y the subject would be easier than x Subst (1) into equation of curve, $x^3+\left(\frac{3}{5}x^2\right)^3-5x\left(\frac{3}{5}x^2\right)=0$ $\frac{27}{125}x^6-2x^3=0$ Using GC, $x=2.0999=2.10\ (3\text{s.f})$ or $x=0$(rejected $\because x>0$) $y=2.6457=2.65\ (3\text{s.f})$ Coordinates of the point is $\left(2.10, 2.65\right)$. **Alternative** (if exact answers needed) $x^3+\frac{27}{125}x^6-3x^3=0$ $\frac{27}{125}x^6-2x^3=0$ $\frac{x^3}{125}\left(27x^3-250\right)=0$ Since $x>0$, $x=\frac{\sqrt[3]{250}}{3}=\frac{5}{3}\sqrt[3]{2}$. $y=\frac{3}{5}\left(\frac{\sqrt[3]{250}}{3}\right)^2=\frac{250^{\frac{2}{3}}}{15}=\frac{5}{3}\sqrt[3]{4}$	Some students had the misconception of $3y^2-5x=0$ but notice gradient of normal would be 0 not undefined Since the question do not require exact answer, students can solve the equation using GC directly

	Coordinates of the point is $\left(\frac{5\sqrt[3]{2}}{3}, \frac{5\sqrt[3]{4}}{3}\right)$.	
(c)	For stationary point, $\frac{dy}{dx}=0$ From part **(a)**, $3x^2+\underline{3y^2\frac{dy}{dx}}-5\left(\underline{x\frac{dy}{dx}}+y\right)=0$. Differentiate wrt x : Apply product rule $6x+\underline{3y^2\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2}-5\left(\underline{x\frac{d^2y}{dx^2}+\frac{dy}{dx}}+\frac{dy}{dx}\right)=0$ $6x+3y^2\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2-5x\frac{d^2y}{dx^2}-10\frac{dy}{dx}=0$ At the stationary point $x=\frac{5\sqrt[3]{2}}{3}$, $y=\frac{5\sqrt[3]{4}}{3}$, $\frac{dy}{dx}=0$. $\frac{d^2y}{dx^2}=\frac{-6x}{3y^2-5x}=-1.2<0$ Hence, maximum point.	Note that finding 2nd derivative is much more efficient than 1st derivative method Note $\frac{d}{dx}\left(3y^2\right)=6y\frac{dy}{dx}$ Ensure clear working is shown **(substitute values of x, y and $\frac{dy}{dx}$ OR calculate correct value of 2nd derivative)** to determine the sign of 2nd derivative and hence nature of stationary point
		Total marks: 7

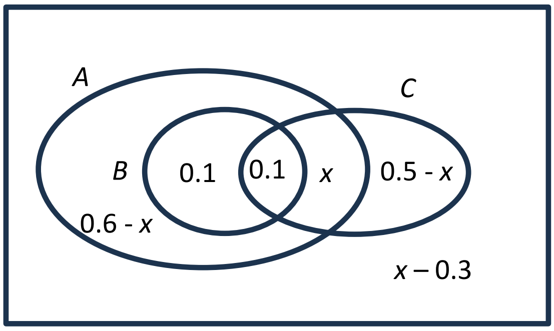
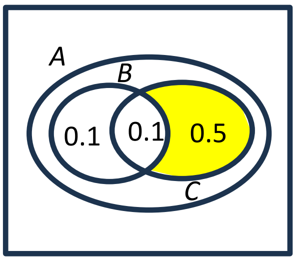
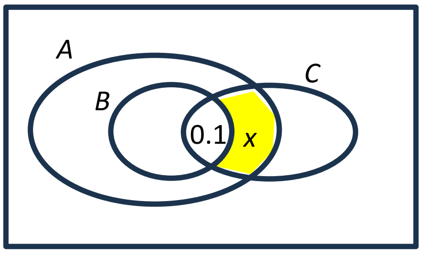
Qn	Suggested Solution	Comments
		
4(a)	Let $\overrightarrow{AD}=k\overrightarrow{AB}$. $\overrightarrow{OD}=\dfrac{k\mathbf{b}+(1-k)\mathbf{a}}{k+(1-k)}=k\mathbf{b}+(1-k)\mathbf{a},\ \ k\in\mathbb{R}$ $l_{OD}:\mathbf{r}=\lambda(k\mathbf{b}+(1-k)\mathbf{a}),\ \lambda\in\mathbb{R}$	- Qn is asking for vector **eqn** of **line** OD. Line OD $\neq\overrightarrow{OD}$ Eqn of line OD is of the form: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ where $\mathbf{a}$ is the position vector of a fixed point on the line (choose vector O or $\overrightarrow{OD}$) and $\mathbf{b}$ is the direction vector of the line (choose $\overrightarrow{OD}$).
(b)	$\overrightarrow{OE}=\frac{1}{2}\left(\overrightarrow{OC}+\overrightarrow{OB}\right)$ $=\frac{1}{2}(3\mathbf{a}+\mathbf{b}+\mathbf{b})$ $=\frac{1}{2}(3\mathbf{a}+2\mathbf{b})$ Since *O, D* and *E* are collinear, point *E* lies on l_{OD}. **Method 1** $\overrightarrow{OE}=\lambda(k\mathbf{b}+(1-k)\mathbf{a})$ $\frac{1}{2}(3\mathbf{a}+2\mathbf{b})=\lambda(k\mathbf{b}+(1-k)\mathbf{a})$ $\frac{3}{2}\mathbf{a}+\mathbf{b}=\lambda(1-k)\mathbf{a}+\lambda k\mathbf{b}$ Since **a** and **b** are non-zero and non-parallel vectors, $\lambda k=1$ ... (1) $\lambda(1-k)=\frac{3}{2}\Rightarrow\lambda-\lambda k=\frac{3}{2}$ ... (2) Subst (1) into (2): $\lambda-1=\frac{3}{2}\Rightarrow\lambda=\frac{5}{2}$ From (1): $\lambda k=1\Rightarrow k=\frac{2}{5}$ **Method 2** Let $\overrightarrow{OD}=\mu\overrightarrow{OE}$ $k\mathbf{b}+(1-k)\mathbf{a}=\frac{\mu}{2}(3\mathbf{a}+2\mathbf{b})=\frac{3\mu}{2}\mathbf{a}+\mu\mathbf{b}$	- $\overrightarrow{OE}\neq\frac{1}{2}\overrightarrow{BC}$ as both $\overrightarrow{OE}$ and $\overrightarrow{BC}$ are not even parallel. - $\overrightarrow{OE}=\frac{1}{2}\left(\overrightarrow{OC}+\overrightarrow{OB}\right)$ using Ratio Thm

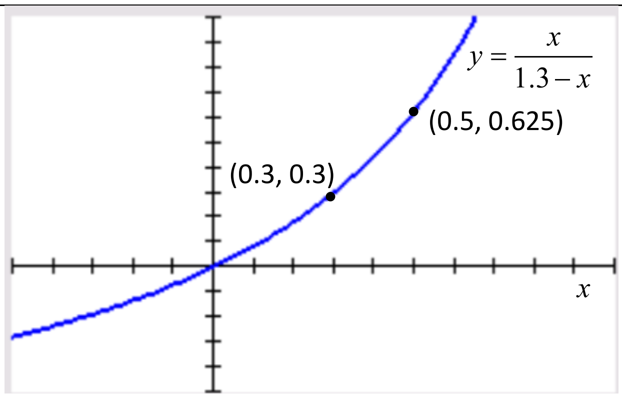
	Since **a** and **b** are non-zero and non-parallel vectors, $k=\mu$ ... (1) $1-k=\frac{3\mu}{2}$ ... (2) Subst (1) into (2): $\Rightarrow 1-\mu=\frac{3\mu}{2} \Rightarrow \mu=\frac{2}{5}$ From (1): $k=\frac{2}{5}$	
(c)	$\lvert\mathbf{a}\rvert=1,\ \lvert\mathbf{b}\rvert=2$ and $\lvert 3\mathbf{a}-2\mathbf{b}\rvert=\sqrt{31}.$ $\lvert 3\mathbf{a}-2\mathbf{b}\rvert=\sqrt{31}$ $\lvert 3\mathbf{a}-2\mathbf{b}\rvert^2=31$ $(3\mathbf{a}-2\mathbf{b}).(3\mathbf{a}-2\mathbf{b})=31$ $9\mathbf{a}.\mathbf{a}-12\mathbf{a}.\mathbf{b}+4\mathbf{b}.\mathbf{b}=31$ $9\lvert\mathbf{a}\rvert^2-12\mathbf{a}.\mathbf{b}+4\lvert\mathbf{b}\rvert^2=31$ $9(1)^2-12\mathbf{a}.\mathbf{b}+4(2)^2=31$ $\mathbf{a}.\mathbf{b}=-\frac{6}{12}=-\frac{1}{2}$ Let θ the angle between **a** and **b**. $\because \cos\theta=\frac{\mathbf{a}.\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}=\frac{-\frac{1}{2}}{1\times 2}=-\frac{1}{4}$ $\therefore \theta=104.5^\circ$ (1.82 rad)	- Concept (derived from scalar product formula): $(3\mathbf{a}-2\mathbf{b}).(3\mathbf{a}-2\mathbf{b})$ $=\lvert 3\mathbf{a}-2\mathbf{b}\rvert^2$ $\neq(3\mathbf{a}-2\mathbf{b})^2$ or $\lvert(3\mathbf{a}-2\mathbf{b})^2\rvert$ Qn did not ask for acute angle hence we cannot put modulus for **a.b**
(d)	$\lvert\mathbf{a}\bullet\mathbf{b}\rvert=\lvert\mathbf{b}\bullet\hat{\mathbf{a}}\rvert$ is the length of projection of **b** onto **a**. (note that **a** is the unit vector)	- Note that **a** is the unit vector. So, it should be length of projection of **b** onto **a** and not the other way round.
		Total marks: 11

Qn	**Suggested Solution**	**Comments**
5(a)	$\frac{\mathrm{d}T}{\mathrm{d}t}=-k\left(T-30\right)$ $\int\frac{1}{T-30}\,\mathrm{d}T=-k\int \mathrm{d}t$ $\ln\lvert T-30\rvert=-kt+c$ $T-30=\pm e^{c}\bullet e^{-kt}=Ae^{-kt} \quad (A=\pm e^{c})$ $T=30+Ae^{-kt}$ Subst $t=0,\ T=160,$ $160=30+A\Rightarrow A=130$ $\therefore T=30+130e^{-kt}$	k is given as a positive constant. Since *T* is decreasing, a negative sign must be applied. You will need to “remove” the modulus before applying the initial conditions

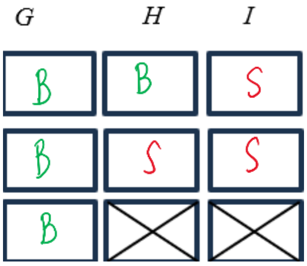
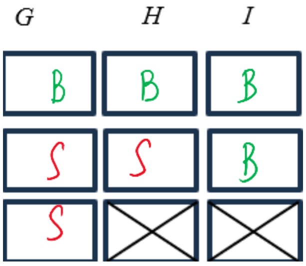
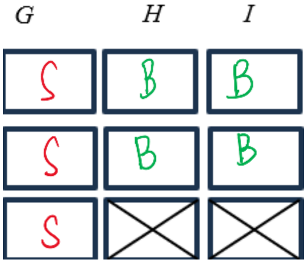
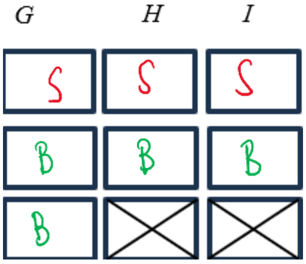
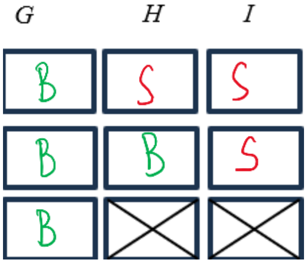
(b)	As the actual cooling starts 5 minutes later, we need to shift the cooling function 5 minutes to the right. Replace t with $t-5 \Rightarrow a=-5$.	
(c)	When $t \geq 5$, $T_p(t)=T(t-5)=30+130\mathrm{e}^{-k(t-5)}$ Subst $t=15,\ T_p=100$ $100=30+130\mathrm{e}^{-10k}$ $\mathrm{e}^{-10k}=\frac{70}{130}$ $\therefore k=-\frac{1}{10}\ln\frac{7}{13}=0.061904$ $T(t-5)=30+130\mathrm{e}^{-0.061904(t-5)}$ $=30+177.16\mathrm{e}^{-0.061904t}$ $=30+177\mathrm{e}^{-0.0619t}$ (shown)	Alternatively, you can think of using part (a), can think of it as 10 mins has lapsed to find k. Subst $t=10,\ T=100$ $100=30+130\mathrm{e}^{-10k}$ $\mathrm{e}^{-10k}=\frac{70}{130}$ $\therefore k=-\frac{1}{10}\ln\frac{7}{13}=0.061904$
(d)	$60=30+177.16\mathrm{e}^{-0.061904t}$ $\mathrm{e}^{-0.061904t}=\frac{30}{177.16}$ $t=28.687=28.7$ min	**Please be aware that you can do part (c) and (d) without attempting the earlier parts.**
(e)	 If there is a horizontal asymptote, you will need draw and label it.	
		Total marks: 10

Section B: Probability and Statistics [60 marks]

Qn	Suggested solutions	Comments
6(ai)	 Note: $\mathrm{P}(B\cap C)=\mathrm{P}(A\cap B\cap C)=0.1$ since $B\subset A$. Max x occurs when the part in C not in B (0.5) is entirely in A. Hence, max x = 0.5. Min x occurs when as much of the part in C not in B is outside of A as possible. $\mathrm{P}(A\cup C)\le 1$ $P(A)+P(C)-P(A\cap C)\le 1$ $0.8+0.6-(0.1+x)\le 1$ $x\ge 0.3$ Hence, min x = 0.3. From the diagram, min x = 0.3, max x = 0.5 **Alternative** $\left.\begin{array}{l}0.6-x\ge 0\Rightarrow x\le 0.6\\ x-0.3\ge 0\Rightarrow x\ge 0.3\\ 0.5-x\ge 0\Rightarrow x\le 0.5\end{array}\right\}\xrightarrow{\text{Taking Intersection}}0.3\le x\le 0.5$ Hence, min x = 0.3, max x = 0.5.	**Venn diagrams are useful for this type of question.** Maximum $\mathrm{P}(A\cap B'\cap C)$:  Minimum $\mathrm{P}(A\cap B'\cap C)$: 
(ii)	$\mathrm{P}(A\cap B'\cap C\mid A\cup C)=\dfrac{\mathrm{P}((A\cap B'\cap C)\cap(A\cup C))}{\mathrm{P}(A\cup C)}$ $=\dfrac{\mathrm{P}(A\cap B'\cap C)}{\mathrm{P}(A\cup C)}=\dfrac{x}{1.3-x}$	**Important** 1) Include **brackets** where appropriate 2) $(A\cap B'\cap C)$ is a subset of $(A\cup C)$ 3) $\mathrm{P}((A\cap B'\cap C)\cap(A\cup C))$ $\ne \mathrm{P}(A\cap B'\cap C)\times\mathrm{P}(A\cup C)$ **as they may not be independent** 4) $\mathrm{P}((A\cap B'\cap C)\cap(A\cup C))$ $\ne \mathrm{P}(A\cap B'\cap C)\cap\mathrm{P}(A\cup C)$ (**Notation error**)

	 For $0.3 \le x \le 0.5 \Rightarrow 0.3 \le \mathrm{P}\left(A \cap B' \cap C \mid A \cup C\right) \le 0.625$ Min $\mathrm{P}\left(A \cap B' \cap C \mid A \cup C\right) = 0.3$ Max $\mathrm{P}\left(A \cap B' \cap C \mid A \cup C\right) = 0.625$ **Alternative** $\mathrm{P}\left(A \cap B' \cap C \mid A \cup C\right) = \dfrac{x}{1.3-x} = \dfrac{1.3}{1.3-x} - 1$ For $0.3 \le x \le 0.5$, $\dfrac{1.3}{1.3-x} - 1$ is **increasing**. Min $\mathrm{P}\left(A \cap B' \cap C \mid A \cup C\right) = \dfrac{0.3}{1.3-0.3} = 0.3$ Max $\mathrm{P}\left(A \cap B' \cap C \mid A \cup C\right) = \dfrac{0.5}{1.3-0.5} = 0.625$	From the graph, we can find the maximum and minimum y value for $0.3 \le x \le 0.5$. Note that $\left(A \cap B' \cap C\right)$ is a subset of $\left(A \cup C\right)$ Use reduced sample space to obtain the fraction. We can only just evaluate the situation at the end points as the function is **increasing**.
(b)	$\mathrm{P}(A \cap C) = \mathrm{P}(A) \cdot \mathrm{P}(C)$ $0.1 + x = (0.8)(0.6)$ $\mathrm{P}(A \cap B' \cap C) = x = 0.38$	Note that **only *A* and *C*** are independent events.
		Total marks: 7

	Suggested Solution	**Comments**
7(a)	There are 13 seats for the remaining 11 people to choose from. Number of ways = $\dbinom{13}{11} 11! = 3113510400$ **Alternative** Number of ways = $\dfrac{13!}{2!} = 3113510400$	It’s a particular student; there is no need to choose the student. There is only one way to sit him/her. Arrangement is needed here. The reason for 2! is due to the 2 empty seats.
(b)	Stage 1: Seat the 2 particular students: $\dbinom{7}{2} \times 2!$	Note that there is a total of 7 non-aisle seats. We settle these 2

	Stage 2: There are 12 remaining seats to be arranged for 10 students. $\binom{12}{10}\times 10!$ or $\frac{12!}{2}$ Number of ways $=\binom{7}{2}\times 2!\times\frac{12!}{2}=1.01\times 10^{10}$ or 10059033600	‘particular’ students to deal with the restrictions. As we are doing the arrangement in stages, we need to use the multiplication principle.
(c)	The 5 from tennis must be in the middle segment: No of ways to arrange them = 6! or $\binom{6}{5}\times 5!=720$ For the bowling and softball students: Number of ways = $5\times 4!\times 3!=720$ Arrange the 4 from bowling Arrange the 3 from softball 5: (from the perspective of bowling first) ①  ②  ③  ④  ⑤  Total number of ways = (720)(720) = 518400	We are doing the arrangement in stages. We ‘settle’ the 5 from tennis first. They can only be in column E and F since they must be together. Students from bowling and softball cannot be here as there are 7 students in total but there are only 6 seats. Multiplication principle is used here since we are doing in stages.
		Total marks: 8

Qn	Suggested soln	Comments
8(a)	$\mathrm{P}(X=2)$ $=\mathrm{P}(2\text{ diff. colours among the 4 discs})$ $=\mathrm{P}(2\text{ discs each of two diff. colours})$ $+\mathrm{P}(3\text{ discs of one colour}+1\text{ disc of diff. colour})$ $=\dfrac{\overset{\text{choose 2 colours from 4}}{\binom{4}{2}}\binom{6}{2}\binom{6}{2}}{\binom{24}{4}}+\dfrac{\overbrace{\binom{4}{2}}^{\text{choose 2 colours from 4}}\binom{6}{1}\binom{6}{3}\times\overbrace{2}^{\text{choose 1 of the 2 colours to repeat}}}{\binom{24}{4}}$ $=\dfrac{465}{1771}$ (shown) **Alternative** $\mathrm{P}(X=2)$ $=\overset{\text{choose 2 colours from 4}}{\binom{4}{2}}\left[\dfrac{6}{24}\cdot\dfrac{5}{23}\cdot\dfrac{6}{22}\cdot\dfrac{5}{21}\cdot\overset{\text{arrange 4 discs, 2 identical of each colour}}{\dfrac{4!}{2!2!}}\right]$ $+\overset{\text{choose 2 colours from 4}}{\binom{4}{2}}\underset{\text{choose 1 of the 2 colours to repeat}}{\binom{2}{1}}\left[\dfrac{6}{24}\cdot\dfrac{5}{23}\cdot\dfrac{4}{22}\cdot\dfrac{6}{21}\cdot\overbrace{\dfrac{4!}{3!}}^{\text{arrange 4 discs, 3 are of identical colour, 1 different}}\right]$ $=\dfrac{465}{1771}$ (shown)	There are other ways to do this. Presented here are the recommended approaches as they are likely easier to understand by most. It is a good idea to indicate the type of cases that you are considering in a systemic, neat manner to make it easier to understand your approach. The combination approach is probably the most efficient. For those that uses the one-by-one probability approach (see **alternative**), you must remember to consider the different permutations that arises.
(b)	$\mathrm{P}(X=1)=\mathrm{P}(4\text{ discs all same colour})$ $=\dfrac{\binom{4}{1}\binom{6}{4}}{\binom{24}{4}}$ $=\dfrac{10}{1771}$	Recall that the last probability can be obtained using **1 minus all the remaining probability values** obtained. You do not need to directly compute the last probability. In this question, the case of $\mathrm{P}(X=3)$ is best done this way. A probability distribution is a listing of all the different $\mathrm{P}(X=x)$ for each x

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$$\mathrm{P}(X=4)=\mathrm{P}(1\text{ disc each of 4 diff. colours})$$

$$=\frac{\binom{6}{1}\binom{6}{1}\binom{6}{1}\binom{6}{1}}{\binom{24}{4}}$$

$$=\frac{216}{1771}$$

$$\mathrm{P}(X=3)=1-\mathrm{P}(X=1)-\mathrm{P}(X=2)-\mathrm{P}(X=4)$$

$$=\frac{1080}{1771}$$

or direct

$$\mathrm{P}(X=3)=\frac{\binom{4}{3}\binom{3}{1}\binom{6}{2}\binom{6}{1}\binom{6}{1}}{\binom{24}{4}}=\frac{1080}{1771}$$

<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>P($X=x$)</td><td>$\frac{10}{1771}$</td><td>$\frac{465}{1771}$</td><td>$\frac{1080}{1771}$</td><td>$\frac{216}{1771}$</td></tr>
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<u>**Alternative**</u>

$$\mathrm{P}(X=1)=\binom{4}{1}\left[\frac{6}{24}\cdot\frac{5}{23}\cdot\frac{4}{22}\cdot\frac{3}{21}\right]=\frac{10}{1771}$$

$$\mathrm{P}(X=4)=\left[\frac{6}{24}\cdot\frac{6}{23}\cdot\frac{6}{22}\cdot\frac{6}{21}\cdot 4!\right]=\frac{216}{1771}$$

$$\mathrm{P}(X=3)=\binom{4}{3}\binom{3}{1}\left[\frac{6}{24}\cdot\frac{6}{23}\cdot\frac{6}{22}\cdot\frac{5}{21}\cdot\frac{4!}{2!}\right]=\frac{1080}{1771}$$

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value. Do include the case of $\mathrm{P}(X=2)$ even if it is done in previous part.

There is no such case for $X = 0$. X has to be at least be 1, the case where all the discs of 1 (same) colour.

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<td>

Let Y be the number of rounds, out of 8, with at most 2 different colour discs. i.e. where Joe wins \$10.

$Y \sim \mathrm{B}(8, \mathrm{P}(X \le 2))$ i.e. $Y \sim \mathrm{B}\left(8, \frac{475}{1771}\right)$

For Joe to win some money at the end, he must win \$10 in **at least 3 rounds**.

$$\mathrm{P}(Y \ge 3)=1-\mathrm{P}(Y \le 2)$$

$$=0.3672845593$$

$$=0.367 \text{ (to 3 s.f.)}$$

<u>**Alternative 1**</u>

Let V be the number of rounds, out of 8, with more than 2 different colour discs. i.e. where Joe loses \$5.

</td>
<td>

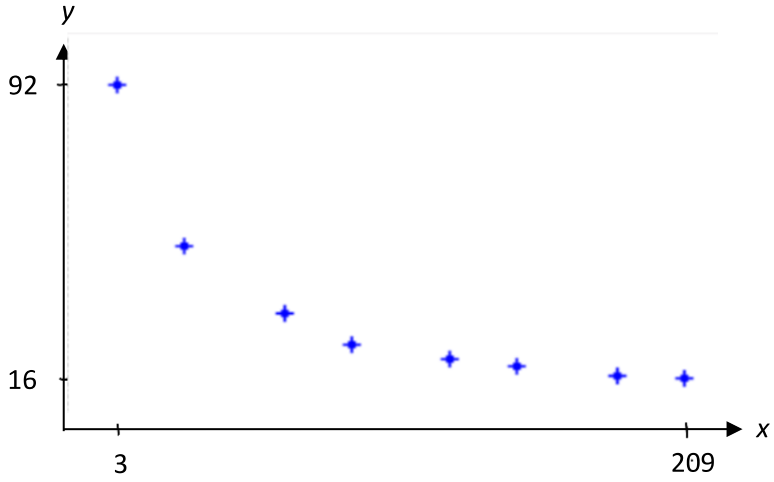
Define any additional random variable(s) used clearly.

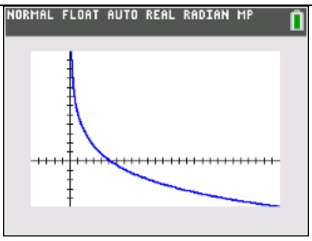
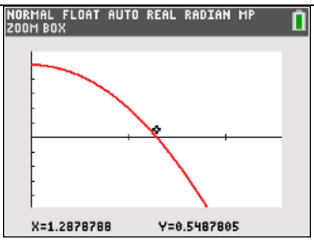
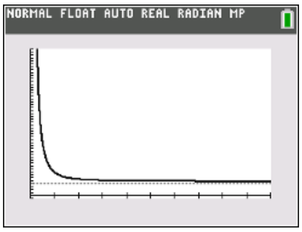
DO NOT use X again which has already been used in the question.

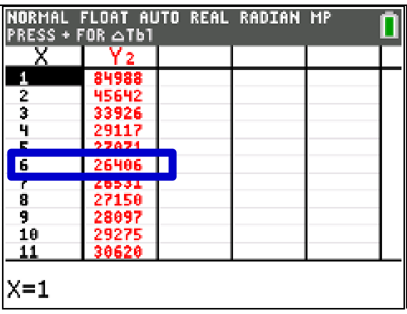
</td>
</tr>
</table>

	$V \sim \text{B}(8, \text{P}(X > 2))$ i.e. $V \sim \text{B}\left(8, \frac{1296}{1771}\right)$ For Joe to win some money at the end, he can **at most lose 5 rounds**. $\text{P}(V \le 5) = 0.3672845593 = 0.367 \text{ (to 3 s.f.)}$ **Alternative 2** $\text{P}(X = 2) = \frac{475}{1771}$ For Joe to win some money at the end, he must win \$10 in **at least 3 rounds**. $\text{P}(\text{Joe wins least 3 rounds})$ $= 1 - \text{P}(\text{Joe wins 2 or fewer rounds})$ $= 1 - \left[\left(\frac{1296}{1771}\right)^8 + \binom{8}{1}\left(\frac{475}{1771}\right)\left(\frac{1296}{1771}\right)^7 + \binom{8}{2}\left(\frac{475}{1771}\right)^2\left(\frac{1296}{1771}\right)^6\right]$ $= 0.3672845593 = 0.367 \text{ (to 3 s.f.)}$	
(d)	Expected "profit" from 1 round $= \left(10 \times \frac{475}{1771}\right) + \left(-5 \times \frac{1296}{1771}\right) = -\frac{1730}{1771} = -0.9768492377$ As Joe is expected to lose \$0.98 per round, and each round is independent, he will be expected to lose money at the end of the game. It is not worthwhile for him to play the game as he will likely lose money. **Alternative 1** Expected "profit" from 8 rounds (1 game) $= \left[\left(10 \times \frac{475}{1771}\right) + \left(-5 \times \frac{1296}{1771}\right)\right] \times 8 = -7.814793902 = -7.81$ Joe will be expected to lose \$7.81 in each complete game. It is not worthwhile for him to play the game as he will likely lose money. **Alternative 2** Using Y defined in **(c)**, $\text{E}(Y) = 8\left(\frac{475}{1771}\right) = \frac{3800}{1771} = 2.145680407 = 2.14 < 3$ This means Joe is expected to win in less than 3 rounds in 1 game. Since he needs to **win at least 3 rounds to win money**, it is not worthwhile for him to play the game as he will likely lose money. **Alternative 3**	Need to account for **both** the probability of winning/losing **AND** the value of the win/lose. Cannot conclude by simply comparing probability values of X or computing the expectation of X only. (These does not account for the value of the win/lose.)

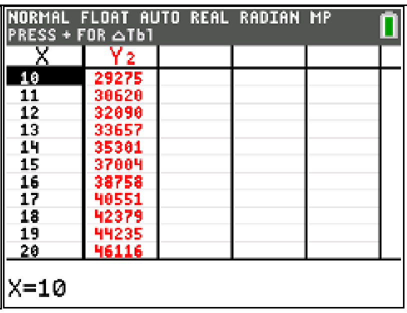
	Let W be the amount he wins in 1 game (8 rounds). $W=10Y-5(8-Y)=15Y-40$ $\mathrm{E}(W)=\mathrm{E}(15Y-40)=15\mathrm{E}(Y)-40$ $=15\left[8\times\frac{475}{1771}\right]-40$ $=-7.814793902=-7.81$ Joe will be expected to lose \$7.81 in each complete game. It is not worthwhile for him to play the game as he will likely lose money.	
		Total marks: 10

Qn	**Suggested Solution**	**Comments**
9(a)	Scatter Diagram I: 0 Scatter Diagram II: 1 Scatter Diagram III: − 0.7	
(b)(i)	 From the scatter diagram in **(i)**, it can be observed that as x increases, y decreases at a decreasing rate. Thus, a linear model would not be appropriate. **OR** Inappropriate to be modelled by a linear model because the data points in the scatter diagram seem to fit a curve that is decreasing at a decreasing rate / decreasing and concave upwards, rather than a line.	For this scatter diagram, there must be **exactly 8 points**. The axes must be labelled with minimum & maximum values shown. The comment must be explained with reference to the scatter diagram.
(ii)	Model **(A)**: $y=a-b\ln x$ Model **(B)**: $y=a-bx^2$	

	  Model **(C)**: $y = a + \frac{b}{x^2}$  From the scatter diagram in **(b)(i)**, it can be observed that as x increases, y decreases at a decreasing rate which is consistent with Models (**A**) and (**C**). However, based on the context of the question, there should be a positive lower limit to the number of working days required for the renovation works to be completed even if infinitely many workers are hired. Model (**A**) predicts that y will decrease indefinitely, while Model (**C**) has a limit to the value y can take ($y = a$ is a horizontal asymptote for Model (**C**)). Thus Model (**C**) is the most appropriate.	The choice & explanation must take reference from the scatter diagram & the context of this question. Note that it's necessary to mention why **C** is the most appropriate; not just why **A** & **B** are not.
(iii)	Equation of regression line for Model (**C**): $y = 17.4880 + \frac{690.745}{x^2} = 17.5 + \frac{691}{x^2}$ $r = 0.992$ (to 3sf)	The final answer should be in 3 s.f. though the intermediate steps give more s.f.
(iv)	$y = 17.4880 + \frac{690.745}{9^2} = 26.0 = 26$ (to nearest whole no.) Since $x = 9$ is within the data range $(3 \le x \le 20)$ and r-value of 0.992 (which is close to 1) indicates that the model used for the estimation is appropriate, the estimate is reliable.	The answer is to be **rounded off** (NOT up as this is an estimation based on the best-fit equation) to an integer. Both reasons must be given to justify its reliability.
(v)	Let T dollars be the total amount of workers' wages to be paid. $T = 120xy = 120x\left(17.4880 + \frac{690.745}{x^2}\right)$ Using GC Table:	*Key Concept*: To **minimise** 120xy or xy



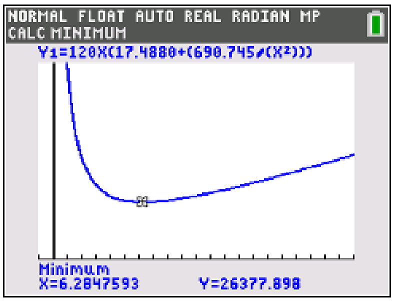
```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl
  X      Y2
  1      84988
  2      45642
  3      33926
  4      29117
  5      27071
  6      26406
  7      26531
  8      27150
  9      28097
  10     29275
  11     30620
X=1
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl
  X      Y2
  10     29275
  11     30620
  12     32090
  13     33657
  14     35301
  15     37004
  16     38758
  17     40551
  18     42379
  19     44235
  20     46116
X=10
```

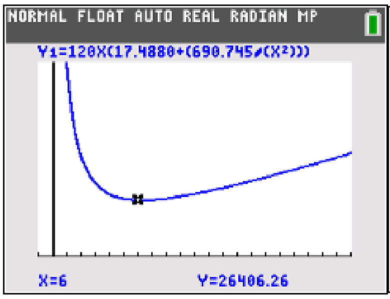
The developer should hire 6 workers to minimise the total amount of workers’ wages to be paid out.

Alternative (Use Graph)

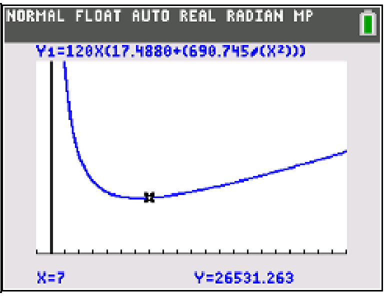


```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC MINIMUM
Y1=120X(17.4880+(690.745/(X²)))
Minimum
X=6.2847593          Y=26377.898
```

The minimum value should be either 6 or 7.



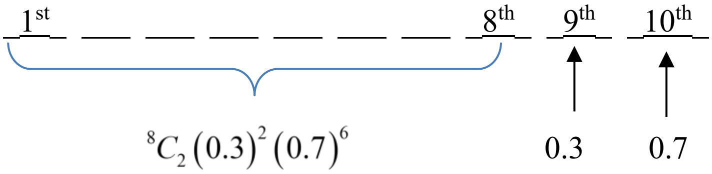
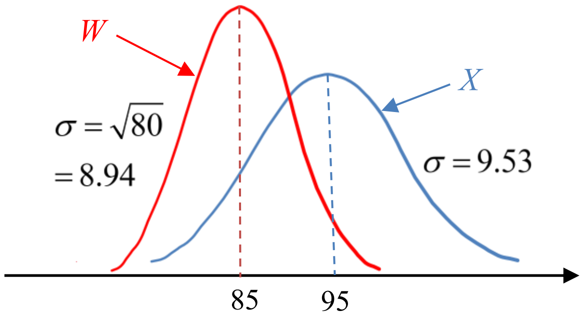
```
NORMAL FLOAT AUTO REAL RADIAN MP
Y1=120X(17.4880+(690.745/(X²)))
X=6                  Y=26406.26
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
Y1=120X(17.4880+(690.745/(X²)))
X=7                  Y=26531.263
```

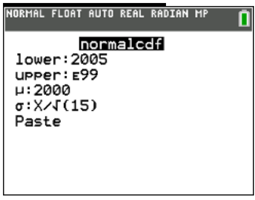
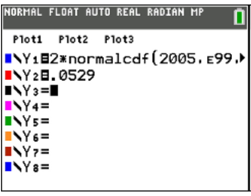
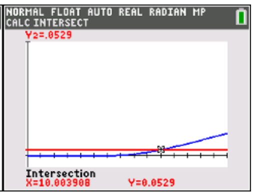
The developer should hire 6 workers to minimise the total amount of workers’ wages to be paid out.

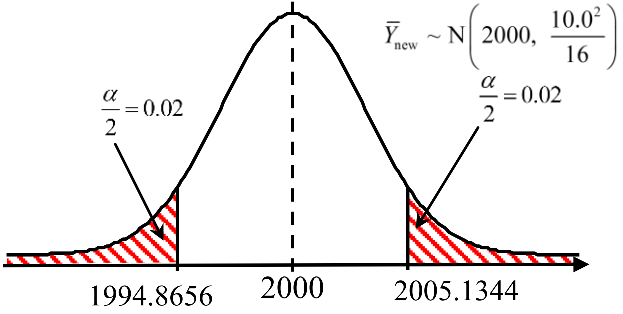
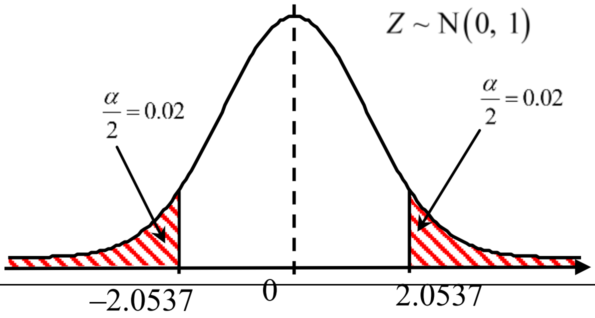
Total = 11 marks

Qn	Suggested Solution	Comments
10(a)	$X \sim \mathrm{N}\left(95,\ \sigma^2\right)$ $\mathrm{P}(X \le 90) = 0.3$ $\mathrm{P}\left(Z \le \frac{90-95}{\sigma}\right) = 0.3$ $\frac{-5}{\sigma} = -0.52440$ $\therefore \sigma = 9.5347 = 9.53$ (3 s.f.) (shown)	- Clear working needs to be shown to get full credit. Use of table is more useful to find unknown integers and not rational numbers like 9.53. Methods like standardising or graphical are preferred in this case.
(b)	9.15 – 11.00 pm = 105 min $\mathrm{P}(X > 105) = 0.147$ (3 s.f.)	
(c)	Prob that member adhere $= \mathrm{P}(X \le 90) = 0.3$ 2 members adhere = $(0.3)^2$ 6 members didn't adhere = $(0.7)^6$ Cases = ${}^8C_2$  Required probability $= {}^8C_2(0.3)^2(0.7)^6(0.3)(0.7)$ $= 0.0623$ (3 s.f.)	
(d)		These are the key features : - pop means 85, 95 - diff height $h_X < h_W$ - spread $\sigma_X > \sigma_W$ - bell shape (asymptotic feature) - label X & W clearly
(e)	Let W denote usage time of a trial member (in min). $W \sim \mathrm{N}(85,\ 80)$ $\mathrm{E}\left(X_1 + X_2 - 2W\right) = 2\mathrm{E}\left(X\right) - 2\mathrm{E}\left(W\right)$ $= 2(95) - 2(85) = 20$ $\mathrm{Var}\left(X_1 + X_2 - 2W\right) = 2\mathrm{Var}\left(X\right) + 4\mathrm{Var}\left(W\right)$ $= 2(9.53^2) + 4(80) = 501.64$ $X_1 + X_2 - 2W \sim \mathrm{N}\left(20,\ 501.64\right)$	

	$\mathrm{P}(\lvert X_1 + X_2 - 2W \rvert \geq 15)$ $= 1 - \mathrm{P}(-15 < X_1 + X_2 - 2W < 15)$ $= 0.647$ (3 s.f.)	• "differs from.." ⇒ modulus
(f)	The usage time of all paying and trial members are independent of each other.	• Insufficient to say - usage time of paying members are independent of trial members • The "independence" applies for both "within" paying members & trial members also
		Total marks: 12

Qn	**Suggested soln**	**Comments**
11(a)	Using GC, unbiased estimate of the population variance, $s^2 = 11.07672684^2 = 122.6938776\text{ g}^2 = 123\text{ g}^2$ (3 s.f.) *Impt Note: **unbiased estimate** of the population variance is defined as s^2, we can only get an "**estimate**" from sample data; this is not the same as the actual population variance σ^2; thus, it is not right to say that $\sigma^2 = \ldots = 123\text{ g}^2$ Similarly, $\mathrm{E}(X) = \mu$ (population mean) $\neq$ $\bar{x}$ (sample mean)	Students can make use of the GC (GC Lists) to do this – refer to Example 8(a)(ii) of Lecture Notes Chapter 18 Sampling **Method 2-GC**. Many students tried to compute s^2 on their own and could not get the right answer.
(b)	A one-tail (lower-tail) test should be conducted as he suspects that his machines now pack Regular-sized packets that are underweight **OR** A one-tail (lower-tail) test should be conducted as he suspects that his machines now pack Regular-sized packets which have <u>mean mass</u> less than the intended 1kg.	We are doing a test for "means", thus if students mention "mass", it should specifically be referred to as "<u>mean</u> mass". Words should be used to explain why – not just math symbols without any definitions.
(c)	Let μ be the **population mean** mass of a Regular-sized packet of flour in g. H$_0$: $\mu = 1000$ H$_1$: $\mu < 1000$ Perform 1-tail test at 5% level of significance Under H$_0$, $\bar{X} \sim \mathrm{N}\left(1000,\ \frac{122.69}{50}\right)$ approximately by Central Limit Theorem, since $n = 50$ is large. $\bar{x} = 997.4$ (exact), $n = 50$ p-value = 0.0484807326 = 0.0485 (3 s.f.) Since p-value < 0.05, we reject H$_0$. There is sufficient evidence at 5% level of significance to conclude that the <u>mean mass</u> of a Regular-sized packet of flour is less than 1000 g. or There is sufficient evidence at 5% level of significance to conclude that his machines now pack Regular-sized packets that are underweight.	μ was often not defined. Some students still made errors in writing the test statistic, $\bar{X} \sim \mathrm{N}\left(1000,\ \frac{122.69}{50}\right)$, sometimes writing X instead of $\bar{X}$ or using $\bar{x} = 997.4$ as the mean instead of 1000 or not dividing s^2 by n. Some students indicated the standardised test statistic instead, but this was often not well done. The correct standardised test statistic should be: $Z = \frac{\bar{X} - 1000}{\sqrt{\frac{122.69}{50}}} \sim \mathrm{N}(0,1)$ Many who did this substituted the value of $\bar{x}$ in and wrote

	It is **not necessary** to assume that the mass of Regular-sized packets of flour, X, follow a normal distribution as the sample size of 50 is large enough for us to use Central Limit Theorem to approximate the distribution of the sample mean mass, $\bar{X}$, to a normal distribution. Students needed to state 3 main points on whether the assumption was necessary: • not necessary • n large • Central Limit Theorem (*spell out instead of just stating as "CLT"*) for the approximation for $\bar{X}$ (sample mean mass) and not X (mass) **Note that Central Limit Theorem (CLT) can only be used to approximate the distributions of* $\bar{X}=\frac{X_1+X_2+\ldots X_n}{n}$ *(e.g. sample mean mass) or* $X_1+X_2+\ldots X_n$ *(total sample mass) for a sufficiently large n; it cannot be used to approximate X (mass); CLT tells us nothing about the distribution of X.*	$\frac{997.4-1000}{\sqrt{\frac{122.69}{50}}}\sim \mathrm{N}(0,1)$; a number/value cannot follow a distribution (it is not a random variable). Explanation on whether the assumption was necessary was not well done and sometimes missed out. Note that the p-value needs to be accurate to obtain credit for the conclusion. Also, the conclusion was sometimes not well written with students missing out the mention of the significance level and "<u>mean</u> mass".
(d)	Let Y g be the mass of a Large-sized packet of flour, and μ_Y be the population mean. H₀: $\mu_Y=2000$ H₁: $\mu_Y\neq 2000$ Perform 2-tail test at α% level of significance Under H₀, $\bar{Y}\sim \mathrm{N}\left(2000,\ \frac{\sigma^2}{15}\right)$. $\bar{y}=\frac{\sum y}{15}=2005$ Since 2005 > 2000, $p\text{-value}=2\mathrm{P}\left(\bar{Y}\geq 2005\right)=0.0529$ **<u>GC Method</u>**  NORMAL FLOAT AUTO REAL RADIAN MP normalcdf lower:2005 upper:E99 μ:2000 σ:X/√(15) Paste  NORMAL FLOAT AUTO REAL RADIAN MP Plot1 Plot2 Plot3 \Y1=2*normalcdf(2005,E99,▸ \Y2=.0529 \Y3= \Y4= \Y5= \Y6= \Y7= \Y8=  NORMAL FLOAT AUTO REAL RADIAN MP CALC INTERSECT Y2=.0529 Intersection X=10.003908 Y=0.0529 Using GC, $\sigma=10.004=10.0$ (to 3 s.f.) (Shown)	The p-value was often not well written and there were some "double-errors" that led to the given $\sigma=10.0$ value. There should be clear working to show how the value of σ was obtained since its value is given and this is a "show" qns – students can show evidence of this by leaving σ to a higher sf value (e.g. $\sigma=10.004$) first if the GC method was used.

	Standardisation Method $z_{\text{cal}}=\dfrac{2005-2000}{\frac{\sigma}{\sqrt{15}}}=\dfrac{5\sqrt{15}}{\sigma}$ $2\text{P}\left(Z\ge z_{\text{cal}}\right)=0.0529$ $\text{P}\left(Z\ge\dfrac{5\sqrt{15}}{\sigma}\right)=\dfrac{0.0529}{2}$ $\dfrac{5\sqrt{15}}{\sigma}=1.935736352$ $\sigma=10.004=10.0$ (to 3.s.f.)	
(e)	H$_0$: $\mu_Y=2000$ H$_1$: $\mu_Y\ne 2000$ Perform 2-tail test at 4% level of significance Under H$_0$, $\bar{Y}_{\text{new}}\sim\text{N}\left(2000,\ \dfrac{10.0^2}{16}\right)$ $\bar{y}_{\text{new}}=\dfrac{\sum y+h}{16}=\dfrac{30075+h}{16}$ **Method 1 (critical-value)**  Since H_0 is rejected: $\bar{y}_{\text{new}}$ lies in critical region $\bar{y}_{\text{new}}\le 1994.8656$ or $\bar{y}_{\text{new}}\ge 2005.1344$ $\dfrac{30075+h}{16}\le 1994.8656$ or $\dfrac{30075+h}{16}\ge 2005.1344$ $h\le 1842.85$ or $h\ge 2007.15$ $\therefore h\le 1842$ (nearest g) or $h\ge 2008$ (nearest g) **Method 2 (standardised critical-value)**  	The critical/rejection region/criteria should not be a strict inequality and should include the "equal sign", i.e. "≤ , ≥", instead of "< , >". Some students carelessly still took the sample size, *n,* as 15. **Method 1** (critical value method) is strongly recommended for such questions. As there are no unknowns in the distribution parameters, **Method 1** is preferred over **Method 2**. **Method 3** (*p*-value method) was generally not well done as it proved to be challenging for most students to write out the *p*-value accurately and to consider the two required cases, thus the critical value method is strongly advised/ preferred for such questions.

Since H_0 is rejected: *z*-value lies in critical region

z-value ≤ –2.0537 or *z*-value ≥ 2.0537

$$\frac{\frac{30075+h}{16}-2000}{\sqrt{10.0^2/16}} \le -2.0537 \quad \text{or} \quad \frac{\frac{30075+h}{16}-2000}{\sqrt{10.0^2/16}} \ge 2.0537$$

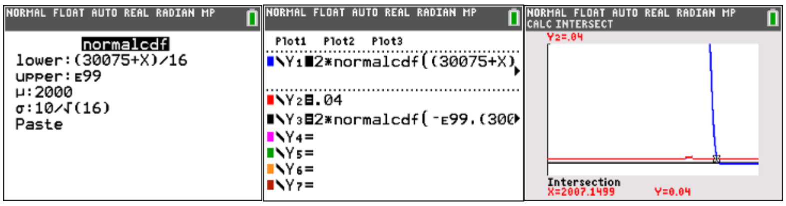
$$h \le 1842.85 \qquad \text{or} \qquad h \ge 2007.15$$

$$\therefore h \le 1842 \text{ (nearest g)} \qquad \text{or} \qquad h \ge 2008 \text{ (nearest g)}$$

<u>Method 3 (*p*-value)</u>

Since H_0 is rejected,

Either $p\text{-value} = 2\text{P}\left(\bar{Y}_{\text{new}} \ge \frac{30075+h}{16}\right) \le 0.04$



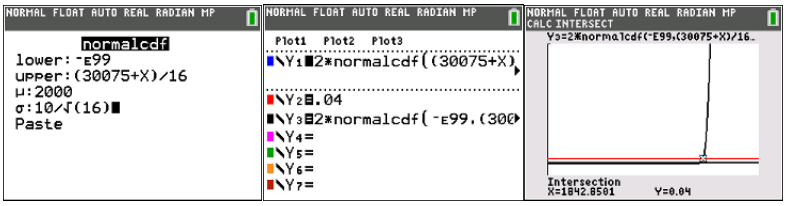
```
NORMAL FLOAT AUTO REAL RADIAN MP
normalcdf
lower:(30075+X)/16
upper:E99
μ:2000
σ:10/√(16)
Paste
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
Plot1  Plot2  Plot3
\Y1■2*normalcdf((30075+X)▸
\Y2■.04
\Y3■2*normalcdf(-E99,(30▸
\Y4=
\Y5=
\Y6=
\Y7=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC INTERSECT
Y2=.04
Intersection
X=2007.1499     Y=0.04
```

From GC, $h \ge 2008$ (nearest g)

Or $p\text{-value} = 2\text{P}\left(\bar{Y}_{\text{new}} \le \frac{30075+h}{16}\right) \le 0.04$



```
NORMAL FLOAT AUTO REAL RADIAN MP
normalcdf
lower:-E99
upper:(30075+X)/16
μ:2000
σ:10/√(16)■
Paste
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
Plot1  Plot2  Plot3
\Y1■2*normalcdf((30075+X)▸
\Y2■.04
\Y3■2*normalcdf(-E99,(30▸
\Y4=
\Y5=
\Y6=
\Y7=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC INTERSECT
Y3=2*normalcdf(-E99,(30075+X)/16…
Intersection
X=1842.8501     Y=0.04
```

From GC, $h \le 1842$ (nearest g)

Total marks: 12