RAFFLES INSTITUTION
H2 Mathematics 9758
2024 Year 6 Term 1 Revision (Summary)

Topic: Discrete Random Variable, Binomial Distribution

Summary for Discrete Random Variable

Probability Distribution

A table or formula giving the values of $\mathrm{P}(X=x)$ for every x in sample space S is called the **probability distribution** of X.

For example, the experiment of tossing a fair coin 3 times where X is the number of heads obtained, the **probability distribution** of X is as follows:

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The probability distribution of X satisfies the following:

1. $0 \le \mathrm{P}(X=x) \le 1$ for all x in S.
2. $\sum_{x \in S} \mathrm{P}(X=x)=1$ where the summation is over all values of x in S.

Expectation of Discrete Random Variable

The expectation (or mean, or expected value) of a discrete random variable X taking values from a set S is given by $\mathrm{E}(X)=\sum_{x \in S} x\,\mathrm{P}(X=x)$.

In the example above, $\mathrm{E}\left(X\right)=0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=\frac{3}{2}$

Independent Discrete Random Variables

Let X and Y be two discrete random variables taking on possible values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ respectively. The random variables X and Y are said to be **independent** if for all i and j,

$$\mathrm{P}(X=x_i \text{ and } Y=y_j)=\mathrm{P}(X=x_i)\mathrm{P}(Y=y_j).$$

In the example above, if T is the event that the outcome of tossing a coin is head, then

$$\mathrm{P}\left(X=0\right)=\mathrm{P}\left(T_1=0, T_2=0, T_3=0\right)=\mathrm{P}\left(T_1=0\right)\mathrm{P}\left(T_2=0\right)\mathrm{P}\left(T_3=0\right)=\left[\mathrm{P}\left(T=0\right)\right]^3=\left(\frac{1}{2}\right)^3=\frac{1}{8}$$

This is possible given that T_1, T_2 and T_3 are independent.

Functions of a Discrete Random Variable

The expectation of $\mathrm{g}(X)$, where g is a function of X, is denoted by $\mathrm{E}(\mathrm{g}(X)) = \sum_{x \in S} \mathrm{g}(x)\,\mathrm{P}(X = x).$

In particular, $\mathrm{E}(X^2) = \sum_{x \in S} x^2\, \mathrm{P}(X = x).$

Variance and Standard Deviation of a Discrete Random Variable

The variance of a discrete random variable X is given by

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X-\mu)^2 = \sum (x-\mu)^2\, \mathrm{P}(X = x) \\ &= \mathrm{E}\left(X^2\right) - \mu^2 = \left(\sum x^2 \mathrm{P}(X = x)\right) - \mu^2\end{aligned}.$$

In the example above, $\mathrm{Var}(X) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

The standard deviation of X, denoted by σ, is defined as $\sqrt{\mathrm{Var}(X)}$.

Properties of Expectation and Variance

(*Note that these properties hold for discrete and continuous random variables*)

Let X and Y be random variables and a and b be constants. We have

Expectation	Variance
(1) $\mathrm{E}(a) = a$ (2) $\mathrm{E}(aX) = a\,\mathrm{E}(X)$ (3) $\mathrm{E}(aX \pm b) = a\mathrm{E}(X) \pm b$ (4) $\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$	(1) $\mathrm{Var}(a) = 0$ (2) $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$ (3) $\mathrm{Var}(aX \pm b) = a^2\mathrm{Var}(X)$ If X and Y are *independent* random variables, then (4) $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

Important Results

In general, $\mathrm{E}(nX) = n\,\mathrm{E}(X)$

$$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = \mathrm{E}\left(X_1\right) + \mathrm{E}\left(X_2\right) + \ldots + \mathrm{E}\left(X_n\right) = n\,\mathrm{E}\left(X\right)$$

but $\mathrm{Var}(nX) = n^2\, \mathrm{Var}(X)$

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}\left(X_n\right) = n\,\mathrm{Var}(X).$$

Summary for Binomial Distribution

Note that Binomial random variable is a special **discrete random variable**.

Conditions for an experiment to follow binomial distribution

1. It consists of n (fixed number of repeated) independent trials.
2. The outcome of each trial is either a success or a failure.
3. The probability of success for each trial, denoted by p, remains constant.

Note that the above must be stated **in context** of a given question.

For example, a biased coin has probability 0.56 of obtaining a head in any toss. Find the probability of getting 6 heads if the coin is tossed 8 times.

Conditions in context as follow

1. The coin is tossed independently 8 times.
2. The outcome of each toss is either a head (success) or tail (failure).
3. The probability of obtaining a head (success) remains constant at 0.56.

Binomial Distribution

If a discrete random variable X follows a **binomial distribution**, we write $X \sim \mathrm{B}(n, p)$,

where the parameters of the distribution n and p refer to the number of trials and the probability of success for each trial respectively.

The probability distribution of X is given by (in MF27)

$$\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, 2, \ldots, n, \text{ where } \binom{n}{x} = {}^nC_x = \frac{n!}{(n-x)!\,x!}$$

The mean and variance of X are given by

(i) $\mathrm{E}(X) = np$ (in MF27)

(ii) $\mathrm{Var}(X) = np(1-p)$ (in MF27)