RAFFLES INSTITUTION
H2 Mathematics 9758
2025 Year 6 Term 4 Post-Prelim Revision
Paper 1 (Source: A level questions)

Total number of marks: 100 **3 hour**

1 **9758/2024/01/Q1**

The graph of $y=\dfrac{x+1}{ax^2+bx+c}$, where a, b and c are non-zero constants, has an asymptote at $x=-\dfrac{1}{2}$. The graph also has a turning point at $\left(-2,\ -\dfrac{1}{9}\right)$. Find the values of a, b and c. [4]

	Suggested Solution	Remarks for Student
[4]	$y=\dfrac{x+1}{ax^2+bx+c}$ Asymptote at $x=-\dfrac{1}{2} \Rightarrow a\left(-\dfrac{1}{2}\right)^2+b\left(-\dfrac{1}{2}\right)+c=0$ $\Rightarrow a-2b+4c=0 \quad \cdots(1)$ Graph passes through $\left(-2,-\dfrac{1}{9}\right) \Rightarrow -\dfrac{1}{9}=-\dfrac{1}{4a-2b+c}$ $\Rightarrow 4a-2b+c=9 \quad \cdots(2)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{ax^2+bx+c-(x+1)(2ax+b)}{\left(ax^2+bx+c\right)^2}$ $=\dfrac{-ax^2-2ax+c-b}{\left(ax^2+bx+c\right)^2}$ Turning point at $\left(-2,-\dfrac{1}{9}\right) \Rightarrow -4a+4a+c-b=0$ $\Rightarrow -b+c=0 \quad \cdots(3)$ Solving: $a=2,\ b=-1$ and $c=-1$	

2 9758/2023/01/Q1

Find the exact equation of the tangent to the curve

$$\ln y = (11-5x)^2$$

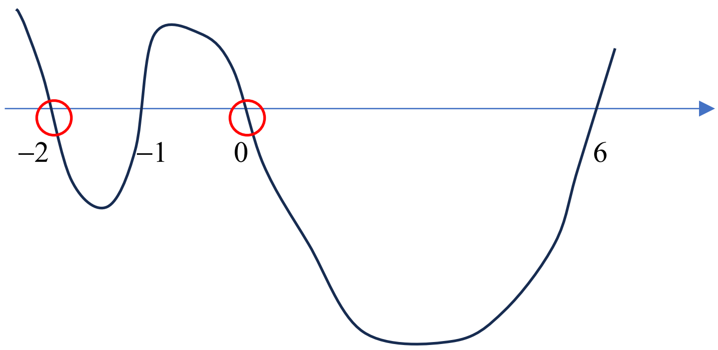
at the point where $x = 2$. [5]

No.	Suggested Solution	Remarks for Student
	$\ln y = (11-5x)^2$ Differentiate with respect to x, $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2(11-5x)(-5) = -10(11-5x)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -10y(11-5x)$ When $x=2$, $\ln y = (11-5(2))^2 = 1 \Rightarrow y = \mathrm{e}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -10y(11-5x) = -10(\mathrm{e})(11-10) = -10\mathrm{e}$ The equation of tangent is $y - \mathrm{e} = -10\mathrm{e}(x-2)$ $y = -10\mathrm{e}x + 21\mathrm{e}$	

3 9758/2024/01/Q4

(a) Without using a calculator, solve the inequality $\frac{4}{x+2} \geq \frac{x-3}{x}$. [4]

(b) Hence, solve the inequality $\frac{4}{\lvert x\rvert+2} \geq \frac{\lvert x\rvert-3}{\lvert x\rvert}$. [2]

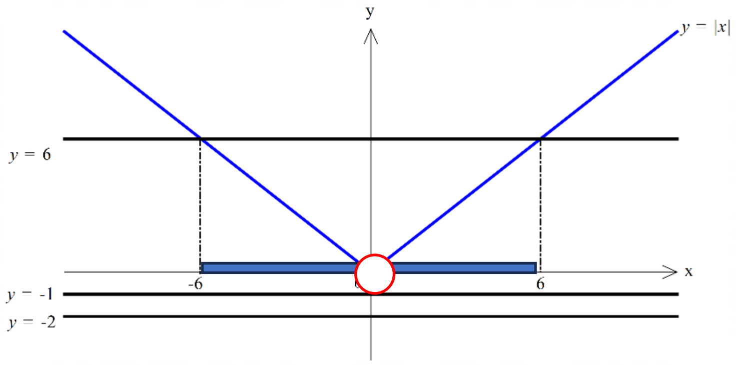
No.	Suggested Solution	Remarks for Student
(a) **[4]**	$\frac{4}{x+2} \geq \frac{x-3}{x}$ $\Rightarrow \frac{4x-(x+2)(x-3)}{x(x+2)} \geq 0$ $\Rightarrow \frac{-4x+(x+2)(x-3)}{x(x+2)} \leq 0$ $\Rightarrow x(x+2)(x^2-5x-6) \leq 0$ and $x \neq 0$ and $x \neq -2$ $\Rightarrow x(x+2)(x-6)(x+1) \leq 0$ and $x \neq 0$ and $x \neq -2$  $\therefore\ -2 < x \leq -1$ or $0 < x \leq 6$	No calculator is allowed so key algebraic steps need to be included in your working Multiplying by "–1" to convert to " $+x^2$ " term results in an easier algebraic manipulation that is less prone to careless mistakes

<table>
<tr><td>(b)
[2]</td><td>
To solve $\dfrac{4}{x+2} \geq \dfrac{x-3}{x}$, we replace x by $|x|$ in part (a):

$-2 < |x| \leq -1$ or $0 < |x| \leq 6$

$|x| \geq 0 \quad \Rightarrow \quad -2 < |x| \leq -1$ has no solution

To solve $0 < |x| \leq 6$, $\quad 0 < |x|$ and $|x| \leq 6$
$\Rightarrow x \neq 0$ and $-6 \leq x \leq 6$
$\therefore -6 \leq x < 0$ or $0 < x \leq 6$

Alternatively, we can consider the graph of $y = |x|$:


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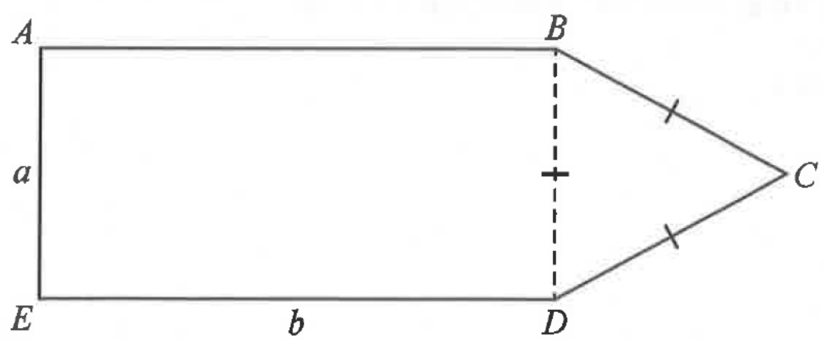
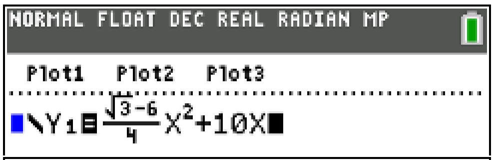
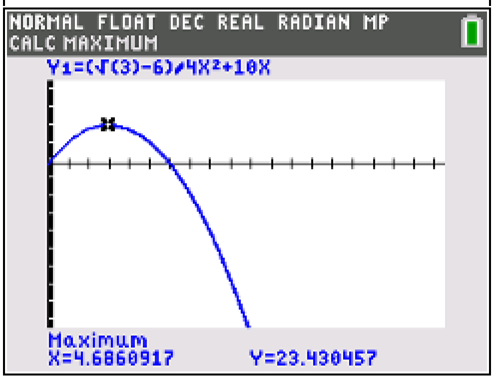
4 **9758/2024/02/Q2**



A gardener designs a flower bed $ABCDE$ in the shape of a rectangle with an equilateral triangle on one of the shorter sides. Side AE is of length a m and side ED is of length b m (see diagram). The total perimeter of the flower bed is 20 m.

Find the maximum possible area of the flower bed, showing that it is a maximum value. Give your answer correct to 4 significant figures. [6]

	Suggested Solution	Remarks for Student
[6]	Considering perimeter of flower bed: $2b+3a=20 \Rightarrow b=10-\frac{3}{2}a \ \cdots(*)$ Area, $A=ab+\frac{1}{2}a^2\sin 60^\circ$ Substitute (*): $A=a\left(10-\frac{3}{2}a\right)+\frac{1}{2}a^2\left(\frac{\sqrt{3}}{2}\right)$ $=\frac{\sqrt{3}-6}{4}a^2+10a$ $\frac{\mathrm{d}A}{\mathrm{d}a}=\frac{\sqrt{3}-6}{2}a+10$ $\frac{\mathrm{d}A}{\mathrm{d}a}=0 \Rightarrow \frac{\sqrt{3}-6}{2}a+10=0$ $\Rightarrow a=\frac{20}{6-\sqrt{3}}$	Since A contains a^2 and a, it is more efficient to express A in terms of a only than in terms of b.
	$\frac{\mathrm{d}^2A}{\mathrm{d}a^2}=\frac{\sqrt{3}-6}{2}\approx -2.13<0$ By 2nd derivative test, maximum area occurs when $a=\frac{20}{6-\sqrt{3}}$. $\therefore$ Maximum area is $\frac{\sqrt{3}-6}{4}\left[\frac{20}{6-\sqrt{3}}\right]^2+10\left[\frac{20}{6-\sqrt{3}}\right]$ $=23.43\ \mathrm{m}^2\,(4\mathrm{sf})$	You can verify the value obtained for maximum area by using a GC as follows:   Interestingly, with the use of the phrase "shorter sides", it is implied that $a<b$. However, when $a=\frac{20}{6-\sqrt{3}}\approx 4.6861$, $b=10-\frac{3}{2}\left(\frac{20}{6-\sqrt{3}}\right)\approx 2.9709$ which suggests that $a>b$.

5 9758/2023/01/Q3

Vectors **a** and **b** are such that $\mathbf{a}.\mathbf{b}=-1$. It is also given that $(\mathbf{a}\times\mathbf{b}+\mathbf{a})$ is perpendicular to $(\mathbf{a}\times\mathbf{b}+\mathbf{b})$.

(a) Show that $|\mathbf{a}\times\mathbf{b}|=1$ [3]

(b) Hence find the angle between the direction of **a** and direction of **b**. [3]

No.	Suggested Solution	Remarks for Student
(a)	$(\mathbf{a}\times\mathbf{b}+\mathbf{a})\perp(\mathbf{a}\times\mathbf{b}+\mathbf{b})$ $(\mathbf{a}\times\mathbf{b}+\mathbf{a}).(\mathbf{a}\times\mathbf{b}+\mathbf{b})=0$ $(\mathbf{a}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{b}).\mathbf{b}+\mathbf{a}.(\mathbf{a}\times\mathbf{b})+\mathbf{a}.\mathbf{b}=0$ $|\mathbf{a}\times\mathbf{b}|^2-1=0$, since $(\mathbf{a}\times\mathbf{b}).\mathbf{b}=\mathbf{a}.(\mathbf{a}\times\mathbf{b})=0$ and $\mathbf{a}.\mathbf{b}=\mathbf{-1}$ $|\mathbf{a}\times\mathbf{b}|^2=1$ $|\mathbf{a}\times\mathbf{b}|=1$, since $|\mathbf{a}\times\mathbf{b}|\geq 0$	$(\mathbf{a}\times\mathbf{b}).\mathbf{b}=\mathbf{a}.(\mathbf{a}\times\mathbf{b})=0$ because $\mathbf{a}\times\mathbf{b}$ is perpendicular to both **a** and **b**. The reason needs to be given.
(b)	Let θ be the angle between the direction of **a** and the direction of **b**. Then, $\mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=-1$ ---- (1) $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta=1$ ---- (2) (2) ÷ (1), $\dfrac{|\mathbf{a}||\mathbf{b}|\sin\theta}{|\mathbf{a}||\mathbf{b}|\cos\theta}=-1$ $\tan\theta=-1$ So, $\theta=135^\circ$	

6 9758/2023/01/Q4bcd

(b) Given that $n \neq 0$, show that $\int x\cos nx\,\mathrm{d}x = \dfrac{x\sin nx}{n} + \dfrac{\cos nx}{n^2} + c$, where c is an arbitrary constant. [3]

(c) Using the result in part **(b)** show that, for all positive integers, n, the value of $\int_0^{\pi} x\cos nx\ \mathrm{d}x$ can be expressed as $\dfrac{k}{n^2}$, where the possible value(s) of k are to be determined [2]

(d) Using the result in part **(b)** find the exact value of $\int_0^{\frac{\pi}{2}} \left|x\cos 2x\right|\ \mathrm{d}x$. [3]

No.	Suggested Solution	Remarks for Student
(b)	$\int x\cos nx\ \mathrm{d}x = \frac{1}{n}x\sin nx - \frac{1}{n}\int \sin nx\ \mathrm{d}x$ $= \frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx + c$	Integration by parts. $u = x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos nx$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad v = \frac{\sin nx}{n}$
(c)	Using the results in part **(b)**, $\int_0^{\pi} x\cos nx\ \mathrm{d}x$ $= \left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$ $= \frac{1}{n}\pi\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{1}{n}(0)\sin n(0) + \frac{1}{n^2}\cos n(0)$ $= \frac{1}{n^2}\cos n\pi - \frac{1}{n^2},$ since $\sin n\pi = 0$ for $n \in \mathbb{Z}$ and $\cos 0 = 1$ When n is even, $\cos n\pi = 1$ and $\int_0^{\pi} x\cos nx\ \mathrm{d}x = \frac{1-1}{n^2} = \frac{0}{n^2} = 0$ When n is odd, $\cos n\pi = -1$ and $\int_0^{\pi} x\cos nx\ \mathrm{d}x = \frac{-1-1}{n^2} = \frac{-2}{n^2}$ Therefore, $\int_0^{\pi} x\cos nx\ \mathrm{d}x = \frac{k}{n^2}$, where $k = \begin{cases} 0 & \text{when } n \text{ is even} \\ -2 & \text{when } n \text{ is odd} \end{cases}$	

(d)

Since $x\cos 2x \ge 0$ when $0 \le x \le \dfrac{\pi}{4}$ and

$x\cos 2x \le 0$ when $\dfrac{\pi}{4} \le x \le \dfrac{\pi}{2}$,

so in the interval $0 \le x \le \dfrac{\pi}{2}$,

$$|x\cos 2x| = \begin{cases} x\cos 2x, & \text{when } 0 \le x \le \dfrac{\pi}{4}; \\ -x\cos 2x, & \text{when } \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2}. \end{cases}$$

$$\int_0^{\frac{\pi}{2}} |x\cos 2x| \, \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} x\cos 2x \, \mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -x\cos 2x \, \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} x\cos 2x \, \mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x \, \mathrm{d}x$$

$$= \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{\frac{\pi}{4}} - \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

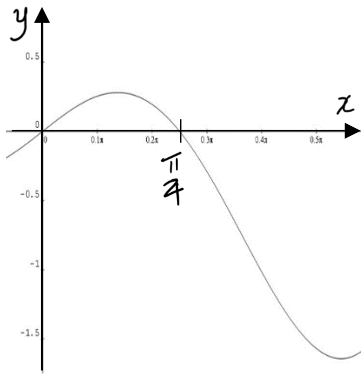
$$= \left[\frac{\frac{\pi}{4}\sin\frac{\pi}{2}}{2} + \frac{\cos\frac{\pi}{2}}{4} - \frac{0}{2} - \frac{\cos 0}{4}\right]$$

$$- \left[\frac{\frac{\pi}{2}\sin\pi}{2} + \frac{\cos\pi}{4} - \frac{\frac{\pi}{4}\sin\frac{\pi}{2}}{2} - \frac{\cos\frac{\pi}{2}}{4}\right]$$

$$= \frac{\pi}{8} + 0 - 0 - \frac{1}{4} - 0 + \frac{1}{4} + \frac{\pi}{8} + 0$$

$$= \frac{\pi}{4}$$

Again, using your GC, sketch $y = x\cos 2x$.



7 9758/2023/01/Q6

(a) Use double angle formula, show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [2]

(b) The region R lies in the first quadrant and is bounded by the curve $y^4 = (9-x^2)^3$, the x-axis and the lines $x = 1.5$ and $x = 3$. R is rotated about the x-axis through 2π radians. Using the substitution $x = 3\sin\theta$, find the exact volume generated. [6]

No.	Suggested Solution	Remarks for Student
(a)	Use double angle formula $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$, we have $\cos^4\theta = \left[\cos^2\theta\right]^2$ $= \left[\frac{1}{2}(\cos 2\theta + 1)\right]^2$ $= \frac{1}{4}\left[\cos^2 2\theta + 2\cos 2\theta + 1\right]$ $= \frac{1}{4}\left[\frac{1}{2}(\cos 4\theta + 1) + 2\cos 2\theta + 1\right]$ since $\cos^2 2\theta = \frac{1}{2}(\cos 4\theta + 1)$ $= \frac{1}{4}\left[\frac{1}{2}\cos 4\theta + 2\cos 2\theta + \frac{3}{2}\right]$ $= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (Shown)	
(b)	Volume generated $= \pi\int_{1.5}^{3} y^2 \, dx = \pi\int_{1.5}^{3} (9-x^2)^{\frac{3}{2}} \, dx$ $x = 3\sin\theta \Rightarrow \frac{dx}{d\theta} = 3\cos\theta$ When $x = 1.5$, $3\sin\theta = 1.5 \Rightarrow \sin\theta = 0.5 \Rightarrow \theta = \frac{\pi}{6}$ When $x = 3$, $3\sin\theta = 3 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$ Therefore, volume generated $= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9 - 9\sin^2\theta)^{\frac{3}{2}} \, 3\cos\theta \, d\theta$ $= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9^{\frac{3}{2}} (1 - \sin^2\theta)^{\frac{3}{2}} \, 3\cos\theta \, d\theta$ $= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 27(\cos^3\theta) \, 3\cos\theta \, d\theta$	

	$=81\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta$ $=81\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{8}\left(\cos 4\theta+4\cos 2\theta+3\right)\mathrm{d}\theta$ $=\frac{81\pi}{8}\left[\frac{\sin 4\theta}{4}+2\sin 2\theta+3\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $=\frac{81\pi}{8}\left[\frac{3\pi}{2}-\frac{\sqrt{3}}{8}-\sqrt{3}-\frac{\pi}{2}\right]$ $=\frac{81\pi}{8}\left[\pi-\frac{9\sqrt{3}}{8}\right]=\frac{81\pi}{64}\left[8\pi-9\sqrt{3}\right]$	

8 (a) 9758/2024/02/Q1b (b) 9758/2024/01/Q3

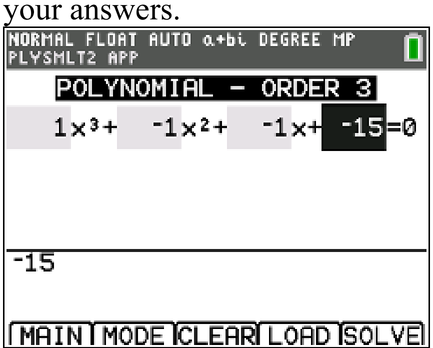
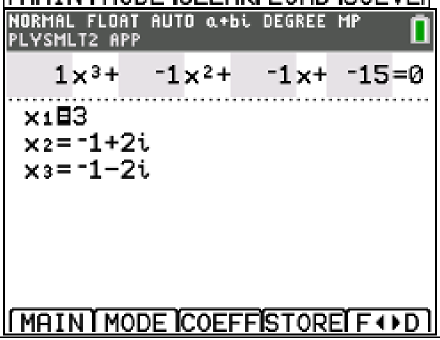
(a) The complex number $-2+\mathrm{i}$ is denoted by z and the complex number $1-3\mathrm{i}$ is denoted by w.

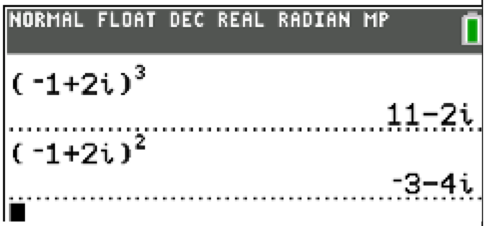
Without using a calculator, evaluate $wz-\dfrac{w}{z}$. . Give your answer in the form $c+d\,\mathrm{i}$, where c and d are real numbers. [4]

(b) The equation $z^3+az^2+bz+c=0$, where a, b and c are constants, has roots 3 and w where w is a complex number.

(i) State a condition on a, b and c for the third root to be w^*. [1]

(ii) Given that the condition in part **(bi)** holds, and that $w=-1+2\mathrm{i}$, find the values of a, b and c. [3]

	Suggested Solution	Remarks for Student
(a) **[4]**	$w=1-3\mathrm{i}$ $wz-\dfrac{w}{z}$ $=(1-3\mathrm{i})(-2+\mathrm{i})-\dfrac{1-3\mathrm{i}}{-2+\mathrm{i}}\times\dfrac{-2-\mathrm{i}}{-2-\mathrm{i}}$ $=\left[-2+3+\mathrm{i}(1+6)\right]-\dfrac{-2-3+\mathrm{i}(-1+6)}{4+1}$ $=1+7\mathrm{i}-\dfrac{1}{5}(-5+5\mathrm{i})$ $=2+6\mathrm{i}$	Detailed working needs to be shown since use of calculator is prohibited in this question (calculator can be used to verify answers).
(bi) **[1]**	a, b and c must be real numbers.	
(bii **[3]**	<u>Method 1 (preferred):</u> $z^3+az^2+bz+c\equiv(z-3)\left[z-(-1+2\mathrm{i})\right]\left[z-(-1-2\mathrm{i})\right]$ $\equiv(z-3)\left[(z+1)^2-(2\mathrm{i})^2\right]$ $\equiv(z-3)\left[z^2+2z+5\right]$ $\equiv z^3-z^2-z-15$ $\therefore a=b=-1$ and $c=-15$	You can use GC Plysmlt2 to check your answers.  

	<u>Method 2:</u> $w=3$ is a root $\Rightarrow 3^3+3^2a+3b+c=0 \quad \Rightarrow 9a+3b+c=-27 \quad \cdots(1)$ $w=-1+2\text{i}$ is a root $\Rightarrow (-1+2\text{i})^3+(-1+2\text{i})^2a+(-1+2\text{i})b+c=0$ $\Rightarrow 11-2\text{i}+(-3-4\text{i})a+(-1+2\text{i})b+c=0$ Comparing real part: $-3a-b+c=-11 \quad \cdots(2)$ Comparing imaginary part: $-4a+2b=2 \quad \cdots(3)$ Solving using GC Plysmlt2: $a=b=-1$ and $c=-15$	Note that we can use GC to evaluate $(-1+2\text{i})^3, (-1+2\text{i})^2$ etc 

9 9758/2024/01/Q8

It is given that $y=\cos\left(1-\mathrm{e}^{2x}\right)$.

(a) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=k\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}-2y\mathrm{e}^{4x}\right)$, where k is a constant to be found. [3]

(b) By differentiation of the result in part **(a)**, find the first three non-zero terms of the Maclaurin expansion of $\cos\left(1-\mathrm{e}^{2x}\right)$. [4]

(c) The first two non-zero terms of the Maclaurin expansion of $\cos\left(1-\mathrm{e}^{2x}\right)$ are equal to the first two non-zero terms of the series expansion of $\dfrac{1}{\sqrt{a+bx^2}}$, where a and b are constants.

Using standard series from the List of Formulae (MF26), find the values of a and b. [2]

No.	Suggested Solution	Remarks for Student
(a) **[3]**	$y=\cos\left(1-\mathrm{e}^{2x}\right)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sin\left(1-\mathrm{e}^{2x}\right)\times\left(-2\mathrm{e}^{2x}\right)$ $=2\mathrm{e}^{2x}\sin\left(1-\mathrm{e}^{2x}\right)$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left[2\mathrm{e}^{2x}\sin\left(1-\mathrm{e}^{2x}\right)+\mathrm{e}^{2x}\cos\left(1-\mathrm{e}^{2x}\right)\times\left(-2\mathrm{e}^{2x}\right)\right]$ $=2\left[\dfrac{\mathrm{d}y}{\mathrm{d}x}-2\mathrm{e}^{4x}y\right]$ where $k=2$ (shown) $\cdots(*)$	
(b) **[4]**	Differentiating (*) wrt x: $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\left[\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(y\left(4\mathrm{e}^{4x}\right)+\mathrm{e}^{4x}\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\right]$ When $x=0$, $y=\cos\left(1-1\right)=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left[0-2\right]=-4$ $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\left[-4-2\left(4\right)\right]=-24$ $y=1+\dfrac{-4}{2!}x^2+\dfrac{-24}{3!}x^3+\cdots$ $=1-2x^2-4x^3+\cdots$ 1st three non-zero terms of Maclaurin expansion of $\cos\left(1-\mathrm{e}^{2x}\right)$ are $1-2x^2-4x^3$ respectively.	

(c) **[2]**	$\dfrac{1}{\sqrt{a+bx^2}}=\left(a+bx^2\right)^{-\frac{1}{2}}$ $=a^{-\frac{1}{2}}\left(1+\dfrac{b}{a}x^2\right)^{-\frac{1}{2}}$ $=a^{-\frac{1}{2}}\left[1+\left(-\dfrac{1}{2}\right)\left(\dfrac{b}{a}x^2\right)+\cdots\right]$ $=a^{-\frac{1}{2}}\left[1+-\dfrac{b}{2a}x^2+\cdots\right]$ $=a^{-\frac{1}{2}}-\dfrac{1}{2}ba^{-\frac{3}{2}}x^2+\cdots$ Comparing the two series: $a^{-\frac{1}{2}}=1\Rightarrow a=1$ $-\dfrac{1}{2}ba^{-\frac{3}{2}}=-2\Rightarrow b=4$	Factorise "a" from $\left(a+bx^2\right)^{-\frac{1}{2}}$ so that we can apply the standard series expansion of $(1+x)^n$ found in the mathematical formulae sheet on $\left(1+\dfrac{b}{a}x^2\right)^{-\frac{1}{2}}$.

10 9758/2023/01/Q9

The line l_1, contains the point A with coordinates $(-3, 1, 2)$ and is parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$ where a is a constant. The line l_2 has equation $\frac{x+2}{3} = \frac{y-1}{2} = z-5$. It is given that l_1 and l_2 cross at the point B.

(a) Find the value of a and the coordinates of B. [5]

(b) The plane π_1 contains the point A and is perpendicular to l_2.

(i) Find the shortest distance from B to π_1. [3]

(ii) Hence find the acute angle between l_1 and π_1. [2]

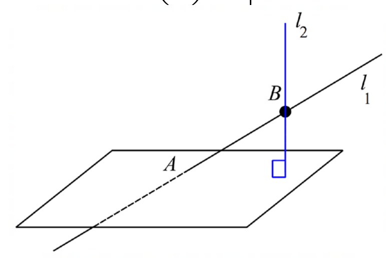
(c) The plane π_2 is perpendicular to π_1 and contains l_1. Find a cartesian equation of π_2. [3]

No.	Suggested Solution	Remarks for Student
(a)	$l_1 : \mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},\ \mu \in \mathbb{R}$ Since l_1 and l_2 cross at the point B, $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, for some $\lambda, \mu \in \mathbb{R}$ $\Rightarrow \begin{cases} 2\lambda - 3\mu = 1 & -(1) \\ \lambda - 2\mu = 0 & -(2) \\ \lambda a - \mu = 3 & -(3) \end{cases}$ Using (1) and (2), $\lambda = 2$ and $\mu = 1$, hence using (3), $a = 2$ Using $\mu = 1$, $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ The coordinates of B is $(1, 3, 6)$.	

(b)(i)

Since π_1 is perpendicular to l_2, then a normal vector of π_1 is $\begin{pmatrix}3\\2\\1\end{pmatrix}$.

$$\overrightarrow{AB}=\begin{pmatrix}1\\3\\6\end{pmatrix}-\begin{pmatrix}-3\\1\\2\end{pmatrix}=\begin{pmatrix}4\\2\\4\end{pmatrix}$$



The shortest distance from B to π_1 is length of projection of $\overrightarrow{AB}$ onto the unit normal vector of π_1

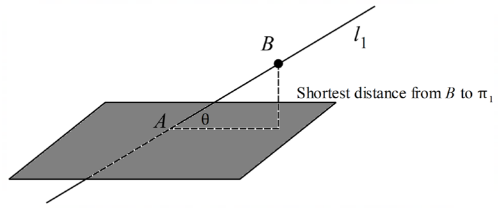
$$=\frac{\left|\begin{pmatrix}4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}3\\2\\1\end{pmatrix}\right|}{\sqrt{3^2+2^2+1^2}}=\frac{20}{\sqrt{14}}$$

(ii)

Let the required angle be θ.

Then, $\sin\theta=\dfrac{\frac{20}{\sqrt{14}}}{\left|\overrightarrow{AB}\right|}=\dfrac{\frac{20}{\sqrt{14}}}{6}=\dfrac{10}{3\sqrt{14}}$

$\theta=62.98^\circ=63.0^\circ$ (correct to 1 d.p.)



(c)

Direction vectors of π_2 are $\begin{pmatrix}4\\2\\4\end{pmatrix}$ and $\begin{pmatrix}3\\2\\1\end{pmatrix}$

Normal vector to $\pi_2=\begin{pmatrix}2\\1\\2\end{pmatrix}\times\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}-3\\4\\1\end{pmatrix}$

The equation of the plane π_2 is $\mathbf{r}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=15$

So, cartesian equation of π_2 is $-3x+4y+z=15$

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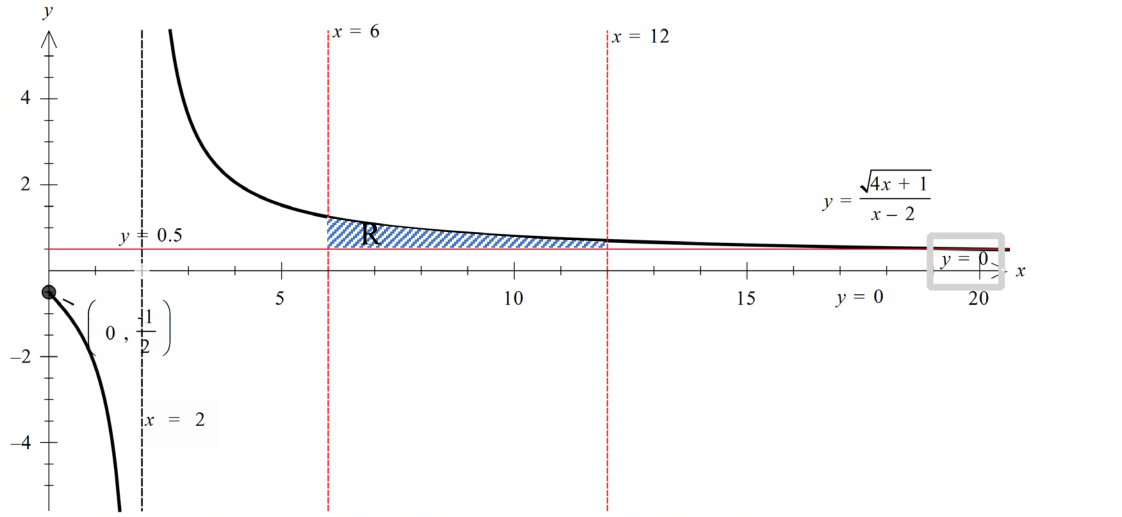
(a) Using the substitution $u=\sqrt{4x+1}$, show that $\int \frac{\sqrt{4x+1}}{x-2}\,\mathrm{d}x=\int \frac{2u^2}{u^2-9}\,\mathrm{d}u$. [3]

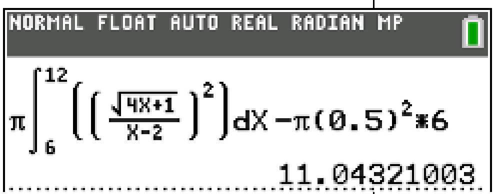
(b) The region R lies in the first quadrant and is bounded by the curve $y=\frac{\sqrt{4x+1}}{x-2}$ and the lines $x=6$, $x=12$ and $y=0.5$.

(i) Sketch the graph of $y=\frac{\sqrt{4x+1}}{x-2}$ for $x\ge 0$, giving the coordinates of any intercepts with the axes and the equations of any asymptotes. Shade the region R on your sketch. [3]

(ii) Find the exact area of R, giving your answer in the form $a+b\ln c$. [5]

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the x-axis. Give your answer correct to 2 decimal places. [3]

No.	Suggested Solution	Remarks for Student
(a)	$u=\sqrt{4x+1}\Rightarrow u^2=4x+1\Rightarrow x=\frac{u^2-1}{4}$ and $\frac{\mathrm{d}x}{\mathrm{d}u}=\frac{2u}{4}=\frac{u}{2}$ $\int \frac{\sqrt{4x+1}}{x-2}\,\mathrm{d}x=\int \frac{u}{\left(\frac{u^2-1}{4}-2\right)}\left(\frac{u}{2}\right)\mathrm{d}u$ $=\int \frac{2u^2}{u^2-1-8}\,\mathrm{d}u$ $=\int \frac{2u^2}{u^2-9}\,\mathrm{d}u$ (shown)	
(bi)		

(bii))	When $x = 6$, $u=5$; When $x = 12$, $u=7$ Required area of R $=\displaystyle\int_6^{12}\frac{\sqrt{4x+1}}{x-2}\,\mathrm{d}x-\left(6\times 0.5\right)$ or $\displaystyle\int_6^{12}\frac{\sqrt{4x+1}}{x-2}-0.5\ \mathrm{d}x$ $=\displaystyle\int_5^{7}\frac{2u^2}{u^2-9}\,\mathrm{d}u-3$ $=\displaystyle\int_5^{7}2+\frac{18}{u^2-9}\,\mathrm{d}u-3$ $=\left[2u+\dfrac{18}{2\times 3}\ln\left\|\dfrac{u-3}{u+3}\right\|\right]_5^7-3$ $=2\left(7\right)+3\ln\left(\dfrac{4}{10}\right)-2\left(5\right)-3\ln\left(\dfrac{2}{8}\right)-3$ $=1+3\ln\left(\dfrac{8}{5}\right)$ units^2 Therefore, $a=1$, $b=3$ and $c=\dfrac{8}{5}$	Exact answer is required
(biii)	Required volume $=\pi\displaystyle\int_6^{12}\left(\frac{\sqrt{4x+1}}{x-2}\right)^2\mathrm{d}x-\pi\left(0.5\right)^2\times 6$ $=11.04$ units^3 (2 d.p.)  	

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The mass of a person depends both on daily rate of energy intake and on daily rate of energy expenditure.

In this question, mass is in kg, time is in days, and energy intake and energy expenditure are measured in Calories per day.

The rate of change of a person's mass with respect to time is proportional to the difference between energy intake and energy expenditure.

Andrew has a mass of M kg and his energy intake is fixed at C Calories per day. For every kg of his mass, he expends 30 Calories per day.

(a) Show that $\dfrac{\mathrm{d}M}{\mathrm{d}t}=k\left(C-30M\right)$, where t is time and k is a constant. [1]

Andrew's initial mass is 110 kg.

(b) Find the energy intake such that he maintains his mass at 110 kg. [1]

As part of a health plan, Andrew fixes his energy intake at 80% of the value found in part **(b)**.

(c) By solving the differential equations in part **(a)**, show that Andrew's mass while he is on the plan satisfies the equation $M=88+22\mathrm{e}^{-30kt}$. [4]

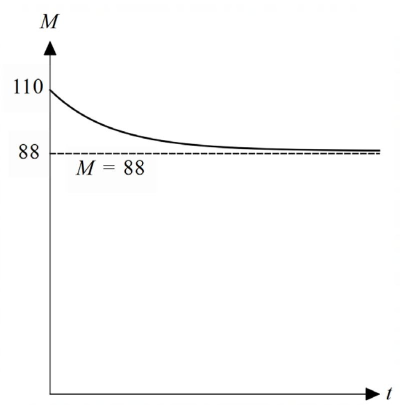
Andrew's mass after 75 days on the plan is 100 kg.

(d) Find the number of additional days required on the plan for Andrew's mass to fall below 96 kg. [4]

(e) **(i)** Sketch a graph of Andrew's mass while on this plan. Explain why Andrew cannot achieve a mass of 80 kg using this plan. [2]

(ii) State the range of possible values of energy intake for which Andrew could achieve a mass of 80 kg. [1]

No.	Suggested Solution	Remarks for Student
(a)	Energy intake by Andrew per day = C Calories Energy expenditure by Andrew per day = $30M$ Calories Difference between energy intake and energy expenditure $= C-30M$ Since $\dfrac{\mathrm{d}M}{\mathrm{d}t}\propto C-30M$, then $\dfrac{\mathrm{d}M}{\mathrm{d}t}=k\left(C-30M\right)$, where k is the constant of proportionality.	
(b)	$\dfrac{\mathrm{d}M}{\mathrm{d}t}=0$ when $M=110$. So, $k\left(C-30(110)\right)=0$ $C=3300$ Calories per day	Note that mass is constant means $\dfrac{\mathrm{d}M}{\mathrm{d}t}=0$
(c)	New $C=0.8\times3300=2640$ Calories per day	

	$\frac{\mathrm{d}M}{\mathrm{d}t} = k\left(2640-30M\right)$ $\int \frac{1}{2640-30M}\ \mathrm{d}M = \int k\ \mathrm{d}t$ $-\frac{1}{30}\ln\left\|2640-30M\right\| = kt + D$ where D is an arbitrary constant $\left\|2640-30M\right\| = \mathrm{e}^{-30(kt+D)}$ $2640-30M = A\mathrm{e}^{-30kt}$ where $A = \pm\mathrm{e}^{-30D}$ $30M = 2640 - A\mathrm{e}^{-30kt}$ $M = 88 + B\mathrm{e}^{-30kt}$ where $B = -\frac{A}{30}$ When $t = 0$, $M = 110$, $110 = 88 + B \Rightarrow B = 22$ Therefore, $M = 88 + 22\mathrm{e}^{-30kt}$ (Shown)	Remember to include the modulus Recall how to remove the modulus
(d)	When $t = 75$, $M = 100$, $100 = 88 + 22\mathrm{e}^{-30k(75)}$ $k = -\frac{1}{2250}\ln\left(\frac{6}{11}\right)$ Note that $k = -\frac{1}{2250}\ln\left(\frac{6}{11}\right) > 0$ For $M < 96$, $88 + 22\mathrm{e}^{-30kt} < 96$ $\mathrm{e}^{-30kt} < \frac{4}{11}$ $-30kt < \ln\left(\frac{4}{11}\right)$ $t > \frac{\ln\left(\frac{4}{11}\right)}{-30k} = \frac{75\ln\left(\frac{4}{11}\right)}{\ln\left(\frac{6}{11}\right)} = 125.17$ Number of additional days needed is $126 - 75 = 51$	The variable t is measured from the time when Andrew's mass is initially 110 kg. But the question is asking for the number of **additional** days to fall from 100 kg to 96 kg. Thus, there is a need to subtract by 75 days.
(e) (i)	 Andrew's mass is reducing and approaching 88 kg in the long run. Thus, Andrew cannot achieve a mass of 80 kg using this plan.	Note that t starts from 0, so do not draw with negative t-values. It is **important** to label the intercepts and the equation of asymptotes.

(ii)	From part **(c)**, it can be observed that for Andrew could achieve a mass of 80 kg, $0 < \frac{C}{30} < 80 \Rightarrow 0 < C < 2400$ The range of possible values of energy intake for which Andrew could achieve a mass of 80 kg is $(0,\ 2400)$	