Source of Question: MI Prelim 9758/2024/01/Q1

1 **(i)** A quadratic curve passes through the point $(-1,-4)$ and has its turning point at $(2,5)$. Find the equation of the curve. [4]

(ii) Given instead that a cubic curve passes through the same point $(-1,-4)$ and has the same turning point as stated in part **(i)**. Explain whether it is possible to obtain a unique equation of the curve based on given information. [1]

Solution:

1(i)

<u>METHOD 1</u>

Let $y = ax^2 + bx + c$.

For point $(-1,-4)$:

$a(-1)^2 + b(-1) + c = -4$

$a - b + c = -4$ ---- (1)

For point $(2,5)$:

$a(2)^2 + b(2) + c = 5$

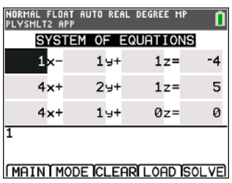
$4a + 2b + c = 5$ ---- (2)

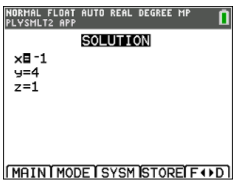
$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b.$

At turning point, when $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$

$2a(2) + b = 0$

$4a + b = 0$ ---- (3)





From GC, $a = -1,\ b = 4,\ c = 1$.

Hence, equation of the curve is $y = -x^2 + 4x + 1$.

<u>METHOD 2 - Preferred</u>

Let $y = a(x-h)^2 + k$.

Using turning point $(2,5)$:

Clearly $h = 2,\ k = 5$

$y = a(x-2)^2 + 5$

For point $(-1,-4)$:

$a(-1-2)^2 + 5 = -4$

$a = -1$

$y = -(x-2)^2 + 5 = x^2 + 4x + 1$

(ii) Not possible.

For a general cubic equation $y = ax^3 + bx^2 + cx + d$, there are 4 unknowns to solve for. But we can only form 3 equations from the given information. Therefore, we will obtain infinitely many solutions.

Source of Question: ACJC Prelim 9758/2024/02/Q1

2 The diagram shows part of the graph $y = x^{\cos 2x}$, for $0 \le x \le 5$, which represents the path of a roller coaster. The horizontal distance travelled by the roller coaster is denoted by x units and its vertical distance travelled is denoted by y units.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^{\cos 2x}\left[\dfrac{\cos 2x}{x} - (2\sin 2x)\ln x\right]$. [2]

(b) At the point on the graph where $x = \pi$, find the rate at which the roller coaster is moving vertically when it is moving horizontally at a rate of 8 units per hour. [2]

(c) Find the acute angle that the tangent to the graph where $x = \pi$ makes with the horizontal. [1]

Solution:

2(a)	$y = x^{\cos 2x}$ $\ln y = (\cos 2x)\ln x$ $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos 2x}{x} + (-2\sin 2x)\ln x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^{\cos 2x}\left[\dfrac{\cos 2x}{x} - (2\sin 2x)\ln x\right]$ (shown)
(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^{\cos 2x}\left[\dfrac{\cos 2x}{x} - (2\sin 2x)\ln x\right] \times 8$ $= \pi^{\cos 2\pi}\left[\dfrac{\cos 2\pi}{\pi} - (2\sin 2\pi)\ln \pi\right] \times 8$ $= \pi\left[\dfrac{1}{\pi} - 0\right] \times 8$ $= 8$
(c)	Since $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\vert_{x=\pi} = 1$, gradient of tangent is 1. Thus the angle that the tangent makes with the horizontal is $\dfrac{\pi}{4}$ or $45°$.

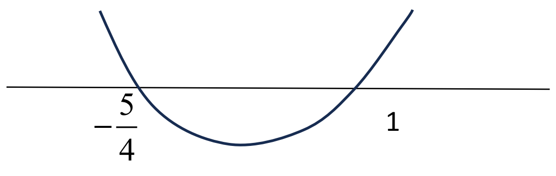
Source of Question: NJC Prelim 9758/2024/01/Q3

3 Do not use a calculator to solve this question.

(i) Solve the inequality $\dfrac{x-6}{4x^2+x-5} \geq 1$. [3]

(ii) Hence solve the inequality $\dfrac{x-6x^2}{4+x-5x^2} \geq 1$. [2]

Solution:

3(i)	**Method 1** $\dfrac{x-6}{4x^2+x-5} \geq 1$ $\Rightarrow (x-6)(4x^2+x-5) \geq (4x^2+x-5)^2 \quad , \quad x \neq -\dfrac{5}{4},\ x \neq 1$ $\Rightarrow (4x^2+x-5)\left[(x-6)-(4x^2+x-5)\right] \geq 0$ $\Rightarrow (4x^2+x-5)(-4x^2-1) \geq 0$ $\Rightarrow (4x+5)(x-1)(-4x^2-1) \geq 0$ $\Rightarrow (4x+5)(x-1) \leq 0$ since $-4x^2-1 \leq 1 < 0$ for all $x \in \mathbb{R}$ **Method 2** $\dfrac{x-6}{4x^2+x-5} \geq 1$ $\Rightarrow (x-6)(4x^2+x-5) \geq (4x^2+x-5)^2$ $\Rightarrow (4x^2+x-5)\left[(x-6)-(4x^2+x-5)\right] \geq 0$ $\Rightarrow (4x^2+x-5)(-4x^2-1) \geq 0$ $\Rightarrow (4x+5)(x-1)(-4x^2-1) \geq 0$ $\Rightarrow (4x+5)(x-1) \leq 0$ since $-4x^2-1 \leq 1 < 0$ for all $x \in \mathbb{R}$  Therefore, $-\dfrac{5}{4} < x < 1$ since $x \neq -\dfrac{5}{4}$ and $x \neq 1$
(ii)	$\dfrac{x-6x^2}{4+x-5x^2} \geq 1$ Replace x with $\dfrac{1}{y}$ and we get

	$\dfrac{\dfrac{1}{y}-6\left(\dfrac{1}{y}\right)^2}{4+\dfrac{1}{y}-5\left(\dfrac{1}{y}\right)^2}\geq 1 \Rightarrow \dfrac{\left(\dfrac{y-6}{y^2}\right)}{\left(\dfrac{4y^2+y-5}{y^2}\right)}\geq 1 \Rightarrow \dfrac{y-6}{4y^2+y-5}\geq 1$ Therefore, $-\dfrac{5}{4}<y<1$, i.e. $-\dfrac{5}{4}<\dfrac{1}{x}<1$. So $-\dfrac{5}{4}<\dfrac{1}{x}<0$ or $0<\dfrac{1}{x}<1$ $x<-\dfrac{4}{5}$ or $x>1$

Source of Question: HCI Prelim 9758/2024/02/Q2

4 A sequence is defined by the recurrence relation

$$u_{n+1}=\frac{1+u_n}{1-u_n},$$

for $n \geq 1$.

(a) State what happens to the sequence when $u_1 = 0$. [1]

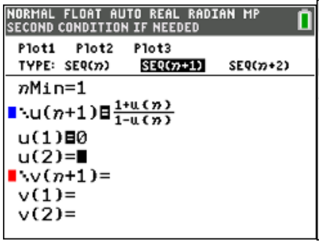
It is now given that $u_1 = 2$.

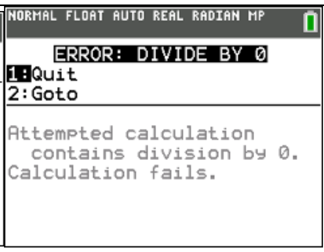
(b) Find u_2, u_3, u_4, u_5 and u_6 . [2]

(c) By observing the pattern in part **(b)**, find $\sum_{r=1}^{4n} u_r$ in terms of n. [2]

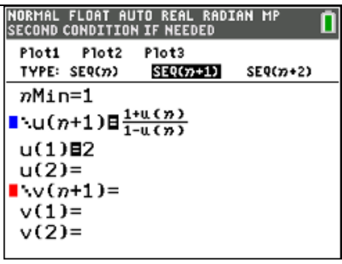
Solution:

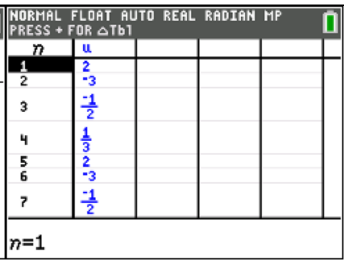
4(a)	$u_2=\dfrac{1+u_1}{1-u_1}=\dfrac{1+0}{1-0}=1$ $u_3=\dfrac{1+u_2}{1-u_2}=\dfrac{1+1}{1-1}$, which is undefined • The sequence ends at $u_2=1$ as the subsequent terms are undefined. OR • There are only two terms and terminate(ends) at the 2nd term.





(b)





From GC,

$u_2 = -3$, $u_3 = -\dfrac{1}{2}$, $u_4 = \dfrac{1}{3}$, $u_5 = 2$, $u_6 = -3$

$u_1 = 2$

$$u_2 = \frac{1+u_1}{1-u_1} = \frac{1+2}{1-2} = -3$$

$$u_3 = \frac{1+u_2}{1-u_2} = \frac{1-3}{1-(-3)} = -\frac{1}{2}$$

$$u_4 = \frac{1+u_3}{1-u_3} = \frac{1-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3}$$

$$u_5 = \frac{1+u_4}{1-u_4} = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$$

$$u_6 = \frac{1+u_5}{1-u_5} = \frac{1+2}{1-2} = -3$$

(c) From observation, the sequence repeats with a period of 4.

$$\sum_{r=1}^{4n} u_r = (u_1+u_2+u_3+u_4)+(u_5+\ldots+u_8)+\ldots+(u_{4n-1}+u_{4n})$$

$$= \left[2-3+\left(-\frac{1}{2}\right)+\frac{1}{3}\right]\times\frac{4n}{4}$$

$$= -\frac{7}{6}n$$

Source of Question: ACJC Prelim 9758/2024/01/Q3

5 Elly started planking as an exercise and she continues the exercise every day to build her core muscles. If she meets her target duration, she increases the target duration of the exercise by an additional 4 seconds on the next day. On any day, she will stop her exercise once she meets her target duration for the day. However, Elly does not always meet her target. Each day when Elly misses her target, she decreases her target duration by 5% on the following day. On Day 1, Elly carries out 20 seconds of planking, and she hopes to reach her target of 2 minutes by the end of 30 days.

(a) Assume that Elly met her targets for the first 11 days but missed her target duration from Day 12 to Day 15. Determine whether Elly will be able to reach her target of 2 minutes by the end of 30 days, if she met all her targets from Day 16 onwards. [3]

Due to the difficulty level, Elly decides to restart the programme by increasing the target duration of the exercise by a% each day, regardless of whether she meets her target.

(b) Find in terms of a, the total target duration Elly has completed by the end of 30 days if she carries out 20 seconds of planking on Day 1. [2]
[You may assume that on any day, she will stop her exercise once she meets her target duration for that day.]

(c) If the total target duration she has completed by the end of 30 days is at least 30 minutes, find, to the nearest integer, the least value of a. [1]

Solution:

5(a)	Let U_n be the Elly's targeted time of nth day $U_{12} = 20 + 11(4) = 64$ $U_{13} = 64(0.95)$ $U_{15} = 64(0.95)^{4-1}$ $U_{16} = 64(0.95)^{5-1}$ $U_{17} = 64(0.95)^4 + 4$ $U_{20} = 64(0.95)^4 + (5-1)(4)$ $\vdots$ $U_{30} = 64(0.95)^4 + 14(4) = 108.1284$ < 120 seconds Since the maximum time Elly can achieve is 108.12 seconds, thus, she is not able to.
(b)	$S_{30} = \dfrac{20\left[1-\left(1+\frac{a}{100}\right)^{30}\right]}{1-\left(1+\frac{a}{100}\right)} = \dfrac{2000}{a}\left[\left(1+\frac{a}{100}\right)^{30} - 1\right]$

(c)	$\dfrac{2000}{a}\left[\left(1+\dfrac{a}{100}\right)^{30}-1\right]\geq 1800$ From GC, $a \geq 6.73$. Thus $a \approx 7$ (to the nearest integer)

Source of Question: HCI Prelim 9758/2024/01/Q5

6 **(a)** By using the substitution $x=\sec\theta$, where $0<\theta<\frac{\pi}{2}$, show that

$$\int_{\sqrt{2}}^{2}\frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \int_{\theta_1}^{\theta_2} \mathrm{g}(\theta)\ \mathrm{d}\theta,$$

where θ_1 and θ_2 are exact constants to be stated, and g is a single trigonometric function to be determined. [4]

(b) Hence find the exact value of $\int_{\sqrt{2}}^{2}\frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$. [3]

Solution:

6(a)	$x=\sec\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sec\theta\tan\theta$ When $x=\sqrt{2}$, $\dfrac{1}{\cos\theta}=\sqrt{2} \Rightarrow \dfrac{1}{\cos\theta}=\dfrac{1}{\sqrt{2}} \Rightarrow \theta=\dfrac{\pi}{4}$ When $x=2$, $\dfrac{1}{\cos\theta}=2 \Rightarrow \dfrac{1}{\cos\theta}=\dfrac{1}{2} \Rightarrow \theta=\dfrac{\pi}{3}$ $\displaystyle\int_{\sqrt{2}}^{2}\frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$ $\displaystyle=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,\mathrm{d}\theta$ $\displaystyle=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\tan\theta}(\sec\theta\tan\theta)\,\mathrm{d}\theta$, Since $\sqrt{\tan^2\theta}=\lvert\tan\theta\rvert=\tan\theta$ where $0<\theta<\dfrac{\pi}{2}$ $\displaystyle=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sec\theta\,\mathrm{d}\theta$

(b)	$\int_{\sqrt{2}}^{2} \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\,\mathrm{d}\theta$ $=\left[\ln\lvert\sec\theta+\tan\theta\rvert\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=\ln\left[2+\sqrt{3}\right]-\ln\left[\sqrt{2}+1\right]$ $=\ln\left[\frac{2+\sqrt{3}}{\sqrt{2}+1}\right]$

Source of Question: DHS Prelim 9758/2024/01/Q7

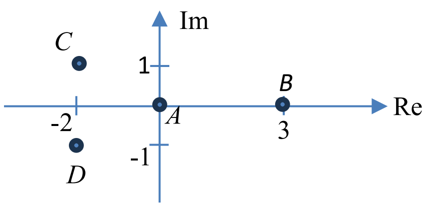
7 The points A, B and C represent the complex numbers a, b and c respectively, such that $a=0$, $b=3$ and $c=-2+\mathrm{i}$. The three complex numbers are roots to the equation $\mathrm{f}(z)=0$ where $\mathrm{f}(z)$ is a quartic polynomial with real coefficients and z is a complex variable.

(a) Express $\mathrm{f}(z)$ as a product of two quadratic factors with real coefficients. [3]

(b) Sketch an Argand diagram showing the roots of the equation $\mathrm{f}(z)=0$. [2]

(c) The point W represents the complex number w, such that $c=\mathrm{i}w$. Find the value of $\overrightarrow{AC}\bullet\overrightarrow{AW}$ and the area of the triangle APW where P represents the complex number $c-b$. [4]

Solution:

7(a)	By conjugate root theorem, if $-2+\mathrm{i}$ is a root, then $-2-\mathrm{i}$ is also a root. $\mathrm{f}(z)$ $=z(z-3)(z-(-2+\mathrm{i}))(z-(-2-\mathrm{i}))$ $=(z^2-3z)((z+2)-\mathrm{i})((z+2)+\mathrm{i})$ $=(z^2-3z)((z+2)^2-\mathrm{i}^2)$ $=(z^2-3z)(z^2+4z+5)$
(b)	

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$a=0$,
$c=-2+\mathrm{i}$.
$c=\mathrm{i}\,w \Rightarrow w=\dfrac{c}{\mathrm{i}}=1+2\mathrm{i}$
$c-b=-2+\mathrm{i}-3=-5+\mathrm{i}$

Area of triangle APW
$=\dfrac{1}{2}\begin{vmatrix} 0 & 1 & -5 & 0 \\ 0 & 2 & 1 & 0 \end{vmatrix}$ ("Shoe-lace method")
$=\dfrac{1}{2}\left|(11)\right|$
$=5.5 \text{ units}^2$

OR

(Using vectors)

$\overrightarrow{AC}\bullet\overrightarrow{AW}=0.$ $\because W$ is obtained by rotating C 90° clockwise so $\angle CAW=90^\circ$

$\overrightarrow{AW}=\begin{pmatrix}1\\2\\0\end{pmatrix}, \overrightarrow{AP}=\begin{pmatrix}-5\\1\\0\end{pmatrix},$

Area of triangle APW
$=\dfrac{1}{2}\left|\overrightarrow{AP}\times\overrightarrow{AW}\right|$
$=\dfrac{1}{2}\left|\begin{pmatrix}-5\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}\right|$
$=\dfrac{1}{2}\left|\begin{pmatrix}0\\0\\-11\end{pmatrix}\right|=5.5 \text{ units}^2$

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Source of Question: VJC Prelim 9758/2024/02/Q4

8 The curve C is defined by the parametric equations

$$x = a\left(1+\frac{1}{t}\right) \text{ and } y = a\left(t-\frac{1}{t^2}\right),$$

where a is a positive constant and $t \neq 0$.

(a) Show that $\frac{dy}{dx} = -\left(\frac{2+t^3}{t}\right)$. [3]

(b) Find the coordinates of the turning point on C, and explain why it is a maximum. [4]

(c) Sketch C. [3]

Solution:

8(a)	$\frac{dx}{dt} = -a\left(\frac{1}{t^2}\right)$ $\frac{dy}{dt} = a\left(1+\frac{2}{t^3}\right)$ $\frac{dy}{dx} = a\left(1+\frac{2}{t^3}\right) \div \left(-a\left(\frac{1}{t^2}\right)\right)$ $= \left(\frac{t^3+2}{t^3}\right) \times \left(-t^2\right)$ $= -\left(\frac{2+t^3}{t}\right)$
(b)	At stationary point, $\frac{dy}{dx} = 0 \Rightarrow \frac{2+t^3}{t} = 0$ $t^3 = -2 \Rightarrow t = -\sqrt[3]{2} = -1.259921$ $x = a\left(1-\frac{1}{\sqrt[3]{2}}\right) = 0.2062995a$ $y = a\left(-\sqrt[3]{2}-\frac{1}{2^{\frac{2}{3}}}\right) = -1.88988a$ Coordinates of turning point are $(0.206a, -1.89a)$.

	To determine that it is a max turning point, use first derivative sign test. Note that since $x=a\left(1+\frac{1}{t}\right)$ with $a>0$, bigger t value gives a smaller x value and vice versa.

t	$\left(-\sqrt[3]{2}\right)^{+}$	$-\sqrt[3]{2}$	$\left(-\sqrt[3]{2}\right)^{-}$
x	$\left[a\left(1-\frac{1}{\sqrt[3]{2}}\right)\right]^{-}$ (left)	$a\left(1-\frac{1}{\sqrt[3]{2}}\right)$	$\left[a\left(1-\frac{1}{\sqrt[3]{2}}\right)\right]^{+}$ (right)
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{2+t^3}{t}\right)>0$ since $t<0$ and $\left(\frac{2+t^3}{t}\right)>0$	0	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{2+t^3}{t}\right)<0$ since $t<0$ and $\left(\frac{2+t^3}{t}\right)<0$
Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$	Positive	Zero	Negative

Therefore, turning point is a maximum.

OR use second derivative test. (Preferred)

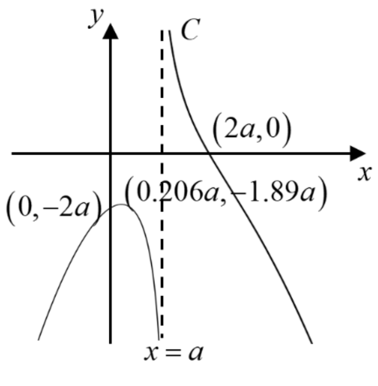
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{2+t^3}{t}\right)\times\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{2}{t}-t^2\right)\times\left(-\frac{t^2}{a}\right)=\left(\frac{2}{t^2}-2t\right)\left(-\frac{t^2}{a}\right)=\frac{2}{a}\left(t^3-1\right)$$

At $t=-\sqrt[3]{2}\Rightarrow t^{\,3}=-2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{2}{a}\left(t^3-1\right)<0$ since $a>0$

Therefore, turning point is a maximum.

(c)



Asymptote at $x=a$:

As $t\to+\infty$, $x\to a^{+}$, $y\to+\infty$

As $t\to-\infty$, $x\to a^{-}$, $y\to-\infty$

Axial intercepts:

At $x=0$, $t=-1$, $y=-2a$

At $y=0$, $t=1$, $x=2a$

Source of Question: ASRJC Prelim 9758/2024/01/Q8

9 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \left|4+2x-x^2\right|,\ x \in \mathbb{R},\ x \geq 3.5,$$

$$\mathrm{g}: x \mapsto 4+\mathrm{e}^{ax},\ x \in \mathbb{R},\ x \geq -1,$$

where $a > 0$.

(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

(b) Find the value of x for which $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(c) Show that the composite function fg exists and express the exact range of fg in the form of $A + B\mathrm{e}^{-a} + C\mathrm{e}^{-2a}$, where A, B and C are real constants. [4]

(d) Without the use of a graphing calculator, solve the inequality $\dfrac{\mathrm{g}(x)}{x^2-2x-2} \geq 0$.

Leave your answer in exact form. [3]

Solution:

9(a)

$y = \left|4+2x-x^2\right|,\ x \in \mathbb{R},\ x \geq 3.5$

$\therefore y = -\left(4+2x-x^2\right) \quad (\because 4+2x-x^2 < 0 \text{ for } x \geq 3.5)$

$y = x^2-2x-4$

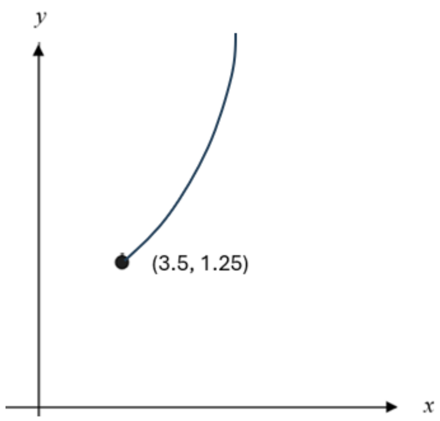
$y = (x-1)^2-1-4$

$y = (x-1)^2-5$

$x = 1+\sqrt{y+5}$ or $x = 1-\sqrt{y+5}$ $\left(\text{rej } \because x \geq 3.5\right)$

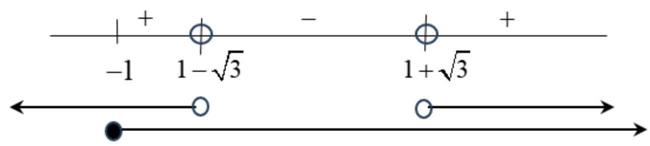
$x = 1+\sqrt{y+5}$

$\mathrm{f}^{-1}(x) = 1+\sqrt{x+5}$



$\mathrm{R}_{\mathrm{f}} = [1.25, \infty)$

$\therefore \mathrm{D}_{\mathrm{f}^{-1}} = [1.25, \infty)$

(b)	$\mathrm{f}^{-1}(x)=\mathrm{f}(x)$, $x\in \mathrm{D}_{\mathrm{f}}\cap \mathrm{D}_{\mathrm{f}^{-1}}$ $x=\mathrm{f}(x)$, $x\in[3.5,\ \infty)$ $x=x^2-2x-4$ If equate f^{-1} found in **(a)** to f, can also give the $x^2-3x-4=0$ $(x-4)(x+1)=0$ $x=-1$ (rej $\because x\geq 3.5$) or $x=4$
(c)	$\mathrm{R_g}=\left[4+\mathrm{e}^{-a},\infty\right)$ $\mathrm{D_f}=[3.5,\infty)$ Since $\mathrm{e}^{-a}>0$, $\therefore 4+\mathrm{e}^{-a}>3.5$ $\therefore \mathrm{R_g}\subset \mathrm{D_f}$, fg exists. $\mathrm{R_g}=\left[4+\mathrm{e}^{-a},\infty\right)$ $\mathrm{f}(4+\mathrm{e}^{-a})=(4+\mathrm{e}^{-a})^2-2(4+\mathrm{e}^{-a})-4$ $=16+8\mathrm{e}^{-a}+\mathrm{e}^{-2a}-8-2\mathrm{e}^{-a}-4$ $=4+6\mathrm{e}^{-a}+\mathrm{e}^{-2a}$ $\mathrm{R_{fg}}=\left[4+6\mathrm{e}^{-a}+\mathrm{e}^{-2a},\infty\right)$
(d)	$\dfrac{\mathrm{g}(x)}{x^2-2x-2}\geq 0$ **and** $x\in \mathrm{D_g}$ $\dfrac{4+\mathrm{e}^{ax}}{(x-1)^2-3}\geq 0$ **and** $x\geq -1$ Since $4+\mathrm{e}^{ax}>0,\ \forall x\in\mathbb{R}$ $\therefore \dfrac{1}{(x-1+\sqrt{3})(x-1-\sqrt{3})}\geq 0$ **and** $x\geq -1$  $\left(x<1-\sqrt{3} \text{ or } x>1+\sqrt{3}\right)$ **and** $x\geq -1$ – for $x<1-\sqrt{3}$ or $x>1+\sqrt{3}$ $\therefore -1\leq x<1-\sqrt{3}$ **or** $x>1+\sqrt{3}$

Source of Question: VJC Prelim 9758/2024/01/Q12

10 Game developers closely monitor the number of people playing their game. Understanding player numbers and behaviour can not only help in optimising in-game purchases, advertisement placements, and other revenue-generating aspects, it can also help the company manage server loads and ensure the game runs smoothly without performance issues.

Two game developers are interested in the number of players playing the mobile game "Mobile Saga". They attempt to model the number of players x, in **hundred thousands**, at time t months after the launch of the game using a differential equation. On the day of the launch, there were 55 000 players.

(a) One game developer suggests that x and t are related by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{5}x-kt^2$, where k is a positive constant.

(i) By substituting $x=u\mathrm{e}^{\frac{3}{5}t}$, show that the differential equation can be written as $\frac{\mathrm{d}u}{\mathrm{d}t}=-kt^2\,\mathrm{e}^{-\frac{3}{5}t}$. [2]

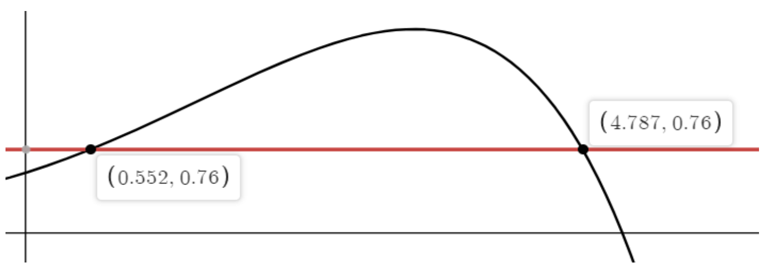
(ii) Hence show that $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)\mathrm{e}^{\frac{3}{5}t}$. [4]

(iii) Company A intends to place an advertisement in the game only if there are more than 76 000 players playing the game. Given that $k=\frac{1}{10}$, find the length of time for which Company A will place an advertisement in "Mobile Saga", giving your answer correct to the nearest month. [2]

(b) The other game developer suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$. Given further that there were 180 000 players playing "Mobile Saga" after 1 month, find x in terms of t. [4]

Solution:

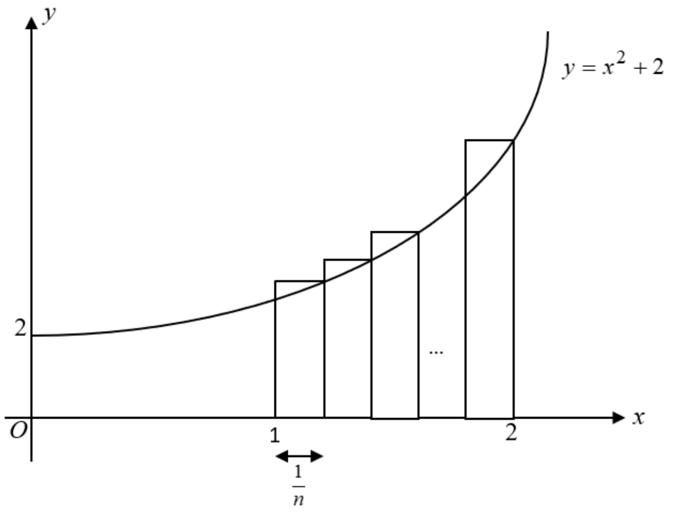
10 (a) (i)	$x=u\,\mathrm{e}^{\frac{3}{5}t}\qquad ---(1)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=0.6x-kt^2\quad ---(2)$ Differentiating (1) w.r.t t $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{5}u\,\mathrm{e}^{\frac{3}{5}t}+\mathrm{e}^{\frac{3}{5}t}\frac{\mathrm{d}u}{\mathrm{d}t}\qquad ---(3)$ Substitute (1) & (3) into (2)

	$\frac{3}{5}u\,e^{\frac{3}{5}t}+e^{\frac{3}{5}t}\frac{du}{dt}=0.6\left(u\,e^{\frac{3}{5}t}\right)-kt^2$ $e^{\frac{3}{5}t}\frac{du}{dt}=-kt^2$ $\frac{du}{dt}=-kt^2\,e^{-\frac{3}{5}t}$
(a) (ii)	$\frac{du}{dt}=-kt^2\,e^{-\frac{3}{5}t}$ Integrating w.r.t. t: $u=-k\int t^2\,e^{-\frac{3}{5}t}\ dt$ $=-k\left[-\frac{5}{3}t^2\,e^{-\frac{3}{5}t}-\int(2t)\left(-\frac{5}{3}e^{-\frac{3}{5}t}\right)dt\right]$ $=\frac{5k}{3}t^2\,e^{-\frac{3}{5}t}-\frac{10k}{3}\int t\,e^{-\frac{3}{5}t}\ dt$ $=\frac{5k}{3}t^2\,e^{-\frac{3}{5}t}-\frac{10k}{3}\left[\left(-\frac{5}{3}t\,e^{-\frac{3}{5}t}\right)-\int-\frac{5}{3}e^{-\frac{3}{5}t}\ dt\right]$ $=\frac{5k}{3}t^2\,e^{-\frac{3}{5}t}+\frac{50k}{9}t\,e^{-\frac{3}{5}t}+\frac{250k}{27}e^{-\frac{3}{5}t}+C$ $x\,e^{-\frac{3}{5}t}=\frac{5k}{3}t^2\,e^{-\frac{3}{5}t}+\frac{50k}{9}t\,e^{-\frac{3}{5}t}+\frac{250k}{27}e^{-\frac{3}{5}t}+C$ $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+C\,e^{\frac{3}{5}t}$ When $t=0$, $x=0.55$ $0.55=\frac{250k}{27}+C \qquad \Rightarrow C=\frac{11}{20}-\frac{250k}{27}$ Hence $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)e^{\frac{3}{5}t}$
(a) (iii)	$x=\frac{1}{10}\left(\frac{5}{3}t^2+\frac{50}{9}t+\frac{250}{27}\right)+\left(-\frac{203}{540}\right)e^{\frac{3}{5}t}$  $\frac{1}{10}\left(\frac{5}{3}t^2+\frac{50}{9}t+\frac{250}{27}\right)+\left(-\frac{203}{540}\right)e^{\frac{3}{5}t}=0.76$ From the GC, longest duration = $4.787-0.552=4.235\ \approx 4$

(b)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$ Integrating w.r.t t: $\frac{\mathrm{d}x}{\mathrm{d}t}=-10\int\frac{1}{(1+t)^3}\,\mathrm{d}t=\frac{5}{(1+t)^2}+C$ Integrating w.r.t. t: $x=5\int\frac{1}{(1+t)^2}\,\mathrm{d}t=-\frac{5}{(1+t)}+Ct+D$ When $t=0$, $x=0.55$ $0.55=-5+D \quad \Rightarrow D=\frac{111}{20}$ When $t=1$, $x=1.8$ $1.8=-\frac{5}{(1+1)}+C+\frac{111}{20} \ \Rightarrow C=-\frac{5}{4}$ $\therefore x=-\frac{5}{1+t}-\frac{5}{4}t+\frac{111}{20}$

Source of Question: CJC Prelim 9758/2024/01/Q9

11 (a)



The diagram shows part of the graph of $y = x^2 + 2$ from $x = 1$ to $x = 2$. The area under the curve in this interval may be approximated by the total area of n rectangles, A, as shown. The width of each rectangle is $\frac{1}{n}$.

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, show that $A = \frac{13}{3} + \frac{3}{2n} + \frac{1}{6n^2}$.

Deduce the exact value of $\int_1^2 \left(2 + x^2\right) \, dx$, justifying your answer. [6]

(b) Region R is bounded by the curve $y = \ln x$, $x = 1$ and $y = \ln 5$.

(i) Find the area of R. [2]

(ii) Find the exact volume of the solid generated when R is rotated through 4 right angles about the y-axis. [4]

Solution:

11 (a)	Width of 1 rectangle = $\frac{1}{n}$ $x_1 = 1 + \frac{1}{n}$: Height of 1st rectangle $= \left(1 + \frac{1}{n}\right)^2 + 2 = \left(\frac{n+1}{n}\right)^2 + 2$ $x_2 = 1 + \frac{2}{n}$: Height of 2nd rectangle $= \left(1 + \frac{2}{n}\right)^2 + 2 = \left(\frac{n+2}{n}\right)^2 + 2$ $x_3 = 1 + \frac{3}{n}$: Height of 3rd rectangle $= \left(1 + \frac{3}{n}\right)^2 + 2 = \left(\frac{n+3}{n}\right)^2 + 2$ …… …… $x_n = 1 + \frac{n}{n} = 2$: Height of nth rectangle $= \left(1 + \frac{n}{n}\right)^2 + 2 = \left(\frac{n+n}{n}\right)^2 + 2$

Total area of n rectangles, A

$$=\frac{1}{n}\left[\left(1+\frac{1}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(1+\frac{2}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(1+\frac{3}{n}\right)^2+2\right]+\ldots+\frac{1}{n}\left[\left(1+\frac{n}{n}\right)^2+2\right]$$

$$=\frac{1}{n}\sum_{r=1}^{n}\left[\left(1+\frac{r}{n}\right)^2+2\right]$$

$$=\frac{1}{n}\sum_{r=1}^{n}\left(3+\frac{2r}{n}+\frac{r^2}{n^2}\right)$$

$$=3+\frac{2}{n^2}\sum_{r=1}^{n}r+\frac{1}{n^3}\sum_{r=1}^{n}r^2$$

$$=3+\frac{2}{n^2}\left[\frac{n}{2}(n+1)\right]+\frac{1}{n^3}\left[\frac{n}{6}(n+1)(2n+1)\right]$$

$$=3+\frac{1}{n}(n+1)+\frac{1}{6n^2}\left(2n^2+3n+1\right)$$

$$=3+\left(1+\frac{1}{n}\right)+\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)$$

$$=\frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2}\quad\text{(Shown)}$$

Or,

Total area of n rectangles, A

$$=\frac{1}{n}\left[\left(\frac{n+1}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(\frac{n+2}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(\frac{n+3}{n}\right)^2+2\right]+\ldots+\frac{1}{n}\left[\left(\frac{n+n}{n}\right)^2+2\right]$$

$$=\frac{1}{n}\sum_{r=1}^{n}\left[\left(\frac{n+r}{n}\right)^2+2\right]$$

$$=\frac{1}{n^3}\sum_{r=n+1}^{2n}r^2+2$$

$$=\frac{1}{n^3}\left(\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2\right)+2$$

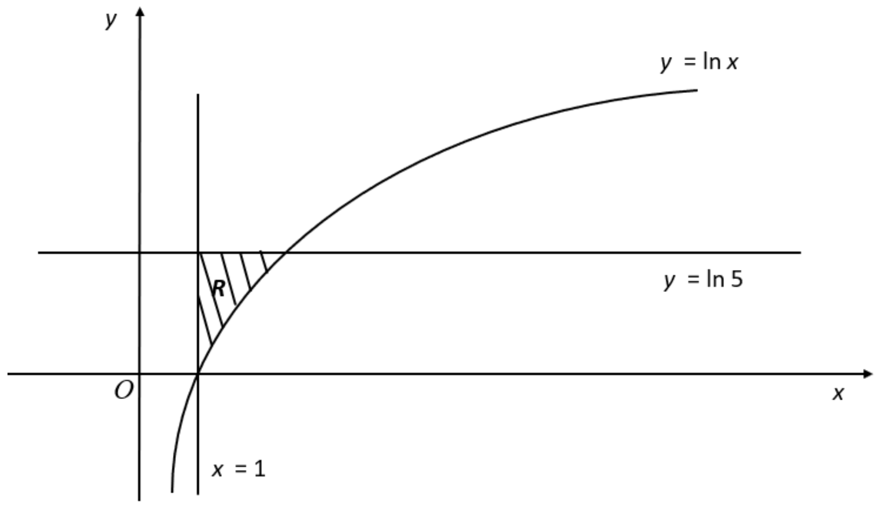
$$=\frac{1}{n^3}\left[\frac{2n}{6}(2n+1)(4n+1)-\frac{n}{6}(n+1)(2n+1)\right]+2$$

$$=\frac{1}{n^3}\left[\frac{n}{6}(2n+1)(7n+1)\right]+2$$

$$=\frac{1}{6n^2}\left[14n^2+9n+1\right]+2$$

$$=\frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2}\quad\text{(Shown)}$$

$$\int_1^2\left(2+x^2\right)\,\mathrm{d}x=\lim_{n\to\infty}A=\lim_{n\to\infty}\left(\frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2}\right)=\frac{13}{3}\quad\text{units}^2$$

(b) (i)	 Area of $R = \int_1^5 (\ln 5 - \ln x)\,\mathrm{d}x = 2.39$ units^2 (to 3 s.f.) Or, Area of $R = (5-1)(\ln 5) - \int_1^5 \ln x\,\mathrm{d}x = 2.39$ units^2 (to 3 s.f.) Or, Area of $R = \int_0^{\ln 5} (\mathrm{e}^y - 1)\,\mathrm{d}y = 2.39$ units^2 (to 3 s.f.) Or, Area of $R = \left(\int_0^{\ln 5} \mathrm{e}^y\,\mathrm{d}y\right) - (1-0)(\ln 5) = 2.39$ units^2 (to 3 s.f.)
(b) (ii)	Volume of solid generated when R is rotated about y-axis $= \pi \int_0^{\ln 5} (\mathrm{e}^{2y} - 1)\,\mathrm{d}y$ $= \pi \left[\frac{1}{2}\mathrm{e}^{2y} - y \right]_0^{\ln 5}$ $= \pi \left[\left(\frac{1}{2}\mathrm{e}^{2\ln 5} - \ln 5 \right) - \left(\frac{1}{2} - 0 \right) \right]$ $= \pi \left[\frac{1}{2}(5^2) - \frac{1}{2} - \ln 5 \right]$ $= \pi (12 - \ln 5)$ units^2

	Or, Volume of solid generated when R is rotated about y-axis $=\pi\int_0^{\ln 5} e^{2y}\,dy-\pi(1)^2(\ln 5)$ $=\pi\left[\frac{1}{2}e^{2y}\right]_0^{\ln 5}-\pi\ln 5$ $=\frac{1}{2}\pi\left[e^{2\ln 5}-1\right]\ -\pi\ln 5$ $=\frac{1}{2}\pi\left[\left(5^2\right)-1\right]-\pi\ln 5$ $=\pi(12-\ln 5)\quad \text{units}^2$

Source of Question: TMJC Prelim 9758/2024/01/Q8

12 Figure 1 shows a triangular prism *OABCDE,* where *O* is the origin. Point *F* lies on *AE* such that $AF{:}FE = 1-k : k$ where $0<k<1$. Figure 2 shows a model of a crystal trophy that is made by cutting away a tetrahedral section *CDEF* from the triangular prism *OABCDE*. With respect to *O*, the coordinates of *A*, *B* and *D* are $(4,3,0)$, $(0,6,0)$ and $(0,0,20)$ respectively. It is given that $\overrightarrow{OD}=\overrightarrow{BC}=\overrightarrow{AE}$.

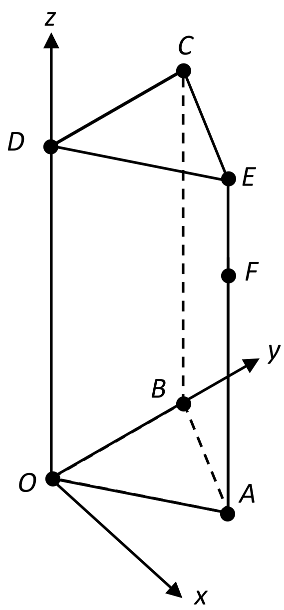


Figure 1

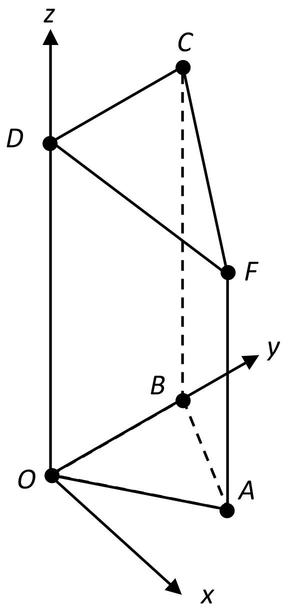


Figure 2

(i) Write down the position vector of *E* and find the position vector of *F* in terms of *k*. [2]

(ii) Show that the equation of the plane *CDF* can be written as $\mathbf{r}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20.$ [3]

(iii) Find the *x*-coordinate of the foot of perpendicular from the point *E* to the plane *CDF* in terms of *k*. [3]

(iv) Hence, find the value of *k* such that the reflection of the plane *CDE* in the plane *CDF* lies on the plane *OBCD*. [4]

Solution:

12 (i)	$\overrightarrow{OE}=\begin{pmatrix}4\\3\\20\end{pmatrix}$ Using Ratio Theorem, $\overrightarrow{OF}=k\begin{pmatrix}4\\3\\0\end{pmatrix}+(1-k)\begin{pmatrix}4\\3\\20\end{pmatrix}=\begin{pmatrix}4\\3\\20(1-k)\end{pmatrix}$

(ii)	$\overrightarrow{DC}=\overrightarrow{OB}=\begin{pmatrix}0\\6\\0\end{pmatrix}$ $\overrightarrow{DF}=\overrightarrow{OF}-\overrightarrow{OD}$ $=\begin{pmatrix}4\\3\\20(1-k)\end{pmatrix}-\begin{pmatrix}0\\0\\20\end{pmatrix}$ $=\begin{pmatrix}4\\3\\-20k\end{pmatrix}$ $n=\overrightarrow{DC}\times\overrightarrow{DF}=\begin{pmatrix}0\\6\\0\end{pmatrix}\times\begin{pmatrix}4\\3\\-20k\end{pmatrix}=\begin{pmatrix}-120k\\0\\-24\end{pmatrix}=-24\begin{pmatrix}5k\\0\\1\end{pmatrix}$ Equation of the plane *CDF* $\mathbf{r}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\20\end{pmatrix}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20$ (Shown)
(iii)	Let *N* be the foot of perpendicular from point *E* to the plane *CDF*. $\overrightarrow{ON}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\lambda\begin{pmatrix}5k\\0\\1\end{pmatrix},\quad \lambda\in\mathbb{R}$ $\begin{pmatrix}4+5k\lambda\\3\\20+\lambda\end{pmatrix}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20$ $5k(4+5k\lambda)+20+\lambda=20$ $20k+25k^2\lambda+\lambda=0$ $(25k^2+1)\lambda=-20k$ $\lambda=\dfrac{-20k}{(25k^2+1)}$ $\overrightarrow{ON}=\begin{pmatrix}4\\3\\20\end{pmatrix}+\left[\dfrac{-20k}{(25k^2+1)}\right]\begin{pmatrix}5k\\0\\1\end{pmatrix}=\begin{pmatrix}4-\dfrac{100k^2}{(25k^2+1)}\\3\\20-\dfrac{20k}{(25k^2+1)}\end{pmatrix}$

	The x-coordinate of $N = 4-\dfrac{100k^2}{\left(25k^2+1\right)}=\dfrac{4}{\left(25k^2+1\right)}$
(iv)	Let E' be the reflection of point E in the plane CDF By ratio theorem, $\overrightarrow{ON}=\dfrac{\overrightarrow{OE}+\overrightarrow{OE}'}{2}$ $\dfrac{4}{\left(25k^2+1\right)}=\dfrac{1}{2}\left(4+x\right)$ $x=\dfrac{8}{\left(25k^2+1\right)}-4$ **Alternatively,** $\overrightarrow{OE}'=2\overrightarrow{ON}-\overrightarrow{OE}=2\begin{pmatrix}\dfrac{4}{\left(25k^2+1\right)}\\ 3\\ 20-\dfrac{20k}{\left(25k+1\right)}\end{pmatrix}-\begin{pmatrix}4\\3\\20\end{pmatrix}=\begin{pmatrix}\dfrac{8}{\left(25k^2+1\right)}-4\\ 3\\ 20-\dfrac{40k}{\left(25k+1\right)}\end{pmatrix}$ x-coordinate of $E' = \dfrac{8}{\left(25k^2+1\right)}-4$ For the reflection of the plane CDE in the plane CDF to lie on the plane $OBCD$, E' has to be a point on the plane $OBCD$. Therefore, x-coordinate of $E' = 0$ $\dfrac{8}{\left(25k^2+1\right)}-4=0$ $\dfrac{8}{\left(25k^2+1\right)}=4$ $8=100k^2+4$ $100k^2=4$ $k^2=\dfrac{1}{25}$ $k=\pm\dfrac{1}{5}$ Since $k>0$, $k=\dfrac{1}{5}$